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ON AN EMBEDDING OF A CLASS OF SEMIGROUPS INTO RELATIONAL ALGEBRAS

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ABSTRACT. In this paper we give a construction of relational algebras in which finite semigroups which are orthogonal sums of groups with zero adjoined could be embedded. It is proved that if two semigroups are isomorphic, then the corresponding relational algebras are also isomorphic.

1. Terminology, notations and basic definitions are taken from [1], [3] and [4].

Definition 1.1. [4] A relational algebra is an algebra

$$\mathcal{A} = (A, +, \bullet, ^{-}, 0, 1, \circ, 1', ^{-1})$$

of type $(2,2,1,0,0,2,0,1)$ which satisfies the following axioms:

(R1) $(A, +, \bullet, ^{-}, 0, 1)$ is a Boollean algebra;
(R2) $(A, \circ, 1')$ is a monoid;
(R3) Operation $^{-1}$ is an involution of the semigroup $(A, \circ)$, i.e. for all $x, y \in A$,

$$(x \circ y)^{-1} = y^{-1} \circ x^{-1}, \quad (x^{-1})^{-1} = x;$$

(R4) For all $x, y \in A$, $(x+y)^{-1} = x^{-1}+y^{-1}$, $x\circ(y+z) = (x\circ y)+(x\circ z)$;
(R5) For all $x, y \in A$, $(x^{-1} \circ (\overline{x \circ y})) \circ y = 0$.

We denote the class of relational algebras by RA. The Boolean part of relational algebra $\mathcal{A}$ we call the *Boolean reduct* of $\mathcal{A}$, and we denote it by $Rd_B(\mathcal{A})$,. The semigroup part will be denoted by $Rd_S(\mathcal{A})$. Therefore $Rd_B(\mathcal{A}) = (A, +, \bullet, ^{-}, 0, 1)$ and $Rd_S(\mathcal{A}) = (A, \circ)$. The set of all atoms of a relational algebra we will denote by $At(\mathcal{A})$. For a relational algebra $\mathcal{A}$ we say that it is atomic (complete), if the corresponding Boolean reduct is an atomic (complete) Boolean algebra.

Definition 1.2. [4] For a relational algebra $\mathcal{A}$ we say that it is:

1) *commutative*, if $x \circ y = y \circ x$, for all $x, y \in A$;
2) *symmetric*, if $x^{-1} = x$, for all $x \in A$;
3) *Boolean*, if $x \circ y = x \bullet y$, for all $x, y \in A$;
4) *integral*, if $x \circ y \neq 0$, for all $x, y \in A - \{0\}$.

An example of an relational algebra on the set $\mathcal{P}(A^2)$ of all binary relations of a set A is $(\mathcal{P}(A^2), \cup, \cap, ^-, \varnothing, A^2, \circ, \triangle_A, ^{-1})$. This relational algebra is called a *full relational algebra*.

2. Let $(G_\alpha, *_\alpha)$, where $\alpha \in I \neq \varnothing$, be groups for which $G_\alpha \cap G_\beta = \varnothing$, if $\alpha \neq \beta$. Define an operation $\bullet$ on the set:

$$S = \bigcup_{\alpha \in I} G_\alpha^0, \qquad \text{where } G_\alpha^0 = G_\alpha \cup \{0\} \text{ and } 0 \notin \bigcup_{\alpha \in I} G_\alpha \tag{1}$$

in the following way:

$$a \bullet b = \begin{cases} a *_\alpha b, & \text{if } a, b \in G_\alpha, \text{ for some } \alpha \in I \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Then $(S, \bullet)$ is a semigroup. Following the terminology from [1] and [2] we will say that S is an orthogonal sum of semigroups G_α^0, i.e. an orthogonal sum of groups with zero adjoined. The class of all such semigroups will be denoted by KG^0, and by KG_n^0 it will be denoted the subclass of KG^0 consisiting of finite semigroups with exactly $n+1$ elements.

Since every group G is isomorphic to a permutation group of a set (G'), for groups $(G_\alpha, *_\alpha)$ from $S = \bigcup_{\alpha \in I} G_\alpha^0$ we have that for every $\alpha \in I$, $G_\alpha \cong G'_\alpha$, where an isomorphism is given by

$$a \mapsto f_a = \varrho_a = \{(x, a *_\alpha x) | x \in G_\alpha\}.$$

Therefore, to an element a from G_α $(\alpha \in I)$ of a semigroup S, there corresponds a function, i.e. the relation ρ_a in the group G'_α $(\alpha \in I)$. The image of 0 is $\varnothing$, i.e. $\rho_0 = \varnothing$. Now in

$$S' = \bigcup_{\alpha \in I} (G_\alpha \cup \{\varnothing\}) \tag{3}$$

for the operation $\circ$ (the composition of relations) we have that

$$\rho_\alpha \circ \rho_\beta = \begin{cases} \rho_{a *_\alpha b}, & \text{if } a, b \in G_\alpha, \text{ for some } \alpha \in I, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{4}$$

Hence, $(S', \circ)$ is a semigroup, belonging to the class KG^0.

Lemma 2.1. *Let S and S' be semigroups from the class KG^0, given by* (1) *and* (3)*. Then they are isomorphic.*

In the sequel, we will consider finite semigroups from the class KG_n^0. Let $S \in KG_n^0$, i.e. let $S = \bigcup_{\alpha\in I} G_\alpha^0 = \{0, a_1, a_2, a_3, \ldots, a_n\}$ be a finite semigroup.

Let $|I| = k$ $(1 \le k \le n)$. From $S \cong S'$ for semigroup $S \in KG_n^0$ it follows that

$$S' = \bigcup_{\alpha\in I} (G_\alpha \cup \{\varnothing\}) = \{\varnothing, \rho_{a_1}, \rho_{a_2}, \ldots, \rho_{a_n}\}.$$

Let us consider the set

$$A_S = \left\{\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_p}} \middle| \rho_{a_{i_\nu}} \in S', a_{i_\nu} \ne 0, 1 \le \nu \le p, 1 \le p \le n\right\} \cup \{0\}.$$

Elements of the set $S' - \{\varnothing\}$ are called atoms. Therefore, elements of the set A_S are unions of atoms and the empty set.

Theorem 2.1. *Let $S = \bigcup_{\alpha\in I} G_\alpha^0$ from KG_n^0 and let $|I| = k$ $(k \le n)$. Then $(A_S, \circ)$, where $\circ$ is the composition of relations, is a semigroup with unit $1'$, where*

$$1' = \rho_{e_1} \cup \rho_{e_2} \cup \cdots \cup \rho_{e_k},$$

and e_i $(1 \le i \le k)$ are the units of groups G_{a_i} $(1 \le i \le k)$.

Proof. It is clear. □

For the unary operation $^{-1}$ in the semigroup $(S', \circ)$ we take the mapping

$$\rho_a \mapsto \rho_a^{-1}$$

where ρ_a^{-1} is the inverse element of ρ_a in the group G'_α, for some $\alpha \in I$. We take that $\varnothing^{-1} = \varnothing$.

Since the operaton $^{-1}$ is the usual inversion of relations, it can be extended to the semigroup $(A_S, \circ)$ where $(x^{-1})^{-1} = x$, $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$, for all $x, y \in A_S$. Therefore, according to Theorem 2.1, we have that $(A_S, \circ, 1', ^{-1})$ is an involutive semigroup with unit $1'$, where $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$ for all $x, y \in A_S$.

Since the elements of the semigroup A_S are relations, the operations $\cup, \cap, ^{-}$(union, intersection and complement) are defined in A_S. The complement is related to 1, where

$$1 = \rho_{a_1} \cup \rho_{a_2} \cup \cdots \cup \rho_{a_n} \quad \text{(union of all atoms)}.$$

So for $x \in A_S$ where $x = \rho_{b_1} \cup \rho_{b_2} \cup \cdots \cup \rho_{b_s}$ $(s \le n)$, will be $\bar{x} = 1 - x = \rho_{c_1} \cup \rho_{c_2} \cup \cdots \cup \rho_{c_{n-s}}$, where $\rho_{c_\nu} \cap \rho_{b_\mu} = \varnothing$ for all ν $(1 \le \nu \le n - s)$ and all μ $(1 \leq \mu \leq s)$. Especially, $\bar{\varnothing} = 1$ and $\bar{1} = \varnothing$. So we get an algebra $(A_S, \cup, \cap, ^-, \varnothing, 1)$ of type $(2, 2, 1, 0, 0)$.

Let us notice Boolean algebra $(\mathcal{P}(At(A_S)), \cup, \cap, ^-, \varnothing, At(A_S))$ and the mapping $f : \mathcal{P}(At(A_S)) \mapsto A_S$ defined by

$$f : \{\rho_{a_{i_1}}, \rho_{a_{i_2}}, \dots, \rho_{a_{i_r}}\} \mapsto \rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \dots \cup \rho_{a_{i_r}} \text{ and } f(\varnothing) = \varnothing,$$

is an isomorphism. Then

$$(A_S, \cup, \cap, ^-, \varnothing, 1) \tag{5}$$

is also Boolean algebra.

Both Boolean algebras have same number of atoms (n-atoms). Therefore

$$\mathcal{P}(At(A_S)) \cong A_S.$$

Using the operations of the set A_S, we obtain:

Theorem 2.2. *The algebra $\mathcal{A} = (A_S, \cup, \cap, ^-, 1, \circ, 1', ^{-1})$ of type $(2, 2, 1, 0, 0, 2, 0, 1)$ is a relational algebra in which the semigroup $(S, \bullet)$ from the class KG_n^0 is embedded.*

Proof. Let $S = \cup_{\alpha \in I} G_\alpha \cup \{0\} = \{0, a_1, a_2, \dots, a_n\}$ and $|I| = k$ $(1 \leq k \leq n)$. The axioms $(R1), (R2), (R3)$ from Definition 1.1 follow by Theorem 2.1, and the axiom $(R4)$ is satisfied for elements from A_S since they are relations. The only thing left to prove is the axiom $(R5)$.

Let $x = \rho_{a_i}$, $y = \rho_{b_j}$ $(1 \leq i, j \leq n)$ be arbitrary atomc A_S. There are two cases:

Case 1: Let $a_i, a_j \in G_\alpha$ for some $\alpha \in I$. Then

$$x \circ y = \rho_{a_i} \circ \rho_{a_j} = \rho_{a_i *_\alpha a_j} = \rho_{a_l}$$

where $a_l \in G_\alpha$, and hence

$$\overline{x \circ y} = 1 - \rho_{a_1} = \rho_{a_1} \cup \rho_{a_2} \cup \dots \cup \rho_{a_{l-1}} \cup \rho_{a_{l+1}} \cup \dots \cup \rho_{a_n},$$

i.e.

$$x^{-1} \circ (\overline{x \circ y}) = \rho_{a_i^{-1}} \circ (\rho_{a_1} \cup \rho_{a_2} \cup \dots \cup \rho_{a_{l-1}1} \cup \rho_{a_{l+1}} \cup \dots \cup \rho_{a_n}).$$

It follows that

$$\begin{aligned} x^{-1} \circ (\overline{x \circ y}) = \rho_{a_i^{-1}} \circ \rho_{a_1} \cup \rho_{a_i^{-1}} \circ \rho_{a_2} \cup \dots \\ \dots \cup \rho_{a_i^{-1}} \circ \rho_{a_{l-1}} \cup \rho_{a_i^{-1}} \circ \rho_{a_{l+1}} \cup \dots \cup \rho_{a_i^{-1}} \circ \rho_{a_n}. \end{aligned} \tag{6}$$

Now we have two different subcases:

(a) $(1 - \rho_{a_l}) \cap G'_\alpha = \varnothing$.
(b) $(1 - \rho_{a_l}) \cap G_\alpha \neq \varnothing$, where $G'_\alpha \cong G_\alpha$.

In the case (a), $x^{-1} \circ (\overline{x \circ y}) = \emptyset$, since all the members of the union (6) are empty sets, and hence $(x^{-1} \circ (\overline{x \circ y})) \cap y = \emptyset \cap y = \emptyset$.

In the case (b), for every $\rho_u \in (1 - \rho_{a_l} \cap G_\alpha)$, $u \neq a_1$ is satisfied. Hence

$$\rho_{a_1^{-1}} \circ \rho_u = \rho_{a_i^{-1} *_\alpha u} = \rho_{a_{p_u}} \qquad (a_{p_u} \in G_\alpha).$$

Therefore $\rho_{a_{p_u}} \neq \rho_{a_j}$ for all u, because in the opposite case from $\rho_{a_{p_u}} = \rho_{a_j}$ i.e. $\rho_{a_i^{-1} *_\alpha u} = \rho_{a_j}$ it follows that $a_i^{-1} *_\alpha u = a_j$, i.e. $a_i *_\alpha a_j = a_l$, which gives a contradiction. Therefore

$$(x^{-1} \circ (\overline{x \circ y})) \cap y = \emptyset.$$

Case 2: Let $a_i \in G_\alpha, a_j \in G_\beta$ and $\alpha \neq \beta$. Then $x \circ y = \emptyset$, and hence

$$\begin{aligned}(x^{-1} \circ (\overline{x \circ y})) \cap y &= (\rho_{a_i^{-1}} \circ 1) \cap \rho_{a_j} \\ &= (\rho_{a_i^{-1}} \circ \rho_{a_1} \cup \rho_{a_i^{-1}} \circ \rho_{a_2} \cup \cdots \cup \rho_{a_i^{-1}} \circ \rho_{a_n}) \cap \rho_{a_j}.\end{aligned}$$

By (4), the members of the union $\rho_{a_i^{-1}} \circ 1$ are for all $a_m \in G_\alpha$ $(1 \leq m \leq n)$ from G'_α, otherwise they are empty sets. Since $\rho_{a_j} \in G'_\beta$ we have that $\rho_{a_j} \neq \rho_{a_i^{-1} *_\alpha a_m}$ for every m for which $a_m \in G_\alpha$. Therefore, from (7) it follows that $(x^{-1} \circ (\overline{x \circ y})) \cap y = \emptyset$. If at least one of the sets x, y is empty, then $(x^{-1} \circ (\overline{x \circ y})) \cap y = \emptyset$ is obviously satisfied. Hence, the theorem is proved. □

Theorem 2.3. *Let $S \in KG_n^0$ where $S = \bigcup_{\alpha \in I} G_\alpha^0$. Then in corresponding relational algebra $\mathcal{A}_S$ we have that $1'_s = 1_s$ if and only if $|G_\alpha| = 1$, for all $\alpha \in I$.*

Proof. If $S = \{0, a_1, a_2, \ldots, a_n\}$, then the atoms in the relational algebra $\mathcal{A}_S$ are $\rho_{a_1}, \rho_{a_2}, \ldots, \rho_{a_n}$ (by Theorem 2.2). The unit 1_S in $\mathcal{A}_S$ is the union of all the atoms, i.e. $1_S = \rho_{a_1} \cup \rho_{a_2} \cup \cdots \cup \rho_{a_n}$. The unit $1'_S$ of the semigroup reduct of $\mathcal{A}_S$ is $1'_S = \rho_{e_1} \cup \rho_{e_2} \cup \cdots \cup \rho_{e_k}$ $(1 \leq k \leq n)$ where $|I| = k$, and e_i $(1 \leq i \leq k)$ are the units of the groups G_{α_i} $(1 \leq i \leq k)$. Let $|G_{\alpha_i}| = n_i$ $(1 \leq i \leq k)$. Suppose that $1'_S = 1_S$. Then for every i $(1 \leq i \leq k)$ we have $\rho_{e_i} \circ 1'_S = \rho_{e_i} \circ 1_S$ i.e.

$$\rho_{e_i} = \rho_i \circ (\rho_{a_1} \cup \rho_{a_2} \cup \cdots \cup \rho_{a_n}) = \bigcup_{1 \leq i \leq n} (\rho_{e_i} \circ \rho_{a_i}) = \bigcup_{1 \leq \nu \leq n_i} (\rho_{e_i} \circ \rho_{a_{j_\nu}}).$$

Since $G_{\alpha_i} = \{a_{j_1}, a_{j_2}, \ldots, a_{j_{n_1}}\}$ and we also have that $\rho_{e_i} = \rho_{a_{j_1}} \cup \rho_{a_{j_2}} \cup \cdots \cup \rho_{a_{j_{n_i}}}$, whence $n_i = 1$, since ρ_{e_i} is the atom. Therefore $|G_\alpha| = 1$, for all $\alpha \in I$.

Convesely, let $|G_\alpha| = 1$, for all $\alpha \in I$. Then $|I| = n$, i.e. $n = k$, hence $1'_S = \rho_{e_1} \cup \rho_{e_2} \cup \cdots \cup \rho_{e_n}$, i.e. $1'_S$ is the union of all the atoms. Therefore $1'_S = 1_S$. Thus, the theorem is proved. □

Theorem 2.4. *Let $S \in KG_n^0$ where $S = \bigcup_{\alpha \in I} G_\alpha^0$. Then the corresponding relational algebra $\mathcal{A}_S$ is integral if and only if $|I| = 1$.*

Proof. Suppose that $|I| > 1$ and that $\mathcal{A}_S$ is an integral relational algebra, i.e. $x \circ y \neq 0$ for all $x, y \in \mathcal{A}_S$ for which $x \neq 0$ and $y \neq 0$ and $|I| = k > 1$. Then there are at least two different groups G_α and G_β ($\alpha \neq \beta$) in s,which follows from the fact that for all $|G_{\alpha_i}| = n_i$ ($1 \leq i \leq k$) we have that $n_1 + n_2 + \cdots + n_k = n$ and the assumption $k > 1$. Now for $a \in G_\alpha$ and $b \in G_\beta, \rho_a \neq \rho_b$, hence in $\mathcal{A}_S$ we have $\rho_a \circ \rho_b = \varnothing$ which contradicts the assumption that A_S is integral.

Conversely, let $|I| = 1$. Then the semigroup S contains only one group G_α, i.e. $S = G_\alpha \cup \{0\}$, and hence, all the elements $x, y \in A_S$ are the unions of the relations from the group G'_α different from $\varnothing$. Since $x \circ y$ is the union of elements of G'_α we have that $x \circ y \neq 0$, and relational algebra A_S is integral, which was to be proved. □

Theorem 2.5. *Let $S \in KG_n^0$ where $S = \cup_{\alpha \in I} S_\alpha \cup \{0\}$. Then the corresponding relational algebra $\mathcal{A}_S$ is Boolean, if and only if $|I| = n$.*

Proof. Since $(\forall \alpha \in I)(|G_\alpha| = 1)$ is equivalent to $|I| = n$, by the property 2.19.[1] and by Theorem 2.3 we have that the assertion holds. □

3. Let a semigroup S be from the class KG_n^0, and let $\mathcal{A}_S$ be the corresponding relational algebra in which the semigroup S is embedded. Denote the elements of $S = \bigcup_{\alpha \in I} G_\alpha^0$ by $0, a_1, a_2, \ldots, a_n$ and the corresponding atoms in the relational algebra A_S by $\rho_{a_1}, \rho_{a_2}, \ldots, \rho_{a_n}$. Before we give an example of an embedding of a semigroup into relational algebra we introduce the following notations. Denote by $0, 1, 2, \ldots, n$ the elements of a semigroup S and by $i_1 i_2 \ldots i_m$ the element $\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_m}}$ from A_S. With these notations we have

$$i_1 i_2 \ldots i_m \circ j_1 j_2 \ldots j_t = (i_1 \circ j_1)(i_1 \circ j_2) \cdots (i_1 \circ j_t)(i_2 \circ j_1)(i_2 \circ j_2) \cdots \\ \cdots (i_2 \circ j_t) \cdots (i_m \circ j_1)(i_m \circ j_2) \cdots (i_m \circ j_t)$$

where

$$i \circ j = \begin{cases} i *_\alpha j, & \text{if } i, j \in G_\alpha \\ 0, & \text{otherwise.} \end{cases}$$

Now, we give an example of an embedding of a semigroup $S \in KG_n^0$ into relational algebra A_S.

Example. Let a semigroup $S = \bigcup_{\alpha \in I} G_\alpha^0 = \{0, 1, 2, 3, 4\}$, where $G_{\alpha_1} = \{1\}$, $G_{\alpha_2} = \{2, 3\}$, $G_{\alpha_3} = \{4\}$ are groups, be given by Table 1 (The operations of groups G_{α_i} ($1 \leq i \leq 3$) are given in the table, i.e. $x *_\alpha y = x \bullet y$).

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	0	0
2	0	0	2	3	0
3	0	0	3	2	0
4	0	0	0	0	4

TABLE 1

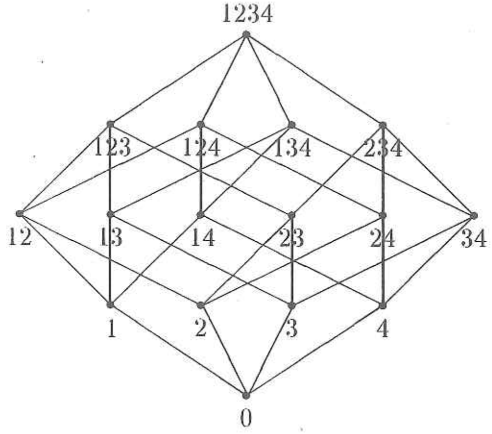

FIGURE 1

Then, for the relational algebra A_S in which the semigroup S is embedded, Boolean reduct is given by the lattice from Figure 1.

The semigroup reduct is presented by the following table

·	0	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
2	0	0	2	3	0	2	3	0	23	2	3	23	2	3	23	23
3	0	0	3	2	0	3	2	0	23	3	2	23	3	2	23	23
4	0	0	0	0	4	0	0	4	0	4	4	0	4	4	4	4
12	0	1	2	3	0	12	13	1	23	2	3	123	12	13	23	123
13	0	1	3	2	0	13	12	1	23	3	2	123	13	12	23	123
14	0	1	0	0	4	1	1	14	0	4	4	1	14	14	4	14
23	0	0	23	23	0	23	23	0	23	23	23	23	23	23	23	23
24	0	0	2	3	4	2	3	4	23	24	34	23	24	34	234	234
34	0	0	3	2	4	3	2	4	23	34	24	23	34	24	234	234
123	0	1	23	23	0	123	123	1	23	23	23	123	123	123	23	123
124	0	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
134	0	1	3	2	4	13	12	14	23	34	24	123	134	124	234	1234
234	0	0	23	23	4	23	23	4	23	234	234	23	234	234	234	234
1234	0	1	23	23	4	123	123	14	23	234	234	123	1234	1234	234	1234

In this algebra, we have that:

$$1'_S = 124, \quad 1_s = 1234.$$

The algebra $\mathcal{A}_S$ is symmetric, since $x^{-1} = x$, for all $x \in \mathcal{A}_S$.

Theorem 3.1. *Let S and S_1 be semigroups from the class KG_n^0. Then the semigroups S and S_1 are isomorphic if and only if the corresponding relational algebras $\mathcal{A}_S$ and $\mathcal{A}_{S_1}$ are isomorphic.*

Proof. Let $S \cong S_1$. Since $S \cong S'$ and $S_1 \cong S_1'$, where $S' \subset \mathcal{A}_S$ and $S_1' \subset \mathcal{A}_{S_1}$, it follows that $S' \cong S_1'$. Let $\varphi : S' \to S_1'$ be an isomorphism. Define a mapping $f : \mathcal{A}_S \to \mathcal{A}_{S_1}$ by

$$f(\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_m}}) = \varphi(\rho_{a_{i_1}}) \cup \varphi(\rho_{a_{i_2}}) \cup \cdots \cup \varphi(\rho_{a_{i_m}}) \tag{8}$$

Since $S' \cong S_1'$, the relational algebras $\mathcal{A}_S$ and $\mathcal{A}_{s1}$ have the same number of atoms, and $Rd_B(\mathcal{A}_S) \cong Rd_B(\mathcal{A}_{S1})$.

Let $x = \rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_r}}$, $y = \rho_{b_{j_1}} \cup \rho_{b_{j_2}} \cup \cdots \cup \rho_{b_{j_s}}$, be arbitrary elements of $\mathcal{A}_S$. Then, by (8), using the isomorphism φ, we have that:

$$\begin{aligned}
f(x \circ y) &= f((\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_r}}) \circ (\rho_{b_{j_1}} \cup \rho_{b_{j_2}} \cup \cdots \cup \rho_{b_{j_s}})) \\
&= f\Big(\bigcup_{\substack{1 \le \mu \le r \\ 1 \le \nu \le s}} (\rho_{a_{i_\mu}} \circ \rho_{b_{j_\nu}})\Big) = \bigcup_{\substack{1 \le \mu \le r \\ 1 \le \nu \le s}} \varphi(\rho_{a_{i_\mu}} \circ \rho_{b_{j_\nu}}) \\
&= \bigcup_{\substack{1 \le \mu \le r \\ 1 \le \nu \le s}} (\varphi(\rho_{a_{i_\mu}}) \circ \varphi(\rho_{b_{j_\nu}})) = \bigcup_{1 \le \mu \le r} \varphi(\rho_{a_{i_\mu}}) \circ \bigcup_{1 \le \nu \le s} \varphi(\rho_{b_{j_\nu}}) \\
&= f(\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_r}}) \circ f(\rho_{b_{j_1}} \cup \rho_{b_{j_2}} \cup \cdots \cup \rho_{b_{j_s}}) \\
&= f(x) \circ f(y).
\end{aligned}$$

It is clear that $f(\varnothing) = \varnothing$. To prove that f is an injection, assume that $x = \rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_r}}$, $y = \rho_{b_{j_1}} \cup \rho_{b_{j_2}} \cup \cdots \cup \rho_{b_{j_s}}$, $r \le s$, and $f(x) = f(y)$. Then

$$f(\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_r}}) = f(\rho_{b_{j_1}} \cup \rho_{b_{j_2}} \cup \cdots \cup \rho_{b_{j_s}}),$$

and hence

$$\varphi(\rho_{a_{i_1}}) \cup \varphi(\rho_{a_{i_2}}) \cup \cdots \cup \varphi(\rho_{a_{i_r}}) = \varphi(\rho_{b_{j_1}}) \cup \varphi(\rho_{b_{j_2}}) \cup \cdots \cup \varphi(\rho_{b_{j_s}}) \tag{9}$$

By (9) for all $\mu (1 \le \mu \le r)$, we obtain that

$$\begin{aligned}
&\varphi(\rho_{a_{i_\mu}}) \cap (\varphi(\rho_{a_{i_1}}) \cup \varphi(\rho_{a_{i_2}}) \cup \cdots \cup \varphi(\rho_{a_{i_r}})) \\
&\qquad = \varphi(\rho_{a_{i_\mu}}) \cap (\varphi(\rho_{b_{j_1}}) \cup \varphi(\rho_{b_{j_2}}) \cup \cdots \cup \varphi(\rho_{b_{j_s}}))
\end{aligned}$$

i.e.

$$(10)\quad \varphi(\rho_{a_{i_\mu}}) = (\varphi(\rho_{a_{i_\mu}}) \cap \varphi(\rho_{b_{j_1}})) \cup (\varphi(\rho_{a_{i_\mu}}) \cap \varphi(\rho_{b_{j_2}})) \cup \dots \cdots \cup (\varphi(\rho_{a_{i_\mu}}) \cap \varphi(\rho_{b_{j_s}})).$$

On the right hand side of relation (10) at least one member of the union is different from the empty set, because in the opposite case $\varphi(\rho_{a_{i_\mu}})$ is not an atom in A_{S1} (which gives a contradiction).

Let $\varphi(\rho_{a_{i_\mu}}) \cap \varphi(\rho_{b_{j_\mu}}) \neq 0$, hence,

$$(11)\qquad \varphi(\rho_{a_{i_\mu}}) = \varphi(\rho_{b_{j_\nu}}).$$

It is clear that for different μ $(1 \leq \mu \leq r)$, we obtain diferent ν $(1 \leq \nu \leq s)$, (11) is satisfied, since φ is an isomorphism. By (9) we have that

$$(12)\quad \varphi(\rho_{a_{i_1}}) \cup \varphi(\rho_{a_{i_2}}) \cup \dots \cup \varphi(\rho_{a_{i_r}}) = (\varphi(\rho_{a_{i_1}}) \cup \varphi(\rho_{a_{i_2}}) \cup \dots \cup \varphi(\rho_{a_{i_r}})) \cup (\varphi(\rho_{b_{j_{r+1}}}) \cup \dots \cup \varphi(\rho_{b_{j_s}}))$$

Therefore, if we make intersection of the right and left hand sides of (12) with $\varphi(\rho_{b_{j_{r+1}}}) \cup \dots \cup \varphi(\rho_{b_{j_s}})$ and since the atoms are differnt, we obtain that

$$\varphi(\rho_{b_{j_{r+1}}}) \cup \dots \cup \varphi(\rho_{b_{j_s}}) = \varnothing.$$

Hence, $r = s$. By (11) we obtain that

$$\rho_{a_{i_\mu}} = \rho_{b_{j_\nu}}, \text{ for all } \mu, \nu \ (1 \leq \mu, \nu \leq r)$$

and hence $x = y$, which means that f is an injection. The mapping f is "onto". Indeed, for an arbitrary $z = \varphi(\rho_{a_1}) \cup \varphi(\rho_{a_2}) \cup \dots \cup \varphi(\rho_{a_m})$, $f(x) = z$ for $x = \rho_{a_1} \cup \rho_{a_2} \cup \dots \cup \rho_{a_m}$.

For $S' = \cup_{a \in I} G'_\alpha \cup \{\varnothing\}$ and $|I| = k$, $(1 \leq k \leq n)$ let $e_i \in G_{\alpha_i}$ $(1 \leq i \leq k)$ be units of groups G_{α_i}, then $1'_s = \rho_{e_1} \cup \rho_{e_2} \cup \dots \cup \rho_{e_k}$. Now

$$f(1') = f(\rho_{e_1} \cup \rho_{e_2} \cup \dots \cup \rho_{e_k}) = \varphi(\rho_{e_1}) \cup \varphi(\rho_{e_2}) \cup \dots \cup \varphi(\rho_{e_k}) = 1'_1.$$

$$\begin{aligned} f(x^{-1}) &= f((\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \dots \cup \rho_{a_{i_r}})^{-1}) = f({\rho_{a_{i_1}}}^{-1} \cup {\rho_{a_{i_2}}}^{-1} \cup \dots \cup {\rho_{a_{i_r}}}^{-1}) \\ &= \varphi({\rho_{a_{i_1}}}^{-1}) \cup \varphi({\rho_{a_{i_2}}}^{-1}) \cup \dots \cup \varphi({\rho_{a_{i_r}}}^{-1}) \\ &= \varphi(\rho_{a_{i_1}})^{-1} \cup \varphi(\rho_{a_{i_2}})^{-1} \cup \dots \cup \varphi(\rho_{a_{i_r}})^{-1} \\ &= (\varphi(\rho_{a_{i_1}}) \cup \varphi(\rho_{a_{i_2}}) \cup \dots \cup \varphi(\rho_{a_{i_r}}))^{-1} \\ &= f(\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \dots \cup \rho_{a_{i_r}})^{-1} = f(x)^{-1}. \end{aligned}$$

$$f(1_S) = f(\rho_{a_{i_1}} \cup \rho_{a_{i_2}} \cup \cdots \cup \rho_{a_{i_n}}) = \varphi(\rho_{a_{i_1}}) \cup \varphi(\rho_{a_{i_2}}) \cup \cdots \cup \varphi(\rho_{a_{i_n}}) = 1_{S_1}.$$

Let us prove that f achieves (8).

It is obvius that for arbitrary $x, y \in \mathcal{A}_S$ it is $f(x \cup y) = f(x) \cup f(y)$. For $x = \rho_{a_{i_1}} \cup \cdots \cup \rho_{a_{i_r}}$ will be

$$f(\bar{x}) = f(\overline{\rho_{a_{i_1}}} \cup \cdots \cup \rho_{a_{i_s}}) = f(\rho_{a_{i_1}} \cup \cdots \cup \rho_{a_{i_{n-s}}})$$

where $\rho_{a_{i_\nu}} \cap \rho_{a_{i_{r\mu}}} = \varnothing$ for all $1 \le \nu \le s$ and $1 \le \mu \le n-s$. From there

$$f(\bar{x}) = \varphi(\rho_{e_{i_1}}) \cup \cdots \cup \varphi(\rho_{e_{i_{nus}}}) = \overline{\varphi(\rho_{a_{i_1}}) \cup \cdots \cup \varphi(\rho_{a_{i_s}})}$$

because of isomorphism φ will be $\varphi(\rho_{e_{i_\nu}}) \cap \varphi(\rho_{e_{i_{mus}}}) = \varnothing$, for every $1 \le \nu \le s$ and $1 \le \mu \le n-s$. So,

$$f(\bar{x}) = \overline{f(\rho_{a_{i_1}} \cup \cdots \cup \rho_{a_{i_s}})} = \overline{f(x)}.$$

Because of $x \cap y = \overline{\bar{x} \cup \bar{y}}$, f is also an isomorphism for $\cap$. Thus, $\mathcal{A}_S \cong \mathcal{A}_{S_1}$.

On the other hand if $f : A_S \to A_{S_1}$ is an isomorphism of algebras $\mathcal{A}_S$ and $\mathcal{A}_{S_1}$, then the restriction $f|_{S'} : S' \to S'_1$ is an isomorphism (since the isomrophism f maps atoms from $\mathcal{A}_S$ in atoms from $\mathcal{A}_{S_1}$), and hence $S \cong S_1$. Thus the theorem is proved. □

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