FILOMAT (Niš) 9:3 (1995), 599–602

Algebra, Logic & Discrete Mathematics
Niš, April 14–16, 1995.

HADAMARD'S INEQUALITY AND FIXED-POINT METHOD

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ABSTRACT. The famous inequality in matrix theory of J. Hadamard has different proofs and extensions [1, 6]. Here given proof is by the method of common fixed-point of mappings monotonic with respect to a functional, which can be applied to many, including all main inequalities [3]. Condition for equality: rows of a matrix are orthogonal or at least one of them is zero, is replaced by proportionality (appearing in numerous other inequalities) between rows of a matrix and corresponding rows of cofactors.

Theorem. *Let $A = (a_{ij})$ be a real square matrix and $|A|$ be it's determinant, then*

$$|A|^2 \leq \prod_{i=1}^{n}\Big(\sum_{j=1}^{n} a_{ij}^2\Big). \tag{1}$$

Equality in (1) *holds if and only if*

$$a_{i1}a_{j1} + a_{i2}a_{j2} + \cdots + a_{in}a_{jn} = 0, \tag{2}$$

for each pair of different i, j, or if at least one factor on the right side of (1) *is equal to zero.*

Condition (2), *including the disjunct, can be replaced by the next one: there are numbers λ_i, μ_i, $\lambda_i^2 + \mu_i^2 \neq 0$, $1 \leq i \leq n$, such that*

$$\lambda_i a_{ij} + \mu_i A_{ij} = 0, \qquad 1 \leq j \leq n, \tag{3}$$

where A_{ij} are cofactors.

Proof. Define the space (product of n-spheres)

$$\mathcal{X} = \Big\{ X = (x_{ij}) \;\Big|\; \sum_{j=1}^{n} x_{ij}^2 = \sum_{j=1}^{n} a_{ij}^2,\ 1 \leq i \leq n \Big\} \tag{a}$$

and the functional $f : \mathcal{X} \to \mathbb{R}$

$$\text{(b)} \qquad f(X) = |X|.$$

Define mappings $F_i : \mathcal{X} \to \mathcal{X}$, $1 \le i \le n$

$$\text{(c)}\ F_i(X) = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{i1} & y_{i2} & \dots & y_{in} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix}, \qquad y_{ij} = \frac{r_i}{R_i} X_{ij}, \qquad 1 \le j \le n,$$

$$r_i = \sqrt{x_{i1}^2 + x_{i2}^2 + \cdots + x_{in}^2}, \qquad R_i = \sqrt{X_{i1}^2 + X_{i2}^2 + \cdots + X_{in}^2}.$$

If $r_i = 0$, then $F_i(X) = X$; define the same if $R_i = 0$. The row $(y_{ij})_j$ is defined to be proportional to the corresponding row of cofactors $(X_{ij})_j$ and that $F_i(X) \in \mathcal{X}$. The mapping F_i is monotonic nondecreasing with respect to the functional f

$$\text{(d)} \qquad f(X) \le f(F_i(X)),$$

by Laplace development (d) is equivalent to

$$x_{i1}X_{i1} + x_{i2}X_{i2} + \cdots + x_{in}X_{in} \le y_{i1}X_{i1} + y_{i2}X_{i2} + \cdots + y_{in}X_{in},$$

$$\text{(4)} \qquad x_{i1}X_{i1} + x_{i2}X_{i2} + \cdots + x_{in}X_{in} \le r_i R_i.$$

The Cauchy inequality (4) is equality [1] if and only if there are numbers λ_i, μ_i, not both 0, such that

$$\lambda_i x_{ij} + \mu_i X_{ij} = 0, \qquad 1 \le j \le n.$$

If $\lambda_i \mu_i \ne 0$, then equality in (d) holds if and only if sets $(x_{ij})_j$ and $(X_{ij})_j$ are proportional, or equivalently $x_{ij} = y_{ij}$, $1 \le j \le n$. Hence, F_i is strictly monotonic with respect to f: equality in (d) holds if and only if X is a fixed point of mapping $F_i(X) = X$, and for non-fixed points strict inequality holds. On the compact set $\mathcal{X}$ the functional f attains maximal value, and, because of strict monotonicity, it is attained on a set $\mathcal{F}$ of common fixed points of mappings F_i, $1 \le i \le n$. The set $\mathcal{F}$ is not empty, since it contains, e.g., diagonal matrix with diagonal r_i, $1 \le i \le n$. If $X \in \mathcal{F}$, $|X| \ne 0$, $X_{ij} = c_i x_{ij}$, $1 \le i, j \le n$, then

$$|X| = c_i r_i^2, \qquad 1 \le i \le n, \qquad |X|^n = \prod_{i=1}^{n} c_i r_i^2.$$

On the other hand

$$|X| = \left|\left(\frac{X_{ij}}{c_i}\right)\right| = \frac{1}{c_1 c_2 \dots c_n}|(X_{ij})| = \frac{1}{c_1 c_2 \dots c_n}||X|X^{-1}| = \frac{1}{c_1 c_2 \dots c_n}|X|^{n-1}$$
$$|X|^n = c_1 c_2 \dots c_n |X|^2,$$

so that

$$|X|^2 = \prod_{i=1}^{n} r_i^2 = \prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right).$$

The equivalence between conditions (2), including the disjunct, and (3), both determining the same set of matrices on which the equality in (1) holds, follows from earlier proofs of the theorem and this one. However, we give a direct proof that (2) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3). If $a_{ij} = 0$, $1 \le j \le n$, then $A_{kj} = 0$, $1 \le k \le n$, $k \ne i$, $1 \le j \le n$ and (3) holds. Let A be an orthogonal matrix with no one zero row, then lineal over rows of the matrix A

$$\mathcal{L}(\{(a_{i1}, a_{i2}, \dots, a_{in}) \mid 1 \le i \le n\})$$

is n-dimensional vector space. Also

$$(a_{i1}, a_{i2}, \dots, a_{in}) \perp \mathcal{L}(\{a_{j1}, a_{j2}, \dots, a_{jn}) \mid 1 \le j \le n,\ j \ne i\}).$$

From

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn} = \delta_{ij}|A|, \tag{5}$$

follows

$$(A_{i1}, A_{i2}, \dots, A_{in}) \perp \mathcal{L}(\{a_{j1}, a_{j2}, \dots, a_{jn}) \mid 1 \le j \le n,\ j \ne i\}).$$

Vectors $(a_{i1}, a_{i2}, \dots, a_{in})$ and $(A_{i1}, A_{i2}, \dots, A_{in})$ in n-dimensional space are orthogonal to the same hyperplane ($(n-1)$-variety) and therefore they are collinear.

(3) $\Rightarrow$ (2). If in (3) some $\mu_i = 0$, then $a_{ij} = 0$, $1 \le j \le n$, that is, the disjunct in (2) holds. Hence, suppose that $\mu_i \ne 0$, $1 \le i \le n$, i. e., rows of A are not zero-vectors. If $\lambda_i \ne 0$, $1 \le i \le n$, then from (5) follows (2). Now, without lose of generality, suppose that $\lambda_k = 0$, $1 \le k \le i$; $\lambda_l \ne 0$, $i < l \le n$. From $A_{kj} = 0$, $1 \le k \le i$, $1 \le j \le n$ follows that $|A| = 0$ and that rows of A are linearly dependent. From (5) follows that rows $(a_{lj})_j$, $i < l \le n$ are orthogonal and, therefore, linearly independent. Using also

$$\mathcal{L}(\{(a_{k1}, a_{k2}, \dots, a_{kn}) \mid 1 \le k \le i\}) \perp \mathcal{L}(\{(a_{l1}, a_{l2}, \dots, a_{ln}) \mid i < l \le n\})$$

obtains that rows $(a_{kj})_j$, $1 \leq k \leq i$ are linearly dependent. Therefore, $A_{lj} = 0$, $i < l \leq n$, $1 \leq j \leq n$, what implies that $i = n$. $\square$

Note that there is an orbit of X

$$|X| \leq |F_1(X)| \leq |F_2(F_1(X))| \leq \cdots \leq |F_n(\ldots F_2(F_1(X))\ldots)|,$$

where $F_n \circ \cdots \circ F_1(\mathcal{X}) = \mathcal{F}$, what gives a direct proof of (1). Geometric interpretation of Hadamard's inequality is that the volume of a parallelepiped in n-dimensional space does not exceed product of lengths of it's edges, equality holds if the edges are orthogonal, or if length of one edge is zero. Also, mention analogy between Hadamard' inequality and generalization of Cauchy inequality, namely, one special case of Hölder's inequality

$$\left(\sum_{j=1}^{m}\left(\prod_{i=1}^{n} a_{ij}\right)\right)^n \leq \prod_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}^n\right),$$

$|A|$ is also a sum of products of elements of matrix A.

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