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ON WEAK CONGRUENCE MODULAR LATTICES

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ABSTRACT. The main result of the paper is a characterization of weak congruence modular varieties (every algebra of which has a modular lattice of weak congruences). Varieties are supposed to have a nullary operation, and every algebra a one element subalgebra. It is proved that such a variety is weak congruence modular if and only if it is polynomially equivalent to the variety of modules over a ring with unit. Some other characterizations of such varieties and of algebras in these varieties having distributive weak congruence lattices, are also given.

1. Introduction

A variety $\mathcal{V}$ whose similarity type contains a nullary operation 0 and every algebra of which has a one element subalgebra is a 0_1-*variety.* An algebra with a nullary operation 0 is 0-*regular* if $\theta = \phi$ for each $\theta, \phi \in Con\mathcal{A}$, whenever $[0]_\theta = [0]_\phi$. A variety $\mathcal{V}$ with 0 is 0-*regular* if each $\mathcal{A} \in \mathcal{V}$ has this property. A 0_1-variety which is 0-regular is a 0_1-*regular* variety. A single algebra $\mathcal{A}$ with a minimal one element subalgebra A_m is A_m-*regular* if every congruence on $\mathcal{A}$ is uniquely determined by the class containing A_m.

An algebra with 0 is *weakly coherent* if for every subalgebra $\mathcal{B}$ of $\mathcal{A}$ and each $\theta \in Con\mathcal{A}$, if $[0]_\theta \subseteq B$, then $[x]_\theta \subseteq B$ for each $x \in B$. A variety $\mathcal{V}$ is *weakly coherent* if each $\mathcal{A} \in \mathcal{V}$ has this property.

Recall that an algebra $\mathcal{A}$ is *Hamiltonian* if every subalgebra of $\mathcal{A}$ is a class of some congruence on $\mathcal{A}$. A variety $\mathcal{V}$ is *Hamiltonian* if each $\mathcal{A} \in \mathcal{V}$ has this property.

The *weak congruence lattice* $Cw\mathcal{A}$ of an algebra $\mathcal{A}$ is the lattice of all symmetric and transitive subalgebras of $\mathcal{A}^2$, i.e. the lattice of all congruences on all subalgebras of $\mathcal{A}$. The congruence lattice $Con\mathcal{A}$ of $\mathcal{A}$ is the filter

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$[\Delta)$ in that lattice generated by the diagonal relation Δ, and $Sub\mathcal{A}$ is isomorphic with the sublattice (ideal) $(\Delta]$ of all diagonal relations. Because of that isomorphism, subalgebras are usually identified with the corresponding diagonal relations; hence, $Sub\mathcal{A}$ is a sublattice of $Cw\mathcal{A}$. In addition, the congruence lattice of every subalgebra of $\mathcal{A}$ is an interval sublattice of $Cw\mathcal{A}$.

An algebra $\mathcal{A}$ has the *Congruence Intersection Property* (the *CIP*), if for all $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$

$$\rho_A \cap \theta_A = (\rho \cap \theta)_A,$$

where ρ_A is the least congruence on $\mathcal{A}$ whose restriction to B^2 is ρ. In the lattice $Cw\mathcal{A}$ the CIP is usually expressed in the following way: for $\rho, \theta \in Cw\mathcal{A}$,

$$(\rho \wedge \theta) \vee \Delta = (\rho \vee \Delta) \wedge (\theta \vee \Delta).$$

Recall that $\mathcal{A}$ has the *Congruence Extension Property* (the *CEP*) if for every congruence ρ on a subalgebra of $\mathcal{A}$ there is a congruence on $\mathcal{A}$ collapsing ρ.

$\mathcal{A}$ has the *Strong Congruence Extension Property* (*SCEP*) if for every $\mathcal{B} \in Sub\mathcal{A}$ and $\rho \in Con\mathcal{B}$ there is $\theta \in Con\mathcal{A}$ such that $[b]_\rho = [b]_\theta$ for every $b \in B$. A variety $\mathcal{V}$ has the SCEP if every $\mathcal{A} \in \mathcal{V}$ has this property.

A variety $\mathcal{V}$ is *weak-congruence modular* (Cw-modular) if the weak congruence lattice of every $\mathcal{A} \in \mathcal{V}$ is modular.

Let $\mathcal{E}$ be some lattice identity

$$p(x_1, ..., x_n) = q(x_1, ..., x_n).$$

We say that $\mathcal{E}$ *implies modularity* if every lattice satisfying $\mathcal{E}$ is modular. Similarly, $\mathcal{E}$ *implies distributivity* if every lattice satisfying $\mathcal{E}$ is distributive.

For more details about weak coherence, 0-regularity and SCEP, see [2,3], and for some other properties of weak congruences see [7,8,9].

2. Results

It is obvious that a variety $\mathcal{V}$ with exactly one nullary operation 0 in its similarity type is a 0_1-variety if and only if the identity

$$f(0, ..., 0) = 0 \qquad (1)$$

holds in $\mathcal{V}$ for every n-ary operational symbol f. Another characterization is the following.

Lemma 1. *A variety $\mathcal{V}$ with a single nullary operation* 0 *is a 0_1-variety if and only if there exists at most unary term g such that for every n-ary term f the identity*

$$f(g(x), \dots, g(x)) = g(x) \qquad (2)$$

holds in $\mathcal{V}$.

Proof. Let $\mathcal{V}$ be a 0_1-variety. Then (1) holds, and the term $g(x) \equiv 0$ satisfies the requirement (2).

On the other hand, if (2) holds, then by Theorem 9 in [1], every congruence on an algebra $\mathcal{A}$ in $\mathcal{V}$ has a class which is a subalgebra of $\mathcal{A}$, and since there is a constant in $\mathcal{A}$, this class is unique. The diagonal relation then provides a one element subalgebra. □

Corollary 1. *A variety $\mathcal{V}$ with 0 is a 0_1-variety if and only if every congruence on an algebra $\mathcal{A}$ in $\mathcal{V}$ has exactly one class which is a subaglebra of $\mathcal{A}$.*

Lemma 2. *(Theorem 1 in [4]) If the lattice of subalgebras of every free algebra in a variety $\mathcal{V}$ is modular, then $\mathcal{V}$ is Hamiltonian.*

Theorem 1. *Let $\mathcal{V}$ be a 0_1-regular variety. If for each $\mathcal{A} \in \mathcal{V}$ the lattice $Sub\mathcal{A}$ satisfies an identity $\mathcal{E}$ which implies modularity, then $Cw\mathcal{A}$ satisfies $\mathcal{E}$.*

Proof. If $Sub\mathcal{A}$ satisfies $\mathcal{E}$ for each $\mathcal{A} \in \mathcal{V}$, then also $Sub\mathcal{F}$ is modular for every free algebra $\mathcal{F}$ of $\mathcal{V}$. By Lemma 2, $\mathcal{V}$ is Hamiltonian. Since $\mathcal{V}$ is 0_1-regular and Hamiltonian, then also each $\mathcal{A} \in \mathcal{V}$ has this property, and by (ii), Theorem 18 in [7], $Con\mathcal{A} \cong Sub\mathcal{A}$. Hence, both $Con\mathcal{A}$ and $Sub\mathcal{A}$ satisfy $\mathcal{E}$. By (iii), Theorem 18 in [7], $\mathcal{A}$ has the CEP and the CIP. Using Theorem 3 in [8], we conclude that also $Cw\mathcal{A}$ satisfies $\mathcal{E}$ for every $\mathcal{A} \in \mathcal{V}$. □

The proof of the preceding theorem shows that the same argument (the use of Theorem 18 in [7] and Theorem 3 in [8]) can be used for a single algebra $\mathcal{A}$ in any variety satisfying $\mathcal{E}$, provided that $\mathcal{A}$ is A_m-regular (in which case it has a unique minimum one element subalgebra, but there should be no constants in the similarity type of $\mathcal{V}$). $\mathcal{E}$ may also imply distributivity.

Theorem 2. *If for each $\mathcal{A}$ in a variety $\mathcal{V}$ the lattice $Sub\mathcal{A}$ satisfies an identity $\mathcal{E}$ which implies modularity (distributivity), then the weak congruence lattice of every A_m-regular algebra in $\mathcal{V}$ also satisfies $\mathcal{E}$.*

Proof. If $\mathcal{A}$ is an A_m-regular algebra in $\mathcal{V}$, then it is Hamiltonian (by Lemma 2), has the CEP and the CIP, and thus, as above, $Sub\mathcal{A} \cong Con\mathcal{A}$. By Theorem 3 in [8], $\mathcal{E}$ holds in $Cw\mathcal{A}$, since it is satisfied on both, $Con\mathcal{A}$ and $Sub\mathcal{A}$. □

By the definition given in Introduction, an algebra $\mathcal{A}$ is weakly coherent whenever every its subalgebra is a union of congruence classes provided it contains a congruence class containing 0. This will be used in the following theorem.

Theorem 3. *Let $\mathcal{V}$ be a Hamiltonian 0_1-variety which is weakly coherent. Then $Cw\mathcal{A}$ is Arguesian (and hence modular) for every $\mathcal{A}$ in $\mathcal{V}$.*

Proof. If $\mathcal{V}$ is weakly coherent then, by Corollary 1 in [2] and by the assumption, $\mathcal{V}$ is 0_1-regular. Since $\mathcal{V}$ is also Hamiltonian, $Sub\mathcal{A} \cong Con\mathcal{A}$ for each $\mathcal{A} \in \mathcal{V}$ directly by Theorem 18 in [7]. By Corollary 2 in [2], $\mathcal{V}$ has permutable congruences and thus, by the famous Jónsson result, $Con\mathcal{A}$ is Arguesian (end hence modular) for each $\mathcal{A} \in \mathcal{V}$. So also $Sub\mathcal{A}$ is Arguesian. By (iii), Theorem 18 in [7], $\mathcal{A}$ satisfies the CIP and the CEP. Using again Theorem 3 in [8], we get that $Cw\mathcal{A}$ is Arguesian for each $\mathcal{A} \in \mathcal{V}$. □

Thus we have obtained some sufficient conditions under which a 0_1-variety is Cw-modular. In the following, we give also the necessary conditions, and we characterize algebras in these varieties with distributive lattices of weak congruences.

First we advance some known results.

Lemma 3. *(Theorem 2.9 in [9]) For an algebra $\mathcal{A}$, $Cw\mathcal{A}$ is a modular lattice if and only if $Con\mathcal{A}$ and $Sub\mathcal{A}$ are modular and $\mathcal{A}$ has the CEP and the CIP.*

Lemma 4. *([3])*

a) *A variety $\mathcal{V}$ has the SCEP if and only if it is Hamiltonian.*
b) *An algebra $\mathcal{A}$ has the SCEP if and only if $\mathcal{A}$ is Hamiltonian and has the CEP.*

If $\mathcal{A}$ is a Hamiltonian algebra with a one element subalgebra, and each subalgebra of $\mathcal{A}$ is 0-regular, then by Theorem 18 in [7] $\mathcal{A}$ has both, CEP and CIP. This result will be used in a characterization of $0_1 - Cw$-modular varieties. For a single algebra, similar problems are solved in the following.

Proposition 1. *Let $\mathcal{A}$ be a Hamiltonian 0-regular algebra which has the CEP. Then,*

a) *$\mathcal{A}$ has the CIP; and*
b) *$\mathcal{A}$ is weakly coherent.*

Proof. a) Let $\rho,\theta \in Cw\mathcal{A}$, $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$, $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$. Then by the CEP, in the lattice $Cw\mathcal{A}$

$$((\rho \wedge \theta) \vee \Delta) \wedge (B \wedge C)^2 = \rho \wedge \theta \text{ and}$$

$$((\rho \vee \Delta) \wedge (\theta \vee \Delta)) \wedge (B \wedge C)^2 = (\rho \vee \Delta) \wedge B^2 \wedge (\theta \vee \Delta) \wedge C^2 = \rho \wedge \theta.$$

By Lemma 4 b), $\mathcal{A}$ has the SCEP, since it is Hamiltonian and has the CEP. Hence, both $(\rho \wedge \theta) \vee \Delta$ and $(\rho \vee \Delta) \wedge (\theta \vee \Delta)$ have the same blocks as $\rho \wedge \theta$. By 0-regularity then

$$(\rho \wedge \theta) \vee \Delta = (\rho \vee \Delta) \wedge (\theta \vee \Delta),$$

and the CIP holds.

b) Let $\mathcal{B} \in Sub\mathcal{A}$, $\theta \in Con\mathcal{A}$, and B contains $[0]_\theta$. Now, $(B^2 \wedge \theta) \vee \Delta$ and θ have the same blok $[0]_\theta$, since $\mathcal{A}$ has the SCEP. By 0-regularity then $(B^2 \wedge \theta) \vee \Delta = \theta$. Again by the SCEP $[0]_\theta \subseteq B$ for each $x \in B$, and $\mathcal{A}$ is weakly coherent. □

Now we can give a characterization theorem for 0_1-Cw-modular varieties.

Theorem 4. *The following are equivalent for a 0_1-variety $\mathcal{V}$:*

(i) *$\mathcal{V}$ is weak congruence modular;*
(ii) *$\mathcal{V}$ is subalgebra modular and 0-regular;*
(iii) *$\mathcal{V}$ is Hamiltonian and 0-regular;*
(iv) *$\mathcal{V}$ is Hamiltonian and weakly coherent;*
(v) *$\mathcal{V}$ is polynomially equivalent to the variety of modules over a ring with unit.*

Proof. (i)$\Longrightarrow$(v). If $\mathcal{A}$ belongs to a Cw-modular variety, then $Sub\mathcal{A}^2$ is a modular lattice and thus (see [4]) $\mathcal{A}^2$ is Hamiltonian. $Con\mathcal{A}$ is also modular and hence, by (vi), Theorem 5.5 in [5], $\mathcal{A}$ is polynomially equivalent to a module over a ring with unit.

(v)$\Longrightarrow$(i). If (v) holds and $\mathcal{A} \in \mathcal{V}$, then $\mathcal{A}$ is 0-regular and $Con\mathcal{A}$ is a modular lattice. By Theorem 1 in [4], $\mathcal{A}^2$ is Hamiltonian, and by the result of E. Kiss ([6]), $\mathcal{A}$ has the CEP. By the similar argument $\mathcal{A}^2$ also has the CEP, and by a), Proposition 1, $\mathcal{A}^2$ has the CIP. In the presence of the CEP, the CIP is hereditary for subalgebras (Corollary 3 in [8]). Hence, $\mathcal{A}$ has the CIP as well (since $\mathcal{A}$ is, up to the isomorphism, a subalgebra of $\mathcal{A}^2$). By Theorem 18 in [7], $Sub\mathcal{A} \cong Con\mathcal{A}$ and $Sub\mathcal{A}$ is also modular. By Lemma 3, $Cw\mathcal{A}$ is a modular lattice.

(ii)$\Longrightarrow$(i). By Theorem 1.

(iv)$\Longrightarrow$(i). By Theorem 3.

(iii)$\Longrightarrow$(iv). By b), Proposition 1.

(v)$\Longrightarrow$(ii). Similarly to (v)$\Longrightarrow$(i), since $\mathcal{A}$ is 0-regular, $Con\mathcal{A}$ is modular, and $Sub\mathcal{A} \cong Con\mathcal{A}$. Hence, $Sub\mathcal{A}$ is modular.

(ii)$\Longrightarrow$(iii). By Lemma 2. □

A consequence of Theorem 17 in [7] is that in the above characterized weak congruence modular varieties, congruence and subalgebra lattices of every algebra in the variety are isomorphic, and the algebra has the CEP and the CIP. Under these conditions, by already used Theorem 3 in [8], lattice identities satisfied on $Con\mathcal{A}$ and $Sub\mathcal{A}$, also hold on $Cw\mathcal{A}$. By these arguments, it is possible to discuss the weak congruence distributivity for algebras in Cw-modular varieties.

Theorem 5. *Let $\mathcal{V}$ be a 0_1-Cw-modular variety. Then, the following are equivalent for an algebra $\mathcal{A} \in \mathcal{V}$:*

(i) *$Cw\mathcal{A}$ is distributive;*
(ii) *$Sub\mathcal{A}$ is distributive;*
(iii) *$Con\mathcal{A}$ is distributive.*

Proof. If $Cw\mathcal{A}$ is distributive, then obviously (ii) and (iii) hold. On the other hand, if $Sub\mathcal{A}$ or $Con\mathcal{A}$ are distributive, then by the above argument, since the variety is weak congruence modular ($Sub\mathcal{A} \cong Con\mathcal{A}$, $\mathcal{A}$ has the CIP and the CEP), the distributivity is transfered to the weak congruence lattice. □

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