FILOMAT (Niš) 9:3 (1995), 619–632
Algebra, Logic & Discrete Mathematics
Niš, April 14–16, 1995.

A CONNECTION BETWEEN CUT ELIMINATION AND NORMALIZATION

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ABSTRACT. Sequent systems for classical and intuitionistic logic and natural deduction systems for these logics are characterized by two important theorems. Sequent systems are characterized by cut- elimination theorems, and natural deduction systems by normalization theorems. In this paper, by means of multicategories and the typed λ-calculus we exhibit some similarities and differences between cut elimination and normalization. We consider the sequent system and the natural deduction system for intuitionistic propositional logic. We define a multicategory corresponding to the sequent system. On the other hand, a typed λ-calculus corresponds to the natural-deduction system. We show how to form a typed λ-calculus out of a multicategory and vice-versa. In some kinds of multicategory, some equations necessary for cut elimination, are not necessary for normalization.

Introduction

In this paper we shall consider Gentzen's sequent system and his system of natural deduction for propositional intuitionistic logic. We shall investigate the connection between cut elimination in the sequent system and normalization in natural deduction.

In their papers Zucker and Pottinger have already described this connection, in a certain way. In this paper this will be done by linking multicategories and typed λ-calculi.

Certain multicategories will correspond to Gentzen's sequent system for intuitionistic propositional logic. Objects of these multicategories will be formulas and operations on objects will be logical connective. We shall take the arrows as proofs and operations on arrows will correspond to inference

I would like to express my gratitude to Professor Kosta Došen for many useful and important comments and advices he made on my manuscript.

rules. The arrow $\Gamma \to A$, which way have been constructed from other arrows by applying operations on arrows, will correspond to a particular derivation of the sequent $\Gamma \vdash A$.

Equations between arrows in multicategories equate arrows which correspond to derivations with the same end-sequent. On the basis of the equality $f = g$ we shall be able to transform the derivation corresponding to the arrow f into the derivation corresponding to the arrow g. Equations between arrows which we shall assume will make cut elimination possible in some multicategories; for example, in multicategories with axioms which are closed for cut. In these multicategories equations can be explained in the following way: there will be two kinds of equation. In the equations of the first kind the arrows on the left-hand side will correspond to a derivation in which cut has to be eliminated and the arrows on the right-hand side will correspond to a derivation in which cuts are eliminated or of smaller degree than the cut on the left-hand side. Equations of the second kind equate arrows which are constructed by application of the same operations on the same arrows and the only difference is in the order of application of these operations.

A typed λ-calculus will correspond to the system of natural deduction of intuitionistic propositional logic. Types of the λ-calculus will correspond to formulas and a term of type A will be considered as a derivation with the end-formula A. We shall define equations on terms that will correspond to the steps of reduction leading to proofs in normal form. We shall postulate equations on terms that will represent reductions in natural deduction such that the middle part of the proof is made of atomic formulas.

In Section 1 four kinds of multicategory will be defined; they differ in equations their arrows. In MG-multicategories with axioms which are closed for cut equations will represent steps of transformation of a derivation into a derivation where no cut appears. In the other kinds of multicategory, called MGI, MN and MNI, we shall require other equations on arrows. The equations of MN and MNI will be closer to normalization of proofs.The polynomial multicategory $\mathcal{M}[X]$ will be formed out of a multicategory a $\mathcal{M}$ in a standard way (cf.[2]). Functional completness will be proved for all those multicategories.

In Section 2 two kinds of typed λ-calculus will be defined. One kind of typed λ-calculus will have equations between terms that correspond to reduction steps in the normalization of proofs. The other kind of typed λ -calculus will have added equations by which formulas of the middle part of the proof are broken into their atomic subformulas.

In Section 3 we shall define how typed λ-calculi can be formed out of objects and arrows of multicategories and vice versa. We shall be able to form only an MN-multicategory and an MNI-multicategory out of the given typed λ-calculus. Then it will be possible to separate some equations which are needed for cut elimination in some extensions of Gentzen's sequent system, but are not needed for normalizing proofs in these systems.

1. Multicategories

Definition 1.1. A **multigraph** is made of a class of objects and a class of arrows together with two mappings: source: $\{\text{arrows}\} \to \{\text{objects}\}^*$, target: $\{\text{arrows}\} \to \{\text{objects}\}$, where $\{\text{objects}\}^*$ is the free monoid generated by the class of objects; $f : \Gamma \to A$ is an arrow, where $\Gamma = A_1 ... A_n$ is string of objects (our *arrows* are sometimes called *multiarrows*).

Definition 1.2. A **context-free recognition grammar** is a multigraph with operations on objects: $\wedge$, $\vee$ and $\Rightarrow$; special object 0. We also have operations on arrows:
structural rules:

$$\frac{f{:}\Gamma AB\Delta \to C}{p_{A,B}(f){:}\Gamma BA\Delta \to C} \quad \text{permutation}$$

$$\frac{f{:}\Gamma \to C}{t_A(f){:}A\Gamma \to C} \quad \text{thinning}$$

$$\frac{f{:}AA\Gamma \to C}{c_A(f){:}A\Gamma \to C} \quad \text{contraction}$$

$$\frac{f{:}\Gamma \to A \quad g{:}A\Delta \to C}{g\langle f\rangle{:}\Gamma\Delta \to C} \quad \text{cut}$$

connective rules:

MI $\dfrac{f{:}A\Gamma \to C}{f_p{:}A\wedge B\Gamma \to C}$ $\qquad \dfrac{g{:}B\Gamma \to C}{g_{p'}{:}A\wedge B\Gamma \to C}$

MII $\dfrac{f{:}\Gamma \to A \quad g{:}\Gamma \to B}{\langle f,g\rangle{:}\Gamma \to A\wedge B}$

MIII $\dfrac{f{:}A\Gamma \to C \quad g{:}B\Gamma \to C}{[f,g]{:}A\vee B\Gamma \to C}$

MIV $\dfrac{f{:}\Gamma \to A}{{}_k f{:}\Gamma \to A\vee B}$ $\qquad \dfrac{g{:}\Gamma \to B}{{}_{k'} g{:}\Gamma \to A\vee B}$

MV $\dfrac{f{:}\Gamma \to \quad g{:}B\Delta \to C}{g[f]{:}A\Rightarrow B\Gamma\Delta \to C}$

MVI $\dfrac{f{:}A\Gamma \to B}{f^*{:}\Gamma \to A\Rightarrow B}$

and special arrows:
MVII $1_A : A \to A$, for each object A from the class of objects.
MVIII $\square^A : 0 \to A$, for each object A from the class of objects.

Definition 1.3.

1. Let $f : \Gamma\Delta\Lambda\Theta \to A$ and $\Delta = C_1 ... C_n$, $\Lambda = D_1 ... D_m$, then $p_{\Delta,\Lambda}(f) =_{def.} p_{C_1,D_m}(...(p_{C_{n-1},D_1}(p_{C_n,D_m} \cdots (p_{C_n,D_1}(f))...))...)$ and $p_{\Delta,\Lambda}(f) : \Gamma\Lambda\Delta\Theta \to A$.

2. Let $f : \Gamma \to A$ and $\Delta = B_1 ... B_n$; then $t_\Delta(f) =_{def.} t_{B_1}(...(t_{B_m}(f))..)$ and $t_\Delta(f) : \Delta\Gamma \to A$.

3. Let $f : \Delta\Delta\Gamma \to A$ and $\Delta = B_1 ... B_n$,
$h : B_i B_{i+1} ... B_n B_i B_{i+1} ... B_n B_1 ... B_{i-1}\Gamma \to A$, $\Delta_i = B_{i+1} ... B_n$, $\Delta_n = \emptyset$, $B_0 = B_{n+1} = \emptyset$, where $\emptyset$ is the empty string and $1 \le i \le n$; then

$c^{B_i}(h) =_{def} p_{B_i,\Delta_i\Delta_i B_1 \ldots B_{i-1}}(c_{B_1}(p_{B_{i+1},B_1}(\ldots(p_{B_n,B_i}(h))\ldots)))$ and
$c_\Delta(f) =_{def} c^{B_n}(c^{B_{n-1}}(\ldots(c^{B_1}(f))\ldots)))$ and
$c_\Delta(f) : \Delta\Gamma \to A$.

Definition 1.4. A **multicategory** is a context-free recognition grammar in which the following equations on arrows hold:

M1. $f\langle 1_A\rangle = f$, where $f : A\Gamma \to B$, $1_A : A \to A$

M2. $1_B\langle g\rangle = g$, where $g : \Delta \to B$, $1_B : B \to B$

M3. $h\langle g\langle f\rangle\rangle = h\langle g\rangle\langle f\rangle$, where $h : B\Lambda \to C$, $g : A\Delta \to B$, $f : \Gamma \to A$

M4. $p_{\Lambda,\Gamma}(p_{\Gamma,B}(h\langle f\rangle)\langle g\rangle) = p_{\Lambda,A}(p_{A,B}(h)\langle g\rangle)\langle f\rangle$,

where $h : AB\Delta \to C$, $g : \Lambda \to B$, $f : \Gamma \to A$

Now we shall define four kinds of multicategory. Each of them will have five families of equations which hold for their arrows.

Definition 1.5. An **MG-multicategory** (M is for "multicategory" and G is for "Gentzen") is a multicategory in which the following families of equations hold:

I PTC-equations:

PP1. $p_{B,A}(p_{A,B}(f)) = f$, for all $f : \Gamma AB\Delta \to C$.

PP2. $p_{A,B}(p_{C,D}(f)) = p_{C,D}(p_{A,B}(f))$, for all $f : \Gamma AB\Delta CD\Lambda \to E$.

PP3. $p_{A,B}(p_{A,C}(p_{B,C}(f))) = p_{B,C}(p_{A,C}(p_{A,B}(f)))$, for all $f : \Gamma ABC\Delta \to C$.

PT1. $p_{A,B}(t_C(f)) = t_C(p_{A,B}(f))$, for all $f : \Gamma AB\Delta \to D$.

PC1. $c_C(p_{A,B}(f)) = p_{A,B}(c_C(f))$, for all $f : CC\Delta AB\Lambda \to D$.

PC2. $c_C(p_{C,C}(f)) = c_C(f)$, for all $f : CC\Delta \to D$.

TC1. $p_{C,B}(t_C(c_B(f))) = c_B(p_{C,BB}(t_C(f)))$, for all $f : BB\Delta \to D$.

TC2. $c_B(t_B(f)) = f$, for all $f : \Delta \to D$.

CC1. $c_A(p_{B,AA}(c_B(p_{AA,BB}(f)))) = p_{B,A}(c_B(p_{A,BB}(c_A(f))))$, for all $f : AABB\Delta \to D$.

Derivations in this family of equations differ only in the order of appearance of the structural rules: of contraction, permutation and thinning.

II CUT-PTC-equations:

CUTP1. $p_{B,C}(g)\langle f\rangle = p_{B,C}(g\langle f\rangle)$, for all $f : \Gamma \to A$, $g : A\Delta BC\Lambda \to D$.

CUTP2. $g\langle p_{C,D}(f)\rangle = p_{C,D}(g\langle f\rangle)$, for all $f : \Gamma CD\Delta \to A$, $g : A\Lambda \to D$.

CUTT1. $t_C(g\langle f\rangle) = p_{\Gamma,C}(p_{C,A}(t_C(g))\langle f\rangle)$, for all $f : \Gamma \to A$, $g : A\Delta \to D$.

CUTT2. $g\langle t_C(f)\rangle = t_C(g\langle f\rangle)$, for all $f : \Gamma \to A$, $g : A\Delta \to D$.

CUTT3. $t_A(g)\langle f\rangle = t_\Gamma(g)$, for all $f : \Gamma \to A$, $g : A\Delta \to D$.

CUTC1. $p_{C,A}(c_C(g))\langle f\rangle = p_{C,\Gamma}(c_C(p_{\Gamma,CC}(p_{CC,A}(g)\langle f\rangle)))$,

for all $f : \Gamma \to A$, $g : CCA\Delta \to B$.

CUTC2. $g\langle c_C(f)\rangle = c_C(g\langle f\rangle)$, for all $f : CC\Gamma \to A$, $g : A\Delta \to B$.

CUTC3. $c_A(g)\langle f\rangle = c_\Gamma(p_{\Gamma,A}(g\langle f\rangle)\langle f\rangle)$, for all $f : \Gamma \to A$, $g : AA\Delta \to B$.

Derivations in this family of equations differ in the order of appearance of cut and other structural rules. For example, in one derivation cut will appear first and permutation will follow and in the other permutation will precede cut. However, in CUTC3 we replace one cut that comes after contraction on the left by two cuts preceding contraction on the right, and in CUTT3 we replace one cut that comes after thinning on the left by zero cuts on the right.

III IMPORTANT CUTS equations:

ICUT1. $g_p\langle\langle f,h\rangle\rangle = g\langle f\rangle$, for all $g : A\Gamma \to C$, $f : \Delta \to A$, $h : \Delta \to B$.

ICUT2. $g_{p'}\langle\langle f,h\rangle\rangle = g\langle h\rangle$, for all $g : B\Gamma \to C$, $f : \Delta \to A$, $h : \Delta \to B$.

ICUT3. $\langle f,h\rangle\langle g\rangle = \langle f\langle g\rangle, h\langle g\rangle\rangle$, for all $g : \Delta \to C$, $f : C\Gamma \to A$, $h : C\Gamma \to B$.

ICUT4. $[f,h]\langle {}_k g\rangle = f\langle g\rangle$, for all $g : \Gamma \to A$, $f : A\Delta \to C$, $h : B\Delta \to C$.

ICUT5. $[f,h]\langle {}_{k'} g\rangle = h\langle g\rangle$, for all $g : \Gamma \to B$, $f : A\Delta \to C$, $h : B\Delta \to C$.

ICUT6. $f[h]\langle g^*\rangle = p_{\Lambda,\Delta}(f\langle g\rangle\langle h\rangle)$, for all $g : A\Delta \to B$, $f : B\Gamma \to C$, $h : \Lambda \to A$.

ICUT7. $h^*\langle g\rangle = (p_{\Delta,A}(p_{A,C}(h)\langle g\rangle))^*$, for all $g : \Delta \to C$, $h : AC\Gamma \to B$.

ICUT8. $f\langle\Box^A\rangle = \Box^B$, for all $\Box^A : 0 \to A, \Box^B : 0 \to B$ and $f : A \to B$.

By equations of this family the following derivations will be equated:

1. a derivation with a cut and a derivation with one or more cuts whose cut formulas are subformulas of the cut formula of the first cut;
2. derivations which differ in the order of application of a conecctive rule and a cut.

IV TROUBLESOME EQUATION: This family consists of a single equation:

T.E. $h\langle[f,g]\rangle = [h\langle f\rangle, h\langle g\rangle]$, for all $h : C\Delta \to D$, $f : A\Gamma \to C$, $g : B\Gamma \to C$.

V PTC-CUT-CON-equations:

P∧1. $p_{A,B}(g_p) = (p_{A,B}(g))_p$, for all $g : C\Delta AB\Lambda \to E$.

P∧2. $p_{A,B}(g_{p'}) = (p_{A,B}(g))_{p'}$, for all $g : D\Delta AB\Lambda \to E$.

P∧3. $p_{A,B}(\langle f,g\rangle) = \langle p_{A,B}(f), p_{A,B}(g)\rangle$, for all $f : \Gamma AB\Delta \to C$, $g : \Gamma AB\Delta \to D$.

P∨1. $p_{A,B}([f,g]) = [p_{A,B}(f), p_{A,B}(g)]$, for all $f : C\Gamma AB\Delta \to E$, $g : D\Gamma AB\Delta \to E$.

P∨2. $p_{A,B}({}_k g) = {}_k(p_{A,B}(g))$, for all $g : \Gamma AB\Delta \to C$.

P∨3. $p_{A,B}({}_{k'} g) = {}_{k'}(p_{A,B}(g))$, for all $g : \Gamma AB\Delta \to D$.

P⇒1. $p_{A,B}(g[f]) = p_{A,B}(g)[f]$, for all $g : D\Delta AB\Lambda \to E$, $f : \Gamma \to C$.

P⇒2. $p_{A,B}(g[f]) = g[p_{A,B}(f)]$, for all $g : D\Lambda \to E$, $f : \Gamma AB\Delta \to C$.

P⇒3. $p_{A,B}(g^*) = (p_{A,B}(g))^*$, for all $g : C\Gamma AB\Delta \to D$.

T∧1. $p_{C,A\wedge B}(t_C(g_p)) = (p_{C,A}(t_C(g)))_p$, for all $g : A\Gamma \to D$.

T∧2. $p_{C,A\wedge B}(t_C(g_{p'})) = (p_{C,B}(t_C(g)))_{p'}$, for all $g : B\Gamma \to D$.

T∧3. $t_C(\langle f,g\rangle) = \langle t_C(f), t_C(g)\rangle$, for all $f : \Gamma \to A$, $g : \Gamma \to B$.

T∨1. $t_C([f,g]) = p_{A\vee B,C}([p_{C,A}(t_C(f)), p_{C,B}(t_C(g))])$,
for all $f : A\Gamma \to D$, $g : B\Gamma \to D$.

T∨2. $t_C({}_k g) =_k (t_C(g))$, for all $g : \Gamma \to A$.

T∨3. $t_C({}_{k'} g) =_{k'} (t_C(g))$, for all $g : \Gamma \to B$.

$T \Rightarrow 1$. $p_{C,A\Rightarrow B}(t_C(g[f])) = g[t_C(f)]$, for all $f : \Gamma \to A$, $g : B\Delta \to D$.

$T \Rightarrow 2$. $p_{C,A\Rightarrow B\Gamma}(t_C(g[f])) = p_{C,B}(t_C(g))[f]$, for all $f : \Gamma \to A$, $g : B\Delta \to D$.

$T \Rightarrow 3$. $t_C(g^*) = (p_{C,A}(t_C(g)))^*$, for all $g : A\Gamma \to B$.

C∧1. $c_C(p_{A\wedge B,CC}(p_{CC,A}(g))_p) = p_{A\wedge B,C}(p_{C,A}(c_C(g)))_p$,
for all $g : CCA\Delta \to D$.

C∧2. $c_C(p_{A\wedge B,CC}(p_{CC,A}(g))_{p'}) = p_{A\wedge B,C}(p_{C,A}(c_C(g)))_{p'}$,
for all $g : CCB\Delta \to D$.

C∧3. $c_C(\langle f,g\rangle) = \langle c_C(f), c_C(g)\rangle$, for all $f : CC\Gamma \to A$, $g : CC\Gamma \to B$.

C∨1. $p_{C,A\vee B}(c_C(p_{A\vee B,CC}([g,f]))) = [p_{C,A}(c_C(p_{A,CC}(g))), p_{C,B}(c_C(p_{B,CC}(f)))]$,
for all $g : ACC\Gamma \to D$, $f : BCC\Gamma \to D$.

C∨2. $c_C({}_k g) = {}_k(c_C(g))$, for all $g : CC\Gamma \to A$.

C∨3. $c_C({}_{k'} g) = {}_{k'}(c_C(g))$, for all $g : CC\Gamma \to B$.

C⇒1. $c_C(p_{A\Rightarrow B,CC}(g[f])) = p_{A\Rightarrow B,C}(g[c_C(f)])$, for all $f : CC\Gamma \to A$, $g : B\Delta \to D$.

C⇒2. $c_C(p_{a\Rightarrow B\Gamma,CC}(p_{CC,B}(g)[f])) = p_{A\Rightarrow B\Gamma,C}(p_{C,B}(c_C(g))[f])$,
for all $f : \Gamma \to A$, $g : CCB\Delta \to D$.

C⇒3. $c_C(f^*) = (p_{C,A}(c_C(p_{A,CC}(f))))^*$, for all $f : ACC\Gamma \to B$.

CUT∧1. $p_{\Gamma,A\wedge B}(p_{A\wedge B,C}(g_p)\langle f\rangle) = (p_{\Gamma,A}(p_{A2,C}(g)\langle f\rangle))_p$
$p_{\Gamma,B\wedge A}(p_{B\wedge A,C}(g_{p'})\langle f\rangle) = (p_{\Gamma,A}(p_{A,C}(g)\langle f\rangle))_{p'}$
for all $f : \Gamma \to C$, $g : AC\Delta \to D$.

CUT∧2. $g\langle f_p\rangle = (g\langle f\rangle)_p$,
$g\langle f_{p'}\rangle = (g\langle f\rangle)_{p'}$, for all $f : A\Gamma \to C$, $g : C\Delta \to D$.

CUT∨1. $p_{\Gamma,A\vee B}(p_{A\vee B,C}([f,g])\langle h\rangle) = [p_{\Gamma,A}(p_{A,C}(f)\langle h\rangle), p_{\Gamma,B}(p_{B,C}(g)\langle h\rangle)]$,
for all $f : AC\Delta \to D$, $g : BC\Delta \to D$, $h : \Gamma \to C$.

CUT∨2. ${}_k g\langle f\rangle = {}_k(g\langle f\rangle)$,
${}_{k'} g\langle f\rangle = {}_{k'}(g\langle f\rangle)$, for all $f : \Gamma \to C$, $C\Delta \to A$.

CUT⇒1. $h\langle g[f]\rangle = h\langle g\rangle[f]$, for all $f : \Gamma \to A$, $g : B\Delta \to C$, $h : C\Lambda \to D$.

CUT⇒2. $p_{A\Rightarrow B,C}(g[f])\langle h\rangle = p_{A\Rightarrow B,\Lambda}(g[f\langle h\rangle])$,

for all $h : \Lambda \to C$, $f : C\Gamma \to A$, $g : B\Delta \to D$.

CUT⇒3. $p_{A\Rightarrow B\Gamma,C}(g[f])\langle h\rangle = p_{A\Rightarrow B\Gamma,\Lambda}(p_{\Lambda,B}(p_{B,C}(g)\langle h\rangle)[f])$,

for all $h : \Lambda \to C$, $g : BC\Delta \to D$, $f : \Gamma \to A$.

Equations of this family equate derivations which differ in the order of applying connective rules and structural rules.

Definition 1.6. An **MN-multicategory** (N stands for "Natural deduction") is a multicategory in which hold all the equations that hold for MG-multicategories except that the TROUBLESOME EQUATION is replaced by the following special cases of this equation:

NE1. $1_{C_p}\langle[f,g]\rangle = [1_{C_p}\langle f\rangle, 1_{C_p}\langle g\rangle]$,

NE2. $1_{D_{p'}}\langle[f,g]\rangle = [1_{D_{p'}}\langle f\rangle, 1_{D_{p'}}\langle g\rangle]$, for all $f : A\Gamma \to C \wedge D$, $g : B\Gamma \to C \wedge D$.

NE3. $1_D[h_1]\langle[f,g]\rangle = [1_D[h_1]\langle f\rangle, 1_D[h_1]\langle g\rangle]$,

for all $h_1 : \Delta \to C$, $f : A\Gamma \to C \Rightarrow D$, $g : B\Gamma \to C \Rightarrow D$.

NE4. $[h_1, h_2]\langle[f,g]\rangle = [[h_1, h_2]\langle f\rangle, [h_1, h_2]\langle g\rangle]$,

for all $h_1 : C\Delta \to E$, $h_2 : D\Delta \to E$, $f : A\Gamma \to C \vee D$, $g : B\Gamma \to C \vee D$.

NE5. $\Box^C\langle[f,g]\rangle = [\Box^C\langle f\rangle, \Box^C\langle g\rangle]$, for all $f : A\Gamma \to 0$, $g : B\Gamma \to 0$.

We shall call these normalizing equations (NOR-equations) because they correspond to steps the normalizing of a derivation.

Definition 1.7. An **MGI-multicategory** is a multicategory in which hold all the equations that hold for MG-multicategories except that PTC-CUT-CON equations are replaced by the following equations:

I∧. $\langle 1_{A_{p'}}, 1_{B_{p'}}\rangle = 1_{A\wedge B}$.

I∨. $[{}_k1_A, {}_{k'}1_B] = 1_{A\vee B}$.

I⇒. $(p_{A\Rightarrow B,A}(1_B[1_A]))^* = 1_{A\Rightarrow B}$.

I0. $\Box^0 = 1_0$.

This family of equations will be called identity equations (ID-equations).

Definition 1.8. An **MNI-multicategory** is a multicategory in which hold all the equations that hold for MN-multicategories except that PTC-CUT-CON-equations are replaced by ID-equations.

It can be easily seen that if the TROUBLESOME EQUATION holds, then the NOR-equations hold, too. This means that each MG-multicategory is an MN-multicategory. In some multicategories the TROUBLESOME EQUATION cannot be derived from the NOR-equations and other equations. For example, let the arrow $h : C\Delta \rightarrow D$ be an axiom, where C is an atomic formula then the TROUBLESOME EQUATION cannot be derived from NE1-NE5 for the arrow h.

It can be shown that from the M1-M4 equations, the equations of the family IMPORTANT CUTS and the ID-equations, we can derive the PTC-CUT-CON-equations. According to that each MG-multicategory is an MGI-multicategory, and in the same way each MN-multicategory is an MNI-multicategory.

From now on in this text, if not stated otherwise, we shall take as multicategory $\mathcal{M}$ a multicategory of any of the four kinds previously defined.

Now we shall form in a standard way a polynomial multicategory $\mathcal{M}[X]$ out of a multicategory $\mathcal{M}$. Suppouse a multicategory $\mathcal{M}$ is given. The polynomial multicategory $\mathcal{M}[X]$ will be of the same kind as the multicategory $\mathcal{M}$ itself. In a usual way, a set of new arrows $X = \{x_A : \ \rightarrow A : A$ is an object of $\mathcal{M}$ $\}$ is added to the multicategory $\mathcal{M}$. Arrows of $\mathcal{M}[X]$ will be arrows of $\mathcal{M}$, arrows from X and arrows obtained with the help of the operations on arrows extending those assumed for $\mathcal{M}$. The arrow of $\mathcal{M}[X]$ will have the form $\varphi(x_1, ..., x_n) : \Gamma \rightarrow A$, where $x_1, ..., x_n$ are arrows from X which can appear in the construction of the arrow the φ. Arrows x_i, $1 \leq i \leq n$ need not occur in this order in construction of φ; they can occur several times, and they need not occur explicitly. The next step is making a multicategory out of $\mathcal{M}[X]$. For this we need some equations that hold on arrows in $\mathcal{M}$. For two arrows of $\mathcal{M}[X]$ which have the some source and target consider all the equivalence relations $\equiv_Y$ for some $Y \subseteq X$ on arrows of $\mathcal{M}[X]$. Which basides some of the families of equations used for defining MG, MGI, MN and MNI-multicategories must satisfy also
X. the equations following form:
if $\varphi \equiv_Y \psi$ then $\varphi_p \equiv_Y \psi_p$, where $\varphi, \psi : A\Gamma \rightarrow C$ in $\mathcal{M}[X]$, and similar by with other operations on arrows.

The relations $\equiv_Y$ will satisfy the families equations assumed for the kind of multicategory to which $\mathcal{M}$ belongs. Then $\cong_Y$ is the smallest of the equivalence relations $\equiv_Y$ (cf. [2], p. 57). For example, let $\mathcal{M}$ be an MG-multicategory; then $\equiv_X$ satisfies the following families of equations: X, M, PTC, CUT-PTC, IMP-CUT, TR-EQ, PTC-CUT-CON. The relation $\cong_X$ is the smallest equivalence relation which satisfies these families of equations. Then $\mathcal{M}[X]$ with $\cong_X$ is a polynomial MG-multicategory.

Let $\mathcal{M}$ be a multicategory of one of the kinds defined above. The functional completness theorem will hold for the multicategory $\mathcal{M}$.

Theorem 1.1. *Let $\varphi(x_1, ..., x_n) : \Gamma \rightarrow A$ be an arbitrary arrow of multicategory $\mathcal{M}[X]$, where $\Gamma = B_1 ... B_m$ and $x_i : \ \rightarrow A_i$, $1 \leq i \leq n$. Then for an arbitrary order of formulas the A_i, $i \leq i \leq n$, for example $A_1 ... A_n$, there is* *a unique arrow* $\overline{\varphi}$ *in the multicategory* $\mathcal{M}$, *such that:*

$$\overline{\varphi} : A_1...A_n\Gamma \to A \quad and \quad \overline{\varphi}\langle x_1\rangle...\langle x_n\rangle \cong_Y \varphi(x_1,...,x_n), \qquad (**)$$

where $Y = \{x_1, ..., x_n\}$.

The arrow $\overline{\varphi}_{A_1...A_n}$ *will be denoted simply by* $\overline{\varphi}$, *and* $A_1...A_n$ *is* Δ.

Proof. Only a sketch of the proof will be given here.

I We shall first show that the arrow $\overline{\varphi}$ of $\mathcal{M}$ exists for each arrow $\varphi(x_1,...,x_n)$ of $\mathcal{M}[X]$ for which hold (**). The proof will be executed by induction on the complexity of the arrow $\varphi(x_1,...x_n)$. The a arrow $\varphi(x_1,...x_n)$ must have one of the following forms:

1. f, where f is an arrow of $\mathcal{M}$; 2. $x_i :\to A_i$; 3. $p_{D,C}(\psi)$; 4. $c_B(\psi)$; 5. $t_E(\psi)$; 6. ψ_p; 7. $\psi_{p'}$; 8. $\langle\psi,\xi\rangle$; 9. $[\psi,\xi]$; 10. ${}_k\psi$; 11. ${}_{k'}\psi$; 12. $\psi[\xi]$; 13. ψ^*; 14. $\psi\langle\xi\rangle$; 15. $\psi\langle g\rangle$; 16. $g\langle\psi\rangle$, where g in 15. and 16. is an arrow of $\mathcal{M}$.

Arrows ψ and ξ are of smaller complexity than arrow φ and then on the basis of inductional hypothesis there are $\overline{\psi}$, $\overline{\xi}$ in $\mathcal{M}$ for which:

$$\overline{\psi}\langle x_1\rangle...\langle x_n\rangle \cong_Y \psi(x_1,...,x_n) \text{ and } \overline{\xi}\langle x_1\rangle...\langle x_n\rangle \cong_Y \xi(x_1,...,x_n)$$

Then $\overline{\varphi}$ is:

1. $\overline{f} = t_{A_1}(...(t_{A_n}(f))...)$; 2. $\overline{x_i} = t_{A_1}...t_{A_{i-1}}(p_{A_{i+1}...A_n,A_i}(t_{A_{i+1}...A_n}(1_{A_i})))$;
3. $p_{D,C}(\overline{\psi})$; 4. $p_{B,A_1...A_n}(c_B(p_{A_1...A_n,BB}(\overline{\psi})))$; 5. $p_{E,A_1...A_n}(t_E(\overline{\psi}))$;
6. $p_{B\wedge C,\Delta}((p_{\Delta,B}(\overline{\psi}))_p)$; 7. $p_{C\wedge B,\Delta}((p_{\Delta,B}(\overline{\psi}))_{p'})$; 8. $\langle\overline{\psi},\overline{\xi}\rangle$;
9. $p_{B\vee C,\Delta}([p_{\Delta,B}(\overline{\psi}),p_{\Delta,C}(\overline{\xi})])$; 10. ${}_k\overline{\psi}$; 11. ${}_{k'}\overline{\psi}$;
12. $c_\Delta(p_{B\Rightarrow C,\Delta\Delta}(p_{B_1...B_k,\Delta}(p_{\Delta,C}(\overline{\psi})[\overline{\xi}])))$; 13. $(p_{\Delta,B}(\overline{\psi}))^*$;
14. $c_\Delta(p_{B_1...B_k,\Delta}(p_{\Delta,B}(\overline{\psi})\langle\overline{\xi}\rangle))$; 15. $p_{B_1...B_k,\Delta}(p_{\Delta,B}(\overline{\psi})\langle g\rangle)$; 16. $g\langle\overline{\psi}\rangle$;

On the basis of the equations that hold in the multicategory $\mathcal{M}[X]$ it can be easily shown that (**) holds in all cases 1-16.

II In order to show the uniqueness of the arrow $\overline{\varphi}$, we first define an equivalence relation $\simeq_Y$ in the following way:

for arrows $\psi(y_1,...y_m)$ and $\xi(y_1,...y_m) : \Gamma \to A$

$$\psi \simeq_Y \xi \quad \text{if and only if} \quad \overline{\psi}_\Theta = \overline{\xi}_\Theta \quad \text{in } \mathcal{M}, \qquad (\circ)$$

where $Y = y_1,...y_m$ and $y_i :\to B_i$ and Θ is $B_1...B_m$ taken in any order.

Then it is proved that the relation $\simeq_Y$ is an equivalence relation $\equiv_Y$. Since $\cong_Y$ is the smallest equivalence relation we get $\psi \cong_Y \xi$ only if $\overline{\psi} = \overline{\xi}$.

Now we suppose that there are two arrows in $\mathcal{M}$, say $\overline{\varphi}$ and $\overline{\varphi}'$ such that $\overline{\varphi}, \overline{\varphi}' : A_1...A_n\Gamma \to A$ and $\overline{\varphi}\langle x_1\rangle...\langle x_n\rangle \cong_Y \varphi$, $\overline{\varphi}'\langle x_1\rangle...\langle x_n\rangle \cong_Y \varphi$. As $\cong_Y$ is transitive we have $\overline{\varphi}\langle x_1\rangle...\langle x_n\rangle \cong \overline{\varphi}'\langle x_1\rangle...\langle x_n\rangle$ and by implication above we get $\overline{\overline{\varphi}\langle x_1\rangle...\langle x_n\rangle} = \overline{\overline{\varphi}'\langle x_1\rangle...\langle x_n\rangle}$ in $\mathcal{M}$. Then from the part I of the proof and from equations which hold in $\mathcal{M}$ we get that $\overline{\varphi} = \overline{\varphi}'$. This means that $\overline{\varphi}$ is the unique arrow of $\mathcal{M}$ for which (**) holds. □

2. Typed λ-calculi

Definition 2.1. A typed λ-calculus ΛC is a formal theory defined by classes of types, terms of each type, and equations between terms. We shal write $t \in A$ to say that t is term of type A.

Types. The class of types contains special a type 0, and is closed under three operations: $A \wedge B$, $A \vee B$, $A \Rightarrow B$, where A, B are types.

Terms 2.1. For each type A there are countably many variables of type A: $x_i \in A$, $i = 1, 2, 3, \ldots$

2.2. If $t \in A \wedge B$ then $Lt \in A$ and $Rt \in B$.

2.3. If $u \in A$ and $t \in B$, then $\Pi(u, t) \in A \wedge B$.

2.4. If $t(x) \in C$ and $x \in A$, $s(y) \in C$, $y \in B$, $u \in A \vee B$, then $\delta_{x,y}(u, t(x), s(y)) \in C$.

2.5. If $t \in A$ and $s \in B$, then $K_B t \in A \vee B$, $K'_A s \in A \vee B$.

2.6. If $u \in A$ and $t \in A \Rightarrow B$, then $tu \in B$.

2.7. If $u(x) \in B$ and $x \in A$, then $\lambda_x u \in A \Rightarrow B$.

2.8. If $t \in 0$, then $\iota_A t \in A$.

Equations. Equations have the form $t =_X s$, where t and s have the same type A, and X is a finite set of variables such that all variables occuring freely in t and s are in X.

3.1. $=_X$ is an equivalence relation.

3.2. If $t =_Y s$ and $Y \subseteq X$, then $t =_X s$.

3.3. $=_X$ satisfies the usual substitution rules for all terms forming operations. For example:

$$\text{if } t_1 =_X t_2 \text{ and } s_1 =_X s_2 \text{ then } \Pi(t_1, s_1) =_X \Pi(t_2, s_2)$$

where $t_1, t_2 \in A$, $s_1, s_2 \in B$.

If $t(x)$ is a term of type A, $x \in B$ and s is a term of type B then $t[x/s]$ is the result of replacing an occurence of x in the term t by the term s.

Definition 2.2. A typed λ-calculus is **ΛC-ND** if the following equations on terms hold in it:

L. $L(\Pi(t, s)) =_X t$, for all $t \in A$, $s \in B$.

R. $R(\Pi(t, s)) =_X s$, for all $t \in A$, $s \in B$.

K. $\delta_{x,y}(K_A u, t(x), s(y)) =_X t[x/u]$, for all $u \in A$, $t, s \in C$, $x \in A$, $y \in B$.

K'. $\delta_{x,y}(K'_B u, t(x), s(y)) =_X s[y/u]$, for all $u \in B$, $t, s \in C$, $x \in A$, $y \in B$.

β. $(\lambda_x t)u =_X t[x/u]$, for all $u \in A$, $t \in B$, $x \in A$,

where in K, K' and β no free variable in u becomes bound in $t[x/u]$ and $s[y/u]$.

0. $t[x/\iota_A(z)] =_X \iota_B z$, for all $x \in A$, $t \in B$, $z \in 0$.

N1. $L\delta_{x,y}(u, t(x), s(y)) =_X \delta_{x,y}(u, Lt(x), Ls(y))$,

for all $u \in A \vee B$, $x \in A$, $y \in B$ $t, s \in C \wedge D$.

N2. $R\delta_{x,y}(u, t(x), s(y)) =_X \delta_{x,y}(u, Rt(x), Rs(y))$,

for all $u \in A \vee B$, $x \in A$, $y \in B$ $t, s \in C \wedge D$.

N3. $\delta_{x_1,x_2}(\delta_{x,y}(u, t(x), s(y)), v_1(x_1), v_2(x_2)) =_X$

$\delta_{x,y}(u, \delta_{x_1,x_2}(t, v_1(x_1), v_2(x_2))(x), \delta_{x_1,x_2}(s, v_1(x_1), v_2(x_2))(y)))$,

for all $u \in A \vee B$, $x \in A$, $y \in B$, $t, s \in C \vee D$, $x_1 \in C$, $x_2 \in D$, $v_1, v_2 \in E$.

N4. $\delta_{x,y}(u, t(x), s(y))v =_X \delta_{x,y}(u, tv(x), sv(y))$,

for all $u \in A \vee B$, $x \in A$, $y \in B$, $t, s \in C \Rightarrow D$, $v \in C$.

N5. $\iota_C(\delta_{x,y}(u, t(x), s(y)) =_X \delta_{x,y}(u, \iota_C t(x), \iota_C s(y))$,

for all $u \in A \vee B$, $x \in A$, $y \in B$, $t, s \in \Lambda$.

The terms of typed a λ-calculus code derivations of the natural deduction system for intuitionistic propositional logic. Equations L, R, K, K', β and 0 correspond to reduction steps in the normalization theorem when a derivation in which maximum formula appears is transformed into derivation without maximum formula. Equations N1, N2, N3, N4, N5 correspond to reduction steps by which a maximum segment in a derivation is eliminated. Terms on the left-hand side of all equations will correspond to derivations in which a maximum formula or a maximum segment occurs. Those derivations are transformed by reductions into derivations corresponding to terms on the right-hand side of equations.
Equations on terms in a typed λ-calculus ΛC-ND equate derivations with their normal forms.

Definition 2.3. A typed λ-calculus ΛC-ND is Λ **C-NDA** if the following equations on terms hold in it:

$\eta\wedge$. $z =_X \Pi(Lz, Rz)$,	for all $z \in A \wedge B$.
$\eta\vee$. $z =_X \delta(z, K_B x, K'_A y)$,	for all $z \in A \vee B$, $x \in A$, $y \in B$.
$\eta\Rightarrow$. $z =_X \lambda_x zx$,	for all $z \in A \Rightarrow B$, $x \in A$.
$\eta 0$. $z =_X \iota_0 z$,	for all $z \in 0$.

Equations on terms $\eta\wedge$, $\eta\vee$, $\eta \Rightarrow$ and $\eta 0$ equate derivations consisting of nonatomic formulas in their middle part with derivations whose middle part consists of atomic formulas.

3. Connection between multicategories and typed λ-calculi

In this section we show how to form a typed λ-calculus out of a multicategory $\mathcal{M}$ and how a multicategory can be formed from a typed λ-calculus $\mathcal{L}$.

First we shall define types and terms LC($\mathcal{M}$) from objects and arrows of some multicategory $\mathcal{M}$.

LCI 1. the types of LC($\mathcal{M}$) will be the objects of $\mathcal{M}$.

2. the operations on types in LC($\mathcal{M}$) will be the operations on objects of $\mathcal{M}$.

LCII Every arrow term $\varphi(x_1, ..., x_n) : \ \rightarrow A$ of $\mathcal{M}[X]$ is a term of type A in LC($\mathcal{M}$), and all the free variables of $\varphi(x_1, ..., x_n)$ are in $X = \{x_1, .., x_n\}$.

0T. $x \in A$, $x =_{def.} x : \ \rightarrow A$.

1T. If $\varphi(x_1, ... x_n) : \ \rightarrow A \wedge B$, then $L\varphi =_{def} 1_{Ap}\langle\varphi\rangle$.

2T. If $\varphi(x_1, ..., x_n) : \ \rightarrow A \wedge B$, then $R\varphi =_{def} 1_{Bp'}\langle\varphi\rangle$.

3T. If $\varphi(x_1, ..., x_n) : \ \rightarrow A$, $\psi(x_1, ..., x_n) : \ \rightarrow B$, then $\Pi(\varphi, \psi) =_{def} \langle\varphi, \psi\rangle$.

4T. If $\varphi(z_1, ..., z_k) : \ \rightarrow A \vee B$, $\psi(x_1, ..., x, ..., x_n) : \ \rightarrow C$,
$\xi(y_1, ..., y, ..., x_m) : \ \rightarrow C$, $x : \ \rightarrow A$, $y : \ \rightarrow B$
then $\delta_{x,y}(\varphi, \psi(x), \xi(y)) =_{def} [\overline{\psi}, \overline{\xi}]\langle\varphi\rangle$,
where $\overline{\psi}$ is defied with respect to $Y = \{x\}$ and $\overline{\xi}$ is defined with respect to $Y = \{y\}$.

5T. If $\varphi(x_1, ..., x_n) : \ \rightarrow A$, then $K_B\varphi =_{def} {}_k\varphi$.

6T. If $\varphi(x_1, ..., x_n) : \ \rightarrow B$, then $K'_A\varphi =_{def} {}_{k'}\varphi$.

7T. If $\varphi(x_1, ..., x_n) : \ \rightarrow A$, $\psi(x_1, ..., x_n) : \ \rightarrow A \Rightarrow B$,
then $\psi\varphi =_{def} 1_B[1_A]\langle\psi\rangle\langle\varphi\rangle$.

8T. If $\varphi(x_1, ..., x_n) : \ \rightarrow B$, $x : \ \rightarrow A$, then $\lambda_x\varphi =_{def} \overline{\varphi}^*$,
where $\overline{\varphi}$ is defined with respect to $Y = \{x\}$.

9T. If $\varphi(x_1, ..., x_n) : \ \rightarrow 0$, then $\iota_A\varphi =_{def} \Box^A\langle\varphi\rangle$.

LCEQ. If $\varphi(x_1, ..., x_n), \psi(x_1, ..., x_n) : \ \rightarrow A$,
then $\varphi =_X \psi$ if and only if $\varphi \cong_X \psi$ in $\mathcal{M}[X]$, where $X = \{x_1, ..., x_n\}$.

Now let one typed λ-calculus, $\mathcal{L}$, be given. We shall define objects and arrows of MC($\mathcal{L}$) from types and terms of the typed λ-calculus $\mathcal{L}$ in the following way:

MCI 1. the objects of MC($\mathcal{L}$) will be the types of $\mathcal{L}$.

2. the operations on objects in MC($\mathcal{L}$)will be the operrations on types of $\mathcal{L}$.

MCII An arrow $f : \ A_1 ... A_n \rightarrow A$ in MC($\mathcal{L}$) will be $(x_1 ... x_n; \varphi(x_1, ..., x_n) \in A)$ where $\varphi(x_1, ..., x_n)$ is the term of $\mathcal{L}$ and $x_1, ..., x_n$ are all free variables of term φ, and $x_1 \in A_1$, ..., $x_n \in A_n$.

0A1. $1_A =_{def} (x; x \in A)$

0A2. $\Box^A =_{def} (x \in 0; \iota_A x \in A)$

1A. If $f = (x_1 ... x_n x y y_1 ... y_m; \varphi \in C)$, $x \in A$, $y \in B$
then $p_{A,B}(f) =_{def} (x_1 ... x_n y x y_1 ... y_m; \varphi \in C)$.

2A. If $f = (x x x_1 ... x_n; \varphi \in C)$, $x \in A$, then $c_A(f) =_{def} (x x_1 ... x_n; \varphi \in C)$.

3A. If $f = (x_1 ... x_n; \varphi \in X)$, $x \in A$, then $t_A(f) =_{def} (x x_1 ... x_n; \varphi \in C)$.

4A. If $f = (x_1 ... x_n; \varphi \in A)$, $g = (x y_1 ... y_m; \psi \in B)$, $x \in A$,
then $g\langle f\rangle =_{def} (x_1 ... x_n y_1 ... y_m; \psi[x/\varphi] \in B)$.

5A. If $f = (x x_1 ... x_n; \varphi \in C)$, $x \in A$, $z \in A \wedge B$,
then $f_p =_{def} (z x_1 ... x_n; \varphi[x/Lz] \in C)$.

6A. If $f = (y x_1 ... x_n; \varphi \in C$, $y \in B$, $z \in A \wedge B$,
then $f_{p'} =_{def} (z x_1 ... x_n; \varphi[y/Rz] \in C)$.

7A. If $f = (x_1 ... x_n; \varphi \in A)$, $g(x_1 ... x_n; \psi \in B)$,
then $\langle f, g\rangle =_{def} (x_1 ... x_n; \Pi(\varphi, \psi) \in A \wedge B)$.

8A. If $f = (xx_1...x_n; \varphi \in C)$, $g = (yx_1...x_n; \psi \in C)$, $x \in A$, $y \in B$ and $z \in A \wedge B$, then $[f, g] =_{def} (zx_1...x_n; \delta_{x,y}(\varphi(x), \psi(y)) \in C)$.
9A. If $f = (x_1...x_n; \varphi \in A)$, then ${}_k f =_{def} (x_1...x_n; K_B\varphi \in A \vee B)$.
10A. If $f = (x_1...x_n; \psi \in B)$, then ${}_{k'} f =_{def} (x_1...x_n; K'_A\psi \in A \vee B)$.
11A. If $f = (x_1...x_n; \varphi \in A)$, $g = (xy_1...y_m; \psi \in C)$, $x \in B$, $y \in A$ $z \in A \Rightarrow B$, then $g[f] =_{def} (x_1...x_n y_1...y_m; \psi[x/zy[y/\varphi]] \in C)$.
12A. If $f = (xx_1...x_n; \varphi \in B)$, then $f^* =_{def} (x_1...x_n; \lambda_x\varphi \in A \Rightarrow B)$.
MCEQ. If $f = (x_1...x_n; \varphi \in A)$, $g = (y_1...y_n; \psi \in A)$, then $f = g$ if and only if 1. $x_i, y_i \in A_i$, $1 \leq i \leq n$, 2. $\varphi =_X \psi[\overline{y}/\overline{x}]$, where $\overline{x} = x_1...x_n$, $\overline{y} = y_1...y_n$ and $X = \{x_1...x_n\}$.

In this way types and terms of LC($\mathcal{M}$) and objects and arrows of MC($\mathcal{L}$) are defined. It is still neccessary to show that LC($\mathcal{M}$) is a typed λ-calculus and that MC($\mathcal{L}$) is a multicategory.

It has to be shown that the corresponding equations hold for terms of LC($\mathcal{M}$). Depending on what kind of multicategory $\mathcal{M}$ belongs to, LC($\mathcal{M}$) will be LC-ND or LC-NDA typed λ-calculus. The following theorem holds:

Theorem 3.1.
1. *If $\mathcal{M}$ is an MN-multicategory, then LC($\mathcal{M}$) is a ΛC-ND typed λ-calculus.*
2. *If $\mathcal{M}$ is an MNI-multicategory, then LC($\mathcal{M}$) is a ΛC-NDA typed λ-calculus.*

Proof. In both cases we have to verify what equations hold on terms in LC($\mathcal{M}$).
1. As an example we show that the equation N1. holds:
$L\delta(u, t(x), s(y)) =_X \delta(u, Lt(x), Ls(y)$, $u : \; \rightarrow A \vee B$, $t, s : \; \rightarrow C \wedge D$, $x : \; \rightarrow A$, $y : \; \rightarrow B$.
$Lt =_{def} 1_{Cp}\langle t\rangle$, $Ls =_{def} 1_{Cp}\langle s\rangle$, $\delta_{x,y}(u, t(x), s(y)) =_{def} [\overline{t}, \overline{s}]\langle u\rangle$,
$\delta_{x,y}(u, Lt(x), Ls(y)) =_{def} [\overline{1_{Cp}\langle t\rangle}, \overline{1_{Cp}\langle s\rangle}]\langle u\rangle$, $L\delta_{x,y}(u, t(x), s(y)) =_{def}$ $1_{Cp}\langle[\overline{t}, \overline{s}]\langle u\rangle\rangle$, $1_{Cp}\langle[\overline{t}, \overline{s}]\langle u\rangle\rangle =_X 1_{Cp}\langle[\overline{t}, \overline{s}]\rangle\langle u\rangle =_X [\overline{1_{Cp}\langle t\rangle}, \overline{1_{Cp}\langle s\rangle}]\langle u\rangle$,
then $L\delta_{x,y}(u, t(x), s(y)) =_X \delta_{x,y}(u, Lt(x), Ls(y))$.

It can be proved in a similar way that all the other equations hold.

2. As an example we will show that the equation $\eta\wedge$ holds and that it depends on the equation I1: $z =_X \Pi(Lz, Rz)$.
$z : \; \rightarrow A \Rightarrow B$, $Lz =_{def} 1_{Ap}\langle z\rangle$, $Rz =_{def} 1_{Bp'}\langle z\rangle$,
$\Pi(Lz, Rz) =_{def} \langle 1_{Ap}\langle z\rangle, 1_{Bp'}\langle z\rangle\rangle$,
$\langle 1_{Ap}\langle z\rangle, 1_{Bp'}\langle z\rangle\rangle =_X \langle 1_{Ap}, 1_{Bp'}\rangle\langle z\rangle =_X 1_{A\wedge B}\langle z\rangle =_X z$,
then $z =_X \Pi(Lz, Rz)$ in LC($\mathcal{M}$).

All the other equations are verified in a similar way. □

Let $\mathcal{M}$ be a multicategory of any of the four kinds previously defined and let its axioms be closed for cut. This means that if the arrows $f : A\Delta \rightarrow B$ and $g : \Gamma \rightarrow A$ are axioms of $\mathcal{M}$ and A is atomic formula, then the arrow $f\langle g\rangle$ is an axiom in $\mathcal{M}$, too. In this multicategory equations on arrows connect a derivation in which cut occurs and a derivation in which cut is eliminated. Equations on terms connect a derivation with the normal form of this derivation. Roughly speaking, equations we need for cut elimination yield normalization.

On the other hand, let us see what we get from equations which we need for normalization. These equations will 'select' from the equations which we need for cut elimination only those really required. We shall show that from LC-ND and LC-NDA typed λ-calculi we obtain only MN-multicategories and MNI-multicategories, which are not necessarily MG and MGI-multicategories.

The following theorem shows this:

Theorem 3.2.
1. If $\mathcal{L}$ is a ΛC-ND typed λ-calculus, then MC($\mathcal{L}$) is an MN-multicategory.
2. If $\mathcal{L}$ is a ΛC-NDA typed λ-calculus, then MC($\mathcal{L}$) is an MNI-multicategory.

Proof. We have to check that the equations which must hold in MN-multicategories, respectively MNI-multicategories, also hold for arrows in MC($\mathcal{L}$).

Some remarks.

The free MG-multicategory corresponds to Gentzen's sequent system for intuitionistic propositional logic. Some MG-multicategories with axioms which are closed for cut are extensions of this system. Equations between arrows which must hold in an MG-multicategory don't make cut elimination possible in an MG-multicategory with arbitrary atomic axioms.

The situation is analogous with MN-multicategories.

We can investigate the connection between some MG-multicategories mentioned above and the corresponding MN-multicategories. This can be another way to link cut elimination and normalization, without going via typed λ-calculi.

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