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ON SOME 4- AND 5-DESIGNS ON $\leq$ 49 POINTS

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ABSTRACT. A search for those t-$(q+1,k,\lambda)$ designs is made, which arise by action of the groups $PSL(2,q)$ and $PGL(2,q)$ on the ground-set $\Omega(q) = \{0,1,...,q-1\} \cup \{\infty\}$. The search is made for $(t,k) = (4,5)$ with prime powers $q \leq 49$ and for $(t,k) \in \{(4,6),(5,6)\}$ with prime powers $q \leq 31$. The group $PSL(2,q)$ is used for $q \equiv 3 \pmod 4$ and the group $PGL(2,q)$ is used otherwise.

The search uses orbit incidence matrices determined by orbits of t-subsets and k-subsets (shortly: t-orbits and k-orbits) of the ground-set, obtained by action of the group used. An element of an orbit incidence matrix is the number of those k-sets within a k-orbit, which contain a fixed t-set (representative) of a t-orbit. Construction of orbit incidence matrices essentially uses 3-homogenicity of the groups.

The total number of distinct quadruples (t,q,k,λ) of parameters, for which t-$(q+1,k,\lambda)$ designs are constructed is equal to 75. It is guaranteed that the obtained values of λ are the only possible, which can be reached by action of the groups used, for the considered triples (t,q,k). It is assumed that most of the obtained quadruples of design parameters are new, in particular those for $q = 19, 25, 27, 31$ and 37.

1. Introduction

Let *n-set* denote a set of cardinality n. A t-(v,k,λ) *design* [5] is an incidence structure on v points, which consists of some k-sets of points (called *blocks*) without repetitions and which satisfies that each t points are contained in exactly λ blocks. $GF(q)$ is the Galois field associated to a prime power $q = p^s$.

The group $GL(2,q)$ is the group of all non-singular 2×2 matrices with elements in $GF(q)$ (= non-singular linear transformations over $(GF(q))^2$), while $SL(2,q)$ is its subgroup consisting of the matrices with determinant 1. The projective general linear group $PGL(2,q)$ and the projective special linear group $PSL(2,q)$ are obtained from $GL(2,q)$ and $SL(2,q)$ respectively, by reduction with the corresponding groups of homoteties.

Both $PGL(2,q)$ and $PSL(2,q)$ act on the common ground-set $\Omega(q) = \{0,1,...,q-1\} \cup \{\infty\}$. It is known that $PGL(2,q)$ acts 3-transitively for all q, while $PSL(2,q)$ acts 3-homogenously for $q \equiv 3 \pmod 4$ and only 2-transitively for other prime powers q. Construction of these two groups is described in [3] and [2] respectively.

The orbit incidence matrix method for searching designs, which will be referred to as "Λ-technique", introduced in [2], can be sketched as follows:

- Let be given a 3-homogenous permutation group G acting on $\Omega(q)$ and a pair (t,k) of natural numbers satisfying $4 \le t < k \le q$.
- Construct the orbits $T_1,...,T_m$ of those t-subsets of $\Omega(q)$, which include the set $\{0,1,\infty\}$. Similarly, construct the orbits $B_1,...,B_n$ of those k-subsets of $\Omega(q)$, which include the set $\{0,1,\infty\}$.
- Construct the orbit incidence matrix $\Lambda = (\lambda_{ij})$, $1 \le i \le m$, $1 \le j \le n$, where λ_{ij} denotes the number k-subsets of $\Omega(q)$ within B_j, which contain a fixed t-subset (representative) of T_i; the sum of all elements in each row of Λ is equal to
$$\lambda_{\text{trivial}} = \binom{q+1-t}{k-t} = \lambda\text{-value of the trivial } t\text{-}(q+1,k,\lambda)\text{-design.}$$
- Try to find for a *proper* subset P of the column set of Λ, which satisfies that the sum of elements within the columns of P is equal to the same constant λ for all the rows $(1 \le \lambda \le \lambda_{\text{trivial}}/2)$.
- If the subset P is found, then all the k-subsets of $\Omega(q)$, which belong to the orbits B_j corresponding to the columns of P, are the blocks of a t-$(q+1,k,\lambda)$ design. The complementary k-subsets of $\Omega(q)$ are the blocks of a t-$(q+1,k,\lambda_{\text{trivial}}-\lambda)$ design.

1.1. A comparision between the use of $PSL(2,q)$ and $PGL(2,q)$

Statement. *If a prime power q is of the form $4k+3$, then the group $PSL(2,q)$ is more suitable for looking for designs than $PGL(2,q)$.*

Namely, as already mentioned, the group $PSL(2,q)$ is 3-homogenous with the values of q of this form. Although 3-transitivity (possessed by $PGL(2,q)$) is a stronger property, it is only 3-homogenicity that matters when the application of the Λ-technique is considered. On the other hand, the group $PSL(2,q)$ is a subgroup (normal, of index 2) of $PGL(2,q)$, which implies that orbits by action of $PSL(2,q)$ are included in orbits by action of $PGL(2,q)$. "Building constituents" of the designs are k-orbits. The smaller are the constituents, the larger is the chance for making equilibrium (suitable sums of λ_{ij}'s), which leads to designs. Therefore we have the following:

Consequence. *If a prime power q is of the form $4k+3$, then each design which can be derived by Λ-technique with application of the group $PGL(2,q)$, can be also derived with application of $PSL(2,q)$.*

However, the group $PGL(2,q)$ is more suitable with other prime powers. It is always 3-transitive (and consequently 3-homogenous), while, when $PSL(2,q)$ is considered, only 2-transitivity is guaranteed.

Conclusion. *The group $PSL(2,q)$ is used for searching for designs with prime powers q of the form $4k+3$, while the group $PGL(2,q)$ is used with other prime powers q.*

2. Results

2.1. A global account of the generated designs

The computer search was performed for prime powers $q \leq 31$ with $k=6$ and for further prime powers $q \leq 49$ with $k=5$.

The search was successful with:
$PSL(2,q)$ and $(t,k)=(4,5)$ for $q=47$;
$PSL(2,q)$ and $(t,k)=(4,6)$ for $q=19$;
$PSL(2,q)$ and $(t,k)=(5,6)$ for $q=11,23,27,31$;
$PGL(2,q)$ and $(t,k)=(4,6)$ for $q=25$;
$PGL(2,q)$ and $(t,k)=(4,5)$ for $q=17,32,37$.

Note that the reported success with $(t,k)=(4,6)$ means that there was no success with $(t,k)=(5,6)$; otherwise, a 4-$(q+1,6,\lambda_2)$ design would be a consequence of a 5-$(q+1,6,\lambda_1)$ design, which corresponds to the same set of columns of the λ_{ij} matrix.

More precisely, the constructed t-$(q+1,k,\lambda)$ designs are summarized in the following table (the numbers of t-orbits and k-orbits by action of the group cited are denoted by m and n respectively):

t	q	k	$\lambda \leq \lambda_{\text{trivial}}/2$	λ_{trivial}	G	m	n
5	11	6	1,2	7	$PSL(2,11)$	2	6
4	17	5	4	14	$PGL(2,17)$	3	4
4	19	6	60	120	$PSL(2,19)$	5	19
5	23	6	1,2,3,4,5,6,7,8,9	19	$PSL(2,23)$	7	34
4	25	6	51,60,81,90,111	231	$PSL(2,25)$	5	28
5	27	6	2,3,4,5,6,7,8,9,10,11	23	$PSL(2,27)$	10	54
5	31	6	6,12	27	$PSL(2,31)$	15	83
4	32	5	4,5,9	29	$PSL(2,32)$	5	11
4	37	5	16	34	$PGL(2,37)$	7	15
4	47	5	8,12,16,20	44	$PSL(2,47)$	10	33

When the design complementations are taken into account, it turns out that the total number of generated designs with distinct parameters is equal to $75 = 2 \cdot 2 + 2 \cdot 1 + 1 \cdot 1 + 2 \cdot 9 + 2 \cdot 5 + 2 \cdot 10 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 1 + 2 \cdot 4$. (note that $\lambda = \lambda_{\text{trivial}}/2$ for $q = 19$).

A global conclusion concerning the generated designs, obtained after a thorough examination of the generated Λ-matrices, is the following:

Statement. *The above listed values of λ (taking in addition the values complementary w.r.t. λ_{trivial} into account), are the only possible values of λ which can be reached by action of the corresponding listed groups.*

However, it is not to say that there may not exist t-(v, k, λ) designs, obtained in another manner, which have some other values of λ and the same values of t, v and k as some of the listed ones.

2.2. Detailed results of application of λ-technique

In this section are listed Λ-matrices corresponding to each one of the ten above cited groups, together with representatives of the underlying orbits and with a representative of the generated designs, for each possible quadruple of parameters. The t-orbits and k-orbits corresponding to successive rows and columns of a Λ-matrix are listed in front of it.

2.2.1. Denotations.

Λ-matrices in this section will be denoted as $\Lambda(G; t, k)$. A Λ-matrix is determined by the corresponding group G and by the values of parameters t and k; it establishes relationship between t-orbits and k-orbits by action of G.

In order to enable precise identification of s-orbits (for $s \in \{4, 5, 6\}$), the following data will be given in the form $(A : B; C)$, where

- $A =$ the ordinal number of the coresponding orbit (= row or column of the (λ_{ij}) matrix).
- $B =$ $s - 2$ elements of the lexicographically the first "special" representative, apart from the compulsory elements 0, 1, ∞.
- $C =$ the number of "special" subsets (supersets of $\{0, 1, \infty\}$) within the orbit.

For example, the denotation $(4 : 2, 3, 7; 10)$ below (that is, $A = 4$; $B = 2, 3, 7$; $C = 10$), used for a 6-orbit by action of $PSL(2, 11)$, means that this orbit is the fourth one among the 6-orbits (corresponds to the 4th column of the λ_{ij} matrix), has the 6-subset $\{0, 1, 2, 3, 7, \infty\}$ as a representative and contains ten "special" 6-subsets.

The design(s) generated from a Λ-matrix are listed after the word "**Design(s)**". A representative design is given in ( )-brackets separately for each possible λ. Designs are denoted by the ordinal numbers of the columns belonging to the set P (cited in the description of Λ-technique); the blocks of the designs are exactly the k-sets belonging to the k-orbits corresponding to the columns of P.

Thus the denotation $(\lambda = 2 : 7, 21, 22, 30)$ after the matrix $\Lambda(PSL(2,23); 5, 6)$ means that the 6-sets of the 7th, 21st, 22nd and 30th orbit of this Λ-matrix constitute a 5-(24,6,2) design.

2.2.2. $PSL(2,11)$, $t = 5$, $k = 6$, $\lambda_{\text{trivial}} = 7$.

5-orbits: (1 : 2, 3; 30) (2 : 3, 4; 6)

6-orbits: (1 : 2, 3, 4; 30) (2 : 2, 3, 5; 12) (3 : 2, 3, 6; 10) (4 : 2, 3, 7; 10) (5 : 2, 3, 8; 10) (6 : 2, 3, 9; 12)

The 2×6 matrix $\Lambda(PSL(2,11); 5, 6)$: $\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

Designs: $(\lambda = 1 : 2)$ $(\lambda = 2 : 2, 6)$

2.2.3. $PGL(2,17)$, $t = 4$, $k = 5$, $\lambda_{\text{trivial}} = 14$.

4-orbits: (1 : 2; 3) (2 : 3; 6) (3 : 4; 6)

5-orbits: (1 : 2, 3; 30) (2 : 2, 5; 15) (3 : 2, 6; 30) (4 : 3, 7; 30)

The 3×4 matrix $\Lambda(PGL(2,17); 4, 5)$: $\begin{pmatrix} 8 & 2 & 4 & 0 \\ 4 & 0 & 4 & 6 \\ 2 & 4 & 4 & 4 \end{pmatrix}$

Designs: $(\lambda = 4 : 3)$

2.2.4. $PSL(2,19)$, $t = 4$, $k = 6$, $\lambda_{\text{trivial}} = 120$.

4-orbits: (1 : 2; 3) (2 : 3; 6) (3 : 4; 6) (4 : 8; 1) (5 : 12; 1)

6-orbits: (1 : 2, 3, 4; 30) (2 : 2, 3, 5; 60) (3 : 2, 3, 6; 60) (4 : 2, 3, 7; 30) (5 : 2, 3, 8; 60) (6 : 2, 3, 9; 60) (7 : 2, 3, 10; 30) (8 : 2, 3, 11; 10) (9 : 2, 3, 12; 30) (10 : 2, 3, 13; 60) (11 : 2, 3, 15; 30) (12 : 2, 5, 6; 30) (13 : 2, 5, 8; 10) (14 : 2, 5, 12; 60) (15 : 2, 5, 15; 30) (16 : 2, 5, 16; 30) (17 : 2, 6, 12; 10) (18 : 2, 6, 16; 30) (19 : 3, 4, 9; 20)

The 5×19 matrix $\Lambda(PSL(2,19); 4, 6)$:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
8	12	16	6	12	8	8	4	8	8	6	4	2	8	2	4	2	2	0
7	12	8	6	12	10	4	2	4	10	6	2	1	10	7	5	1	7	6
4	8	10	5	8	14	5	1	5	14	5	9	2	12	4	4	2	4	4
0	12	12	0	12	12	12	0	0	0	6	6	6	12	12	12	0	6	0
0	12	12	6	12	0	0	0	12	12	0	6	0	12	6	12	6	12	0

Design: $(\lambda = 60 : 4, 5, 7, 9, 10, 11, 13, 14, 15)$.

2.2.5. $PSL(2,23)$, $t = 5$, $k = 6$, $\lambda_{\text{trivial}} = 19$.

5-orbits: (1 : 2, 3; 30) (2 : 2, 5; 30) (3 : 2, 6; 30) (4 : 2, 8; 30) (5 : 3, 4; 30) (6 : 3, 7; 30) (7 : 3, 14; 30)

6-orbits: (1 : 2, 3, 4; 30) (2 : 2, 3, 5; 60) (3 : 2, 3, 6; 60) (4 : 2, 3, 7; 60) (5 : 2, 3, 8; 60) (6 : 2, 3, 9; 60) (7 : 2, 3, 10; 60) (8 : 2, 3, 11; 60) (9 : 2, 3, 12; 30) (10 : 2, 3, 13; 10) (11 : 2, 3, 14; 30) (12 : 2, 3, 15; 60) (13 : 2, 3, 18; 60) (14 : 2, 3, 19; 60) (15 : 2, 5, 6; 30) (16 : 2, 5, 7; 30) (17 : 2, 5, 8; 30) (18 : 2, 5, 10; 30) (19 : 2, 5, 11; 60) (20 : 2, 5, 14; 30) (21 : 2, 5, 15; 30) (22 : 2, 5, 17; 30) (23 : 2, 5, 18; 30) (24 : 2, 5, 19; 60) (25 : 2, 6, 8; 60) (26 : 2, 6, 10; 10) (27 : 2, 6, 14; 60) (28 : 2, 6, 19; 30) (29 : 2, 8, 14; 10) (30 : 3, 4, 9; 20) (31 : 3, 4, 11; 30) (32 : 3, 4, 16; 30) (33 : 3, 7, 10; 10) (34 : 3, 7, 21; 10)

The 7×34 matrix $\Lambda(PSL(2,23);5,6)$:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
2	2	2	1	1	2	2	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	1	1	1	0	0	0	0	0	0	1	1	1	2	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	1	1	3	0	0	1	2	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	1	1	1	1	0	0	0	0	0	0
0	0	1	0	1	1	1	0	0	0	1	2	0	3	0	0	1	0	0	1	1	0	1	1	1	0	1	1	1	0	0	0	0	0
1	1	0	0	2	1	0	1	0	0	0	1	2	0	0	0	0	1	1	1	0	0	0	1	2	0	0	0	0	2	1	1	0	0
0	1	0	1	0	1	2	1	0	0	0	1	0	1	0	0	0	0	2	0	0	0	0	2	0	0	2	1	0	0	1	1	1	1
0	0	0	0	1	0	0	1	1	0	1	1	1	0	1	1	1	0	1	0	1	1	0	1	2	0	2	0	0	0	1	1	0	0

Designs: ($\lambda = 1$: 9, 20, 32) ($\lambda = 2$: 7, 21, 22, 30) ($\lambda = 3$: 2, 11, 18, 20, 27) ($\lambda = 4$: 5, 7, 10, 18, 20, 22, 27) ($\lambda = 5$: 3, 5, 10, 13, 24, 26, 27, 28) ($\lambda = 6$: 1, 5, 7, 11, 18, 19, 20, 21, 23, 27) ($\lambda = 7$: 1, 7, 10, 12, 13, 18, 19, 20, 21, 22, 23, 27) ($\lambda = 8$: 1, 6, 7, 10, 13, 16, 18, 21, 23, 24, 25, 26, 27, 28) ($\lambda = 9$: 1, 5, 7, 8, 12, 13, 14, 16, 18, 19, 22, 23, 27)

2.2.6. $PGL(2,25)$, $t = 4$, $k = 6$, $\lambda_{\text{trivial}} = 231$.

4-orbits: (1 : 2; 3) (2 : 5; 6) (3 : 6; 6) (4 : 7; 6) (5 : 8; 2)

6-orbits: (1 : 2, 3, 4; 1) (2 : 2, 3, 5; 120) (3 : 2, 5, 6; 120) (4 : 2, 5, 7; 120) (5 : 2, 5, 8; 60) (6 : 2, 5, 9; 120) (7 : 2, 5, 10; 60) (8 : 2, 5, 11; 60) (9 : 2, 5, 13; 20) (10 : 2, 5, 15; 60) (11 : 2, 5, 16; 60) (12 : 2, 5, 17; 120) (13 : 2, 5, 18; 60) (14 : 2, 5, 19; 30) (15 : 2, 5, 20; 60) (16 : 2, 5, 21; 120) (17 : 2, 5, 22; 30) (18 : 2, 5, 23; 60) (19 : 2, 5, 24; 120) (20 : 2, 6, 8; 60) (21 : 2, 6, 9; 60) (22 : 2, 6, 10; 60) (23 : 2, 6, 11; 30) (24 : 2, 6, 12; 60) (25 : 2, 6, 21; 30) (26 : 2, 10, 12; 30) (27 : 5, 7, 12; 20) (28 : 6, 7, 15; 20)

The 5×28 matrix $\Lambda(PGL(2,25);4,6)$ the first part:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	40	16	16	8	8	8	8	4	8	8	8	12	4
0	12	16	16	8	20	6	8	2	12	10	20	6	5
0	12	16	16	8	8	12	8	2	6	4	20	6	2
0	12	16	20	6	20	8	6	4	8	10	12	8	6
0	12	12	0	12	24	0	12	0	0	6	12	12	0

The 5×28 matrix $\Lambda(PGL(2,25);4,6)$ the second part:

15	16	17	18	19	20	21	22	23	24	25	26	27	28
8	8	4	8	16	4	4	8	6	8	4	4	0	0
4	8	2	4	16	12	6	10	4	10	2	6	6	0
4	20	5	10	16	6	12	10	4	10	2	6	0	6
12	20	6	10	16	6	6	2	2	2	5	0	4	4
18	24	0	6	12	12	12	12	6	12	12	3	0	0

Designs: ($\lambda = 51 : 1, 3, 14, 17, 18, 20, 23, 25, 26$) ($\lambda = 60 : 3, 9, 11, 13, 18, 21, 22, 27$) ($\lambda = 81 : 1, 3, 7, 10, 13, 14, 15, 17, 21, 22, 25, 26, 27$) ($\lambda = 90 : 3, 4, 10, 13, 15, 18, 20, 21, 22, 23$) ($\lambda = 111 : 1, 3, 5, 7, 9, 11, 12, 13, 14, 15, 17, 18, 22, 23, 25, 26, 27$)

2.2.7. $PSL(2, 27)$, $t = 5$, $k = 6$, $\lambda_{\text{trivial}} = 23$.

5-orbits: $(1 : 2, 3; 30)$ $(2 : 2, 6; 30)$ $(3 : 3, 4; 30)$ $(4 : 3, 5; 30)$ $(5 : 3, 7; 30)$ $(6 : 3, 10; 30)$ $(7 : 3, 12; 30)$ $(8 : 3, 15; 30)$ $(9 : 4, 6; 30)$ $(10 : 4, 11; 30)$

6-orbits: $(1 : 2, 3, 4; 60)$ $(2 : 2, 3, 6; 30)$ $(3 : 2, 3, 7; 30)$ $(4 : 2, 3, 8; 30)$ $(5 : 2, 3, 9; 60)$ $(6 : 2, 3, 10; 30)$ $(7 : 2, 3, 11; 60)$ $(8 : 2, 3, 12; 60)$ $(9 : 2, 3, 13; 30)$ $(10 : 2, 3, 14; 60)$ $(11 : 2, 3, 15; 30)$ $(12 : 2, 3, 17; 60)$ $(13 : 2, 3, 18; 60)$ $(14 : 2, 3, 19; 30)$ $(15 : 2, 3, 20; 60)$ $(16 : 2, 3, 21; 30)$ $(17 : 2, 3, 22; 60)$ $(18 : 2, 3, 23; 60)$ $(19 : 2, 3, 24; 30)$ $(20 : 2, 6, 7; 60)$ $(21 : 2, 6, 12; 30)$ $(22 : 2, 6, 13; 60)$ $(23 : 2, 6, 14; 60)$ $(24 : 2, 6, 18; 60)$ $(25 : 2, 6, 20; 30)$ $(26 : 2, 6, 21; 30)$ $(27 : 3, 4, 9; 20)$ $(28 : 3, 4, 10; 60)$ $(29 : 3, 4, 11; 60)$ $(30 : 3, 4, 12; 30)$ $(31 : 3, 4, 15; 60)$ $(32 : 3, 4, 16; 60)$ $(33 : 3, 4, 17; 30)$ $(34 : 3, 4, 19; 30)$ $(35 : 3, 4, 23; 60)$ $(36 : 3, 4, 26; 30)$ $(37 : 3, 5, 12; 30)$ $(38 : 3, 5, 14; 30)$ $(39 : 3, 5, 15; 60)$ $(40 : 3, 5, 17; 60)$ $(41 : 3, 5, 18; 30)$ $(42 : 3, 5, 19; 30)$ $(43 : 3, 7, 10; 60)$ $(44 : 3, 7, 11; 30)$ $(45 : 3, 7, 17; 20)$ $(46 : 3, 7, 18; 60)$ $(47 : 3, 7, 20; 30)$ $(48 : 3, 10, 12; 60)$ $(49 : 3, 10, 14; 20)$ $(50 : 3, 10, 15; 30)$ $(51 : 3, 10, 18; 30)$ $(52 : 3, 12, 15; 30)$ $(53 : 3, 12, 18; 30)$ $(54 : 4, 6, 13; 20)$

The 10×54 matrix $\Lambda(PSL(2, 27); 5, 6)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
2	1	1	1	2	1	2	1	1	1	1	2	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	0	0	0	1	1	1	0	0	1	1	1	1	1	1	0	2	1	2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	1	0	0	0	0	0	1	0	0	1	1	0	0	1	0	1	0	1	0	0	2	2	1	1	2	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	1	1	1	1	0	1	0	0	0	0	0	1	1	1	1	0	1	1	0	1	1	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0	1	0	1	1	0	0	0	0	0	2	0	0	0	1	0	2	1	0	0	0	2	0	0	1	1	2	1	1	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0	0	0	1	0	2	0	2	0	0	2	1	0	0	0	0	0	0	0	0	1	1	0	0	1	0	1	0	0	0	0	0	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	2	0	2	0	0	0	0	0	0	2	0	0	0	1	1	0	0	1	1	0	1	0	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	1	0	1	1	0	0	1	1	0
1	0	0	0	1	0	0	0	1	0	1	0	1	0	0	0	1	0	0	1	0	0	0	1	0	1	0	2	0	0	1	1	0	0	0	0	1	0	2	0	1	0	1	0	0	1	0	0	0	1	1	1	1	0
0	0	0	1	1	0	1	1	0	0	0	1	0	0	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	1	1	1	0	0	1	0	0	0	0	0	0	2	0	0	2	0	2	0	0	1	0	1	2
0	1	0	0	1	1	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	1	1	0	0	0	1	0	0	0	0	0	1	0	0	1	2	0	0	1	1	1	0	1	1	2	0	1	0	1	0	0

Designs: ($\lambda = 2 : 7, 22, 39, 49$) ($\lambda = 3 : 2, 9, 14, 28, 33, 40, 48$) ($\lambda = 4 : 6, 7, 11, 22, 25, 26, 30, 34, 39, 45, 54$) ($\lambda = 5 : 2, 7, 9, 14, 22, 28, 33, 39, 40, 48, 49$) ($\lambda = 6 : 3, 4, 6, 11, 16, 19, 21, 25, 26, 27, 30, 33, 34, 39, 44, 45, 47, 51, 53, 54$) ($\lambda = 7 : 2, 6, 7, 9, 11, 14, 22, 25, 26, 28, 30, 33, 34, 39, 40, 45, 48, 54) ($\lambda = 8$: 3, 4, 6, 7, 11, 16, 19, 21, 22, 25, 26, 27, 29, 32, 37, 38, 39, 43, 45, 49, 52, 54) ($\lambda = 9$: 2, 3, 4, 6, 9, 11, 14, 16, 19, 21, 25, 26, 27, 28, 30, 33, 34, 36, 39, 40, 44, 45, 47, 48, 51, 53, 54) ($\lambda = 10$: 1, 5, 8, 10, 13, 15, 17, 18, 20, 24, 30, 33, 34, 36, 38, 39, 42, 44, 45, 47, 51, 53, 54) ($\lambda = 11$: 2, 3, 4, 6, 7, 9, 11, 14, 16, 19, 21, 22, 25, 26, 28, 29, 31, 32, 35, 39, 40, 43, 46, 48, 49).

2.2.8. $PSL(2,31)$, $t = 5$, $k = 6$, $\lambda_{\text{trivial}} = 27$.

5-orbits: (1 : 2, 3; 30) (2 : 2, 5; 30) (3 : 2, 6; 30) (4 : 2, 8; 30) (5 : 2, 9; 30) (6 : 2, 18; 30) (7 : 3, 4; 30) (8 : 3, 7; 30) (9 : 3, 8; 30) (10 : 3, 10; 30) (11 : 4, 6; 30) (12 : 4, 9; 30) (13 : 5, 6; 10) (14 : 5, 7; 30) (15 : 12, 13; 6)

6-orbits: (1 : 2, 3, 4; 30) (2 : 2, 3, 5; 60) (3 : 2, 3, 6; 30) (4 : 2, 3, 7; 60) (5 : 2, 3, 8; 60) (6 : 2, 3, 9; 60) (7 : 2, 3, 10; 60) (8 : 2, 3, 11; 60) (9 : 2, 3, 13; 60) (10 : 2, 3, 14; 60) (11 : 2, 3, 15; 60) (12 : 2, 3, 16; 30) (13 : 2, 3, 17; 10) (14 : 2, 3, 18; 30) (15 : 2, 3, 19; 60) (16 : 2, 3, 20; 60) (17 : 2, 3, 21; 60) (18 : 2, 3, 22; 60) (19 : 2, 3, 24; 60) (20 : 2, 3, 25; 60) (21 : 2, 3, 26; 60) (22 : 2, 3, 27; 60) (23 : 2, 3, 28; 30) (24 : 2, 5, 6; 60) (25 : 2, 5, 7; 60) (26 : 2, 5, 8; 60) (27 : 2, 5, 9; 30) (28 : 2, 5, 10; 60) (29 : 2, 5, 12; 60) (30 : 2, 5, 13; 60) (31 : 2, 5, 14; 60) (32 : 2, 5, 15; 30) (33 : 2, 5, 18; 60) (34 : 2, 5, 19; 20) (35 : 2, 5, 21; 60) (36 : 2, 5, 23; 60) (37 : 2, 5, 24; 30) (38 : 2, 5, 25; 60) (39 : 2, 5, 27; 60) (40 : 2, 5, 28; 30) (41 : 2, 6, 7; 30) (42 : 2, 6, 8; 60) (43 : 2, 6, 9; 60) (44 : 2, 6, 10; 30) (45 : 2, 6, 12; 30) (46 : 2, 6, 18; 60) (47 : 2, 6, 21; 60) (48 : 2, 6, 23; 60) (49 : 2, 6, 26; 30) (50 : 2, 6, 27; 30) (51 : 2, 6, 28; 60) (52 : 2, 8, 10; 30) (53 : 2, 8, 13; 60) (54 : 2, 8, 18; 60) (55 : 2, 8, 21; 60) (56 : 2, 8, 26; 30) (57 : 2, 9, 13; 60) (58 : 2, 9, 21; 30) (59 : 2, 9, 27; 60) (60 : 2, 9, 28; 30) (61 : 2, 18, 21; 30) (62 : 2, 18, 26; 30) (63 : 3, 4, 9; 30) (64 : 3, 4, 10; 60) (65 : 3, 4, 11; 30) (66 : 3, 4, 12; 30) (67 : 3, 4, 15; 60) (68 : 3, 4, 23; 30) (69 : 3, 4, 24; 30) (70 : 3, 4, 25; 60) (71 : 3, 4, 26; 30) (72 : 3, 7, 8; 30) (73 : 3, 7, 15; 12) (74 : 3, 7, 20; 10) (75 : 3, 7, 23; 20) (76 : 3, 8, 12; 30) (77 : 3, 8, 14; 30) (78 : 3, 8, 18; 60) (79 : 3, 8, 22; 30) (80 : 3, 10, 18; 10) (81 : 4, 6, 17; 10) (82 : 5, 7, 23; 10) (83 : 5, 7, 29; 12)

The 15×83 matrix $\Lambda(PSL(2,31); 5, 6)$:

```
         11111111112222222222333333333344444444445555555555666666666677777777778888
12345678901234567890123456789012345678901234567890123456789012345678901234567890123
22111112111111111211111000000000000000000000000000000000000000000000000000000000000
01100000011000011011000111111211121111110000000000000000000000000000000000000000000
01121000201100000000000110001001100000001121211111100000000000000000000000000000000
00001112100000001011100102120000000101000101000100021111000000000000000000000000000
00000110110000110100001101111100110001010020000000010100021110000000000000000000000
00000000000001102100121000001000102001100000011011000111011012000000000000000000000
10001110100000001011100000000000000110000100020000000101000010011112111100000000000
00010001001000110001010000000111000000001010100100002020001000001011010111100000000
01011000010000010100110010000200000110100000000110100000011000000101000100011210000
00000020000011001001001000010201000000100000011100000110010000110000011000000221000
01001000001100000002000001000100010000100010101100010010020000110000110000020200100
00000100010000010001000020200100010001000100021001101001000000110020011100001000000
00000000000000000000000030000000000000030030000000330000300000003003000000300000000
00010101011000100000010010000100001200000200000000001001100110101101000000101000011
00000000000000000000000050000000000000055000000000005000000050000000000010000000001
```

Designs: ($\lambda = 6$: 8, 13, 15, 16, 21, 25, 26, 29, 30, 38, 45, 46, 49, 67, 71, 76, 80, 83) ($\lambda = 12$: 1, 2, 5, 6, 9, 10, 13, 15, 16, 22, 25, 26, 28, 29, 31, 33, 36, 37, 41, 42, 46, 47, 49, 53, 54, 55, 58, 59, 70, 73, 78, 83).

2.2.9. $PGL(2,32)$, $t = 4$, $k = 5$, $\lambda_{\text{trivial}} = 29$.

4-orbits: (1 : 2; 6) (2 : 4; 6) (3 : 6; 6) (4 : 14; 6) (5 : 16; 6)

5-orbits: (1 : 2, 3; 15) (2 : 2, 5; 60) (3 : 2, 6; 60) (4 : 2, 8; 60) (5 : 2, 9; 60) (6 : 2, 11; 60) (7 : 2, 12; 15) (8 : 4, 5; 15) (9 : 4, 17; 60) (10 : 6, 14; 15) (11 : 14, 22; 15)

The 5×11 matrix $\Lambda(PGL(2,32); 4, 5)$:
$$\begin{pmatrix} 4&4&8&4&4&4&1&0&0&0&0\\ 1&4&0&8&4&4&0&4&4&0&0\\ 0&0&4&4&4&4&4&0&8&1&0\\ 0&8&4&4&4&0&0&0&4&4&1\\ 0&4&4&0&4&8&0&1&4&0&4 \end{pmatrix}$$

Designs: ($\lambda = 4$: 5) ($\lambda = 5$: 1, 7, 8, 10, 11) ($\lambda = 9$: 1, 5, 7, 8, 10, 11)

2.2.10. $PGL(2,37)$, $t = 4$, $k = 5$, $\lambda_{\text{trivial}} = 34$.

4-orbits: (1 : 2; 3) (2 : 3; 6) (3 : 4; 6) (4 : 5; 6) (5 : 6; 6) (6 : 8; 6) (7 : 11; 2)

5-orbits: (1 : 2, 3; 30) (2 : 2, 5; 60) (3 : 2, 6; 60) (4 : 2, 7; 15) (5 : 2, 8; 60) (6 : 3, 4; 30) (7 : 3, 7; 60) (8 : 3, 12; 60) (9 : 3, 14; 30) (10 : 3, 15; 30) (11 : 3, 26; 60) (12 : 4, 5; 30) (13 : 4, 11; 10) (14 : 4, 17; 30) (15 : 5, 8; 30)

The 7×15 matrix $\Lambda(PGL(2,37); 4, 5)$:

$$\begin{pmatrix} 8&8&8&2&8&0&0&0&0&0&0&0&0&0&0\\ 4&0&4&0&0&4&8&4&4&2&4&0&0&0&0\\ 2&4&0&0&4&4&0&8&0&0&4&4&2&2&0\\ 0&8&4&0&0&2&4&0&0&4&4&4&0&0&4\\ 0&0&4&4&4&0&4&4&0&4&4&2&0&4&0\\ 0&4&4&0&4&0&0&4&6&0&4&0&0&4&4\\ 0&0&0&0&12&0&12&0&0&0&0&0&4&0&6 \end{pmatrix}$$

Design: ($\lambda = 16$: 2, 3, 7, 8, 13, 14)

2.2.11. $PSL(2,47)$, $t = 4$, $k = 5$, $\lambda_{\text{trivial}} = 44$.

4-orbits: (1 : 2; 3) (2 : 3; 6) (3 : 4; 6) (4 : 5; 3) (5 : 6; 6) (6 : 7; 6) (7 : 10; 3) (8 : 11; 6) (9 : 13; 3) (10 : 22; 3)

5-orbits: (1 : 2, 3; 30) (2 : 2, 5; 30) (3 : 2, 6; 30) (4 : 2, 7; 30) (5 : 2, 8; 30) (6 : 2, 10; 30) (7 : 2, 12; 30) (8 : 2, 13; 30) (9 : 2, 14; 30) (10 : 2, 16; 30) (11 : 3, 4; 30) (12 : 3, 7; 30) (13 : 3, 8; 30) (14 : 3, 11; 30) (15 : 3, 12; 30) (16 : 3, 13; 30) (17 : 3, 14; 30) (18 : 3, 15; 30) (19 : 3, 17; 30) (20 : 3, 19; 30) (21 : 3, 20; 30) (22 : 3, 22; 30) (23 : 3, 26; 30) (24 : 3, 39; 30) (25 : 4, 9; 30) (26 : 4, 13; 30) (27 : 4, 19; 30) (28 : 4, 20; 30) (29 : 4, 21; 30) (30 : 4, 27; 30) (31 : 5, 8; 30) (32 : 6, 10; 30) (33 : 7, 11; 30)

The 10×33 matrix $\Lambda(PSL(2,47); 5, 6)$:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
8	4	4	4	8	4	4	4	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	2	0	0	0	0	0	0	2	4	8	8	2	2	2	4	2	2	2	2	4	2	2	0	0	0	0	0	0	0	0	0
2	2	0	0	4	0	0	2	0	0	4	0	0	0	4	0	4	0	0	0	2	0	2	0	4	4	2	2	4	2	0	0	0
0	4	4	0	0	4	4	0	0	0	0	0	0	8	4	4	0	4	4	0	0	0	0	0	0	0	0	4	4	0	4	0	0
0	0	2	4	4	0	0	0	0	2	0	4	4	0	0	4	4	0	0	0	2	0	2	0	0	0	4	4	0	4	2	2	0
0	0	0	4	4	2	2	0	2	0	2	4	4	2	0	0	0	4	4	4	0	0	0	2	2	2	0	0	2	4	0	0	4
0	4	0	0	0	4	4	0	0	4	0	0	0	0	0	0	0	4	4	0	0	4	8	0	0	4	4	0	0	0	0	4	4
0	2	0	0	4	2	2	2	2	0	0	4	4	2	0	0	8	0	0	0	0	2	0	2	4	0	0	0	0	0	4	4	2
0	0	0	0	0	0	0	4	4	4	0	0	0	0	4	4	0	0	0	4	0	0	0	8	0	4	4	0	0	0	4	0	0
0	0	4	0	0	0	0	4	4	0	0	0	0	0	0	0	0	0	0	4	8	4	0	0	0	0	0	4	4	0	0	4	4

Designs: ($\lambda = 8 : 5, 14, 21, 23, 24$) ($\lambda = 12 : 2, 4, 10, 14, 17, 20, 21, 26$) ($\lambda = 16 : 3, 6, 8, 9, 11, 14, 17, 20, 23, 27, 30$) ($\lambda = 20 : 3, 4, 5, 8, 11, 14, 18, 21, 22, 23, 24, 25, 27, 31$)

2.3. Some observations on the constructed designs

In this section we give some miscalleneous data concerning the constructed designs and the construction itself.

The designs for $q \in \{17, 32\}$ were considered in more detail in [2]; just a few data are mentioned here. The construction for $q = 17$ is due to Alltop and was described in [5], Example 8.5, pp. 186-187; Λ-technique is an improvement of the Alltop's construction. The design constructed for $q = 32$ and $\lambda = 5$ is the first member of an Alltop's infinite class of 4-designs. It is likely that all the constructed designs for $q = 32$ can arise ([8]) by action of the 4-homogeneous group $PGamaL(2, 32)$.

The designs with $q = 11$ and $q = 23$ are related to the well-known ([5]) Steiner systems $S(5, 6; 12)$ and $S(5, 6; 24)$ (that is, to the 5-$(12, 6, 1)$ design and to the 5-$(24, 6, 1)$ design). The first one of these Steiner systems is, as stated in [7], Theorem 2.26., the uniquely determined Steiner system $S(5, 6; 12)$, with the automorphism group isomorphic to the famous Mathieu 5-transitive group M_{12} of cardinality $8 \cdot 9 \cdot 10 \cdot 11 \cdot 12$.

The brute-force search over the colums of Λ-matrices was applicable on a PC-386 computer in the cases when the number n of columns was restricted to 30 ($n = 30$ required one week of computing time and each added unit to n would double the time required). The following shortcut was used for $q = 23$ and $q = 47$, where n is equal to 34 and 33 respectively:

It is observed that there exist in both cases several pairs of duplicate columns within the Λ-matrix (exactly four pairs with $q = 23, n = 34$ and three pairs with $q = 47, n = 33$). The search is performed over the reduced 30-column matrices, which are obtained from the Λ-matrices by discarding one of the columns from each duplicate pair. Such a reduction does not guarantee completeness of the search; it might happen that some of the existing designs require combinations of columns which include both columns in a duplicate pair. However, the arguments related to the specific coefficients of the two Λ-matrices show that no set of design parameters is missed in this way.

For example, the set of λ values with $q = 23$ is complete (all the values in the interval $[1,...,18 = \lambda_{\text{trivial}} - 1]$ are present). Similarly, all the elements in the first row of the Λ-matrix for $q = 47$ are divisible by 4, which implies that all the corresponding values of λ must be divisible by 4; an additional argument shows that $\lambda = 4$ is impossible.

The Λ-matrices with $q = 27$ and $q = 31$ have very large numbers of columns (54 and 83 respectively), so there is no chance for a full search. However, ad hoc designed heuristic approaches ([3]) have given designs with all the possible values of λ in these cases.

The number of successful (that is, design-corresponding) combinations of columns is very large with some of the Λ-matrices (several hundreds with $q = 19$ and $q = 47$ and several thousands with $q = 23$).

Some of the obtained parameters seem to be particularly interesting. For example, the designs constructed for $q = 37$ seem ([8]) to be the first 4-designs known on 38 points.

A design isomorphism search was performed ([6]) among the constructed 560 4-designs on 48 points for $\lambda \in \{8, 12, 16, 20\}$. Auxiliary graphs were attached to the designs so that non-isomorphism of some two attached graphs implies non-isomorphism of the corresponding designs. Global results of this search seem to be very interesting. All the equivalence classes of isomorphic attached graphs are of cardinality 2; this implies that at least one half of the total number of the constructed designs are pairwise non-isomorphic. Moreover, the unique and involutory (a product of transpositions) isomorphism maps onto each other the two graphs of each one of the equivalence classes; this means that the recognized isomorphism is a global symmetry of the whole found class of 4-designs.

Finally, it seems worth-while to try an isomorphism search for $q = 19$. It is only in this case that there exists a unique (and self-complementary) value $\lambda = 60$. Is the 4-(20,6,60) design unique up to an isomorphism within the class of designs with these parameters generated by $PSL(2, 19)$? The isomorhism search in this case might use attached hypergraphs with edges containing three vertices each.

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