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A REMARK ON CONVOLUTION POLYNOMIALS

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ABSTRACT. A family of polynomials $\{P_i(x), i = 0, 1, \dots, m\}$ $(m \in \mathcal{N}_0)$ of degree i is a convolution one if satisfies the functional equation

$$\sum_{i=0}^{m} P_{m-i}(x)P_i(y) = P_m(x+y), \tag{1}$$

for every $x, y \in \mathcal{R}$. The generalization of (1) is the functional equation

$$\sum_{i=0}^{m} P_{m-i,i}(a,p,q;x,y) = P_m(a,p+q;x+y), \tag{2}$$

where $P_{j,k}(a,p,q;x,y)$ is polynomial of degree $j+k=m$ in two variables, x and y, and a, p, q are real parameters. The n-dimensional generalization of (1) is

$$\sum_{m_1+\dots+m_n=m} P_{m_1,\dots,m_n}(a,p_1,\dots,p_n;x_1,\dots,x_n) \\ = P_m(a,p_1+\dots+p_n;x_1+\dots+x_n). \tag{3}$$

1. Introduction

In the paper *Convolution Polynomials* [1] D. E. Knuth systematized the known identities regarding convolution polynomials.

The identities involving not only convolution of variables but also convolution of parameters are presented in this paper.

A family of polynomials $\{P_i(x), i = 0, 1, \dots, m\}$ $(m \in \mathcal{N}_0)$ of degree i is a convolution one if satisfies the functional equation (convolution condition)

$$\sum_{i=0}^{m} P_{m-i}(x)P_i(y) = P_m(x+y), \tag{1}$$

for every $x, y \in \mathcal{R}$.

The polynomials $P_m(x) = \dfrac{x^m}{m!}$ and $P_m(x) = \dfrac{(x)_m}{m!}$, which are necessary for further work, satisfy the convolution condition (1).

The generalization of (1) is a functional equation

$$\sum_{i=0}^{m} P_{m-i,i}(a,p,q,x,y) = P_m(a,p+q,x+y), \tag{2}$$

where $P_m(a,p+q,x+y)$ is the polynomial in one variable of degree m and $P_{m-i,i}\ (a,p,q,x,y)$ is the polynomial in two variables, x and y, of degree m, and a,p,q are real parameters. The polynomial $P_{m-i,i}(a,p,q,x,y)$ can not be factorized to two polynomials in x and y, respectively.

The equality (2) is satisfied by two families:

$$\text{(3)}\quad \left\{P_m(a,p+q,x+y) = \frac{1}{m!}G_m(a;p+q;x+y)\right\},$$
$$\left\{P_{m-i,i}(a,p,q,x,y) = \frac{1}{(m-i)!i!}G_{m-i,i}(a;p,q;x,y),\right.$$
$$\left. m = 0,1,\dots,\ i = 0,1,\dots,m\right\}.$$

The monic Gauss hypergeometric polynomial $G_m(a;p+q;x+y)$ in variable $(x+y)$ is defined by Gauss hypergeometric function F in the following way:

$$\text{(4a)}\quad G_m(a;p+q;x+y) := (-1)^m\frac{(p+q)_m}{(a)_m}F(a,-m;p+q;x+y)$$
$$= (-1)^m\frac{(p+q)_m}{(a)_m}\sum_{j=0}^{m}\frac{(-m)_j(a)_j}{(p+q)_j}\frac{(x+y)^j}{j!},$$

i.e.,

$$\text{(4b)}\quad G_m(a;p+q;x+y) := \sum_{j=0}^{m}(-1)^{m+j}\binom{m}{j}\frac{(p+q+j)_{m-j}}{(a+j)_{m-j}}(x+y)^j.$$

The polynomials $G_{m-i,i}(a;p,q;x,y)$ are monic Appell's hypergeometric polynomials in two variables defined by the Appell hypergeometric function F_2

$$\text{(5a)}\quad G_{m-i}(a;p,q;x,y) := (-1)^m\frac{(p)_{m-i}(q)_i}{(a)_m}F_2(a;-m+i,-i;p,q;x,y)$$
$$= (-1)^m\frac{(p)_{m-i}(q)_i}{(a)_m}\sum_{j=0}^{m-i}\sum_{k=0}^{i}\frac{(a)_{j+k}(-m+i)_j(-i)_k}{(p)_j(q)_k}\frac{x^jy^k}{j!k!},$$

i.e.,

(5b) $G_{m-i,i}(a;p,q;x,y) :=$
$$\sum_{j=0}^{m-i}\sum_{k=0}^{i}(-1)^{m+j+k}\binom{m-i}{j}\binom{i}{k}\frac{(p+j)_{m-i-j}(q+k)_{i-k}}{(a+j+k)_{m-j-k}}x^j y^k.$$

The following recurrence relations for monic hypergeometric polynomials in one and two variables, necessary for proving the Theorem 1, are given. Based on Gauss relations [2, pp.3,8]

$$\text{(6)}\qquad \begin{aligned}&\frac{\beta}{\gamma}tF(\alpha+1,\beta+1;\gamma+1;t) = F(\alpha+1,\beta;\gamma;t) - F(\alpha,\beta;\gamma;t),\\ &\alpha F(\alpha+1,\beta;\gamma;t) - \beta F(\alpha,\beta+1;\gamma;t) = (\alpha-\beta)F(\alpha,\beta;\gamma;t),\end{aligned}$$

and putting $\alpha = a$, $\beta = -m-1$, $\gamma = s$, $F(a,-m;s;t) = (-1)^m(a)_m/(s)_m$ $G_m(a;s;t)$, by elimination of $F(a+1,-m-1;s;t)$, one yields the recurrence relation

$$\text{(7)}\qquad G_{m+1}(a;s;t) = tG_m(a+1;s+1;t) - \frac{s+m}{a+m}G_m(a;s;t).$$

Starting from the relations for Appell hypergeometric function F_2 [2, pp.20–21]

$$\text{(8)}\quad \begin{aligned}&\alpha F_2(\alpha+1,\beta,\beta';\gamma,\gamma';x,y) - \beta F_2(\alpha,\beta+1,\beta';\gamma,\gamma';x,y)\\ &\quad - \beta' F_2(\alpha,\beta,\beta'+1;\gamma,\gamma';x,y) = (\alpha-\beta-\beta')F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y),\\ &\qquad \frac{\beta x}{\gamma}F_2(\alpha+1,\beta+1,\beta';\gamma+1,\gamma';x,y)\\ &\qquad\quad + \frac{\beta' y}{\gamma'}F_2(\alpha+1,\beta,\beta'+1;\gamma,\gamma'+1;x,y)\\ &\qquad\quad = F_2(\alpha+1,\beta,\beta';\gamma,\gamma';x,y) - F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y),\end{aligned}$$

by eliminating $F_2(\alpha+1,\beta,\beta';\gamma,\gamma';x,y)$, and by setting $\alpha = a$, $\beta = -m-1+i$, $\beta' = -i$, $\gamma = p$, $\gamma' = q$,

$$G_{m-i,i}(a;p,q;x,y) := (-1)^m\frac{(p)_{m-i}(q)_i}{(a)_m}F_2(a,-m+i,-i;p,q;x,y),$$

the recurrence relation for monic Appell hypergeometric polynomials $G_{m+1-i,i}(a;p,q;x,y)$ of degree $m+1$ as a sum of degrees of variables x and y

$$G_{m+1-i,i}(a;p,q;x,y) = \frac{m+1-i}{m+1} x G_{m-i,i}(a+1;p+1,q;x,y) + \frac{iy}{m+1} G_{m+1-i,i-1}(a+1;p,q+1;x,y) - \frac{(m+1-i)(p+m-i)}{(m+1)(a+m)} G_{m-i,i}(a;p,q;x,y) - \frac{i(q+i-1)}{(m+1)(a+m)} G_{m+1-i,i-1}(a;p,q;x,y). \tag{9}$$

Now, by replacing (3) in (2), one obtains the following

Theorem 1. *Let a,p,q be real numbers $(a,p,q>0)$ and $m \in N_0$. Then*

$$\sum_{i=0}^{m} \binom{m}{i} G_{m-i,i}(a;p,q;x,y) = G_m(a;p+q;x+y). \tag{10}$$

Proof. We implement again the mathematical induction. Using (4b) and (5b) one can prove (10) by simple testing for $m=1,2,3$. Suppose (10) holds for $k=m$, Then we prove that (10) holds for $k=m+1$, i.e.,

$$\sum_{i=0}^{m+1} \binom{m+1}{i} G_{m+1-i,i}(a;p,q;x,y) = G_{m+1}(a;p+q;x+y). \tag{11}$$

Starting from the recurrence relation (9), the left-hand size of (11) becomes

$$\sum_{i=0}^{m+1} \binom{m+1}{i} \Big[\tfrac{m+1-i}{m+1} x G_{m-i,i}(a+1;p+1,q;x,y) + \tfrac{i}{m+1} y G_{m+1-i,i-1}(a+1;p,q+1;x,y) - \tfrac{(m+1-i)(p+m-i)}{(m+1)(a+m)} \times G_{m-i,i}(a;p,q;x,y) - \tfrac{i(q+i-1)}{(m+1)(a+m)} G_{m+1-i,i-1}(a;p,q;x,y) \Big] \tag{12}$$

$$= \sum_{i=0}^{m} \binom{m}{i} \Big[x G_{m-i,i}(a+1;p+1,q;x,y) - \tfrac{(p+m-i)}{(a+m)} G_{m-i,i}(a;p,q;x,y) \Big] + \sum_{i=1}^{m+1} \binom{m}{i-1} \Big[y G_{m+1-i,i-1}(a+1;p,q+1;x,y) - \tfrac{(q+i-1)}{(a+m)} G_{m+1-i,i-1}(a;p,q;x,y) \Big]$$

$$= \sum_{i=0}^{m} \binom{m}{i} \Big[xG_{m-i,i}(a+1;p+1,q;x,y)$$

$$+ yG_{m-i,i}(a+1;p,q+1;x,y) - \tfrac{p+q+m}{a+m} G_{m-i,i}(a;p,q;x,y) \Big]$$

$$= xG_m(a+1;p+q+1;x+y)$$

$$+ yG_m(a+1;p+q+1;x+y) - \tfrac{p+q+m}{a+m} G_m(a;p+q;x+y)$$

$$= (x+y)G_m(a+1;p+q+1;x+y) - \tfrac{p+q+m}{a+m} G_m(a;p+q;x+y).$$

According to (7), the expression (12) is $G_{m+1}(a;p+q;x+y)$, i.e. the right-hand size of (11).

Remark 1. The equality (11) leads to the relation between the orthogonal polynomilas in two and one variable. At first we have the relations between orthogonal and hypergeometric polynomilas.

The monic Jacobi polynomial $\mathcal{P}_m^{(a,b-1)}(t)$ on $[0,1]$ and weight function $t^{b-1}(1-t)^a$ and monic Gauss polynomial $G_m(a+b+m,b,t)$ have a relation

$$\mathcal{P}_m^{(a,b-1)}(t) := \sum_{j=0}^{m} (-1)^{m+j} \binom{m}{j} \frac{(b+j)_{m-j}}{(a+b+m+j)_{m-j}} t^j = G_m(a+b+m;b;t).$$

The monic Appell hypergeometric polynomials $G_{m-i,i}(a+p+q+m;p,q;x,y)$ and basic Appell orthogonal polynomials $E_{m-i,i}(a,p,q;x,y)$ on the triangle $T_2 := \{(x,y) \mid x \geq 0,\, y \geq 0,\, x+y \leq 1\}$ and weight $x^{p-1}y^{q-1}(1-x-y)^a$ are connected by the relation

$$E_{m-i,i}(a,p,q;x,y) = G_{m-i,i}(a+p+q+m;p,q;x,y).$$

The basic orthogonal polynomials $V_{m-i,i}(a,p,q;x,y)$ on the circle $C_2 := \{(x,y) \mid x^2+y^2 \leq 1\}$ and weight function $|x|^p|y|^q(1-x^2-y^2)^a$ and basic Appell orthogonal polynomials $E_{m-i,i}(a,p,q;x;y)$ are connected by the following four equalities:

$$V_{2m-2i,2i}(a,p,q;x,y) = E_{m-i,i}(a,(p+1)/2,(q+1)/2;x^2,y^2),$$
$$V_{2m-2i+1,2i}(a,p,q;x,y) = xE_{m-i,i}(a,(p+3)/2,(q+1)/2;x^2,y^2),$$
$$V_{2m-2i,2i+1}(a,p,q;x,y) = yE_{m-i,i}(a,(p+1)/2,(q+3)/2;x^2,y^2),$$
$$V_{2m-2i+1,2i+1}(a,p,q;x,y) = xyE_{m-i,i}(a,(p+3)/2,(q+3)/2;x^2,y^2).$$

The corollary of Theorem 1 are the following equalities which connect the orthogonal polinomials in two and one variable:

$$\sum_{i=0}^{m}\binom{m}{i}E_{m-i,i}(a,p,q;x,y)=\mathcal{P}_m^{(a,p+q-1)}(x+y),$$

$$\sum_{i=0}^{m}\binom{m}{i}V_{2m-2i,2i}(a,p,q;x,y)=\mathcal{P}_m^{(a,(p+q)/2)}(x^2+y^2),$$

$$\sum_{i=0}^{m}\binom{m}{i}V_{2m-2i+1,2i}(a,p,q;x,y)=x\mathcal{P}_m^{(a,(p+q)/2+1)}(x^2+y^2),$$

$$\sum_{i=0}^{m}\binom{m}{i}V_{2m-2i,2i+1}(a,p,q;x,y)=y\mathcal{P}_m^{(a,(p+q)/2+1)}(x^2+y^2),$$

$$\sum_{i=0}^{m}\binom{m}{i}V_{2m-2i+1,2i+1}(a,p,q;x,y)=xy\mathcal{P}_m^{(a,(p+q)/2+2)}(x^2+y^2).$$

For a family of polynomials $\{P_m(r,x)\}$, where r is real parameter and x is real variable, the particular case of (2) is a functional equation

$$\sum_{i=0}^{m}P_{m-i}(p,x)P_i(q,y)=P_m(p+q,x+y). \tag{13}$$

The polynomial family $\{P_m(r,x)=\frac{1}{m!}G_m(r;x)\}$, where

$$G_m(r;x):=(-1)^m(r)_mF(-m;r;t)=(-1)^m(r)_m\sum_{j=0}^{m}\frac{(-m)_j}{(r)_j}\frac{x^j}{j!}, \tag{14}$$

i.e.,

$$G_m(r;x):=\sum_{j=0}^{m}(-1)^{m+j}\binom{m}{j}(r+j)_{m-j}x^j,$$

satisfies (13).

The monic confluent hypergeometric polynomial in two variables, x,y, of degree m (as a sum of degees in variables x and y) is denoted as

$$G_{m-i,i}(p,q;x,y):=G_{m-i}(p;x)\cdot G_i(q;y), \tag{15}$$
$$(m=0,1,\dots,i=0,1,\dots,m).$$

Replacing the polynomials defined in (14) into the (13), we obtain the following

Theorem 2. *Let p, q be real numbers $(p, q > 0)$ and $m \in N_0$. Then holds*

$$\sum_{i=0}^{m} \binom{m}{i} G_{m-i,i}(p, q; x, y) = G_m(p+q; x+y). \tag{16}$$

Proof. Using the recurrence relation

$$G_{m+1}(s;t) = tG_m(s+2;t) - sG_m(s+1;t) \tag{17}$$

proof is identical of proof of Theorem 1.

Remark 2. The relation among the generalized Laguerre polynomial $L_l^{s-1}(t)$ on $[0, +\infty)$ and weight $t^{s-1}e^{-t}$, generalized Hermite polynomial $H_m(s,t)$ on $(-\infty, +\infty)$ with weight $|t|^p e^{-t^2}$, and confluent hypergeometric polynomial $G_l(s;t)$ is

$$H_m(s,t) = (-1)^m 2^m t^\delta L_l^{(s+2\delta+1)/2-1}(t^2) = 2^m t^\delta G_l((s+2\delta+1)/2; t^2), \tag{18}$$

where $l = [m/2], \delta = m - 2l$. The direct corollary of equality (16) and (18) are the following equalities, for the cases when degrees of both variables are even, then when degree of one variable is odd, and when degrees of both variables are odd, respectively:

$$\sum_{i=0}^{m} \binom{m}{i} H_{2m-2i}(p,x)H_{2i}(q,y) = (-1)^m 2^{2m} L_m^{(p+q)/2}(x^2+y^2), \tag{19}$$

$$\sum_{i=0}^{m} \binom{m}{i} H_{2m+1-2i}(p,x)H_{2i}(q,y) = (-1)^m 2^{2m+1} x L_m^{(p+q)/2+1}(x^2+y^2),$$

$$\sum_{i=0}^{m} \binom{m}{i} H_{2m-2i}(p,x)H_{2i+1}(q,y) = (-1)^m 2^{2m+1} y L_m^{(p+q)/2+1}(x^2+y^2),$$

$$\sum_{i=0}^{m} \binom{m}{i} H_{2m-2i+1}(p,x)H_{2i+1}(q,y) = (-1)^m 2^{2m+2} xy L_m^{(p+q)/2+2}(x^2+y^2).$$

The polynomial generalization

Like

$$\sum_{m_1+\cdots+m_n=m} P_{m_1}(x_1)\cdots P_{m_n}(x_n) = P_m(x_1+\cdots+x_n)$$

generalizes (1), one can define the n-dimensional ($n \geq 2$) generalizations of (2) and (4),

$$(20) \qquad \sum_{m_1+\cdots+m_n=m} P_{m_1,\ldots,m_n}(a,p_1,\ldots,p_n,x_1,\ldots,x_n) = P_m(a,p_1+\cdots+p_n,x_1+\cdots+x_n),$$

and

$$(21) \qquad \sum_{m_1+\cdots+m_n=m} P_{m_1}(p_1,x_1)\cdots P_{m_n}(p_n,x_n) = P_m(p_1+\cdots+p_n,x_1+\cdots+x_n),$$

respectively.

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