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INVITED LECTURE

ON THE MAXIMAL ORDER OF CERTAIN ARITHMETIC FUNCTIONS

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ABSTRACT. An upper bound for $f(f(n))$ is obtained when $f(n)$ belongs to a certain class of multiplicative functions. Also the maximal and average order of $Q(n)$ and $Q(Q(n))$ are determined, where $Q(n)$ denotes the number of distinct exponents in the canonical decomposition of n.

It is well-known (see e.g. Hardy and Wright [3]) that

$$\limsup_{n\to\infty} \frac{\log d(n) \log\log n}{\log n} = \log 2 \tag{1}$$

where $d(n)$ denotes the number of divisors of n. A more difficult problem is to determine the maximal order of $d(d(n))$. In [1] P. Erdős and the author have shown that

$$\log d(d(n)) \ll \left(\frac{\log n \log_2 n}{\log_3 n}\right)^{1/2}, \tag{2}$$

where $\log_k x = \log(\log_{k-1} x)$ is the k-fold iterated natural logarithm of x, and $f(x) \ll g(x)$ (same as $f(x) = O(g(x))$) means that $\|f(x)\| \le Cg(x)$ for some $C > 0$, $g(x) > 0$, $x \ge x_0$. The upper bound in (2) is certainly close to being best possible. Namely if one takes

$$N = p_1^{p_1 - 1} p_2^{p_2 - 1} \cdots p_r^{p_r - 1}, \qquad r \to \infty,$$

where p_j is the j-th prime number, then

$$d(N) = p_1 p_2 \cdots p_r, \quad d(d(N)) = 2^r.$$

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But since from the prime number theorem (see [3]) it follows that

$$p_k = k(\log k + O(\log\log k)),$$

we have, with $\theta(x) = \sum_{p\le x} \log p$,

$$\log N = \sum_{k\le r} \log p_k - \theta(p_k) = \sum_k k\log^2 k + O(r^2\log r\log\log r)$$
$$= \frac{1}{2}r^2\log^2 r(1 + O\left(\frac{\log\log r}{\log r}\right)).$$

Therefore

$$r = \omega(N) = \frac{2(2\log N)^{1/2}}{\log_2 n}\left(1 + O\left(\frac{\log_3}{\log_2}\right)\right), \tag{3}$$

where $\omega(n)$ denotes the number of distinct prime factors of n. This gives

$$\log d(d(N)) = \frac{2\log 2(2\log N)^{1/2}}{\log_2 n}\left(1 + O\left(\frac{\log_3}{\log_2}\right)\right), \tag{4}$$

which was already known to S. Ramanujan (see [5]).

P. Erdős and I. Kátai [2] proved that for every $\varepsilon > 0$

$$\log d^{(r)}(n) \ll (\log n)^{1/\ell_r+\varepsilon}$$

and that

$$\log d^{(r)}(n) > (\log n)^{1/\ell_r-\varepsilon}$$

for infinitely many n, where is the r-fold iterated divisor function and is the r-th Fibonacci number: Their method, however, does not seem to yield any improvement of (2). ℓ_r is r-th Fibnacci number: ℓ_{-1}, ℓ_0, $\ell_r = \ell_{r-2}$ $(r \ge 1)$. Their method, however, does not seem to yield any improvement of (2).

The argument in [1] that led to (2) depended on an upper bound for

$$Q = Q(S,n) := \sum_{a_i\ge S} 1, \tag{5}$$

where $n > 1$, $1 \le S \le \log n/\log 2$ and

$$n = p_{j_1}^{a_1}p_{j_2}^{a_2}\cdots p_{j_r}^{a_r} \tag{6}$$

is the canonical decomposition of n. As one trivially has $n \geq 2^{QS}$, it follows that

$$Q \leq \frac{\log n}{S \log 2} \qquad (n > 1),$$

but a slightly better bound also holds. Namely (6) yields

$$\log n \geq \sum_{a_i \geq S} a_i \log p_{j_i} \geq S \sum_{p \leq p_Q} \log p = S\theta(p_Q) \geq \frac{1}{2} SQ \log Q$$

for $Q \geq Q_0$. Thus $Q \leq Q_1 = Q_1(S, n)$ where $(2 \log n)/S = Q_1 \log Q_1$. If $S \leq \log^A n$, $0 < A < 1$, then

$$2 \log Q_1 \geq \log Q_1 + \log\log Q_1 \geq \log 2 + (1 - A) \log\log n \gg \log\log n,$$

hence $logQ_1 \gg \log\log n$, which gives

$$Q(S, n) \ll \frac{\log n}{S \log\log n} \qquad (1 \leq S \leq log^A n\, 0 < A < 1). \tag{7}$$

If $a(n)$ denotes the number of non-isomorphic abelian (i.e. commutative) groups with n elements, then $a(n)$ is a multiplicative function (meaning $a(mn) = a(m)a(n)$ if m, n are coprime natural numbers) and $a(pk) = P(k)$, where $P(k)$ is the number of partitions of k. It was shown in [1] that with n elements, then $a(n)$ is a multiplicative function (meaning $a(mn) = a(m)a(n)$ if m, n are coprime natural numbers) and $a(p^k) = P(k)$, where $P(k)$ is the number of partitions of k. It was shown in [1] that

$$\omega(a(n)) \ll (\log n)^{3/4}(\log_2 n)^{-8}, \log a(a(n)) \ll (\log n)7/8(log_2 n)^{-C} \tag{8}$$

with $B = 11/8$,$C = 19/16$. In what follows a variation of the method developed in [1] will be used to prove a general result for iterates of certain arithmetic functions, which in the case of the function a(n) yields the slightly better values B = 7/4, C = 11/8 in (8). Perhaps the correct values of the exponents of the logarithms in (8) are both 1/2 (they cannot be smaller than 1/2). If true, this conjecture seems difficult to prove. [1] will be used to prove a general result for iterates of certain arithmetic functions, which in the case of the function $a(n)$ yields the slightly better values $B = 7/4$, $C = 11/8$ in (8). Perhaps the correct values of the exponents of the logarithms in (8) are both 1/2(they cannot be smaller than 1/2). If true, this conjecture seems difficult to prove.

The functions $a(n)$ and $d(n)$ belong to the class of arithmetic functions F, which contains all multiplicative, prime-independent functions $f(n) : N \to N$ such that

$$f(p^k) = g(k),\ g(k) \le e^{Ak^c} \qquad (0 < c < 1,\ A > 0) \tag{9}$$

for all integers $k \ge 1$ and primes p, where $g(k) \in N$. As we have $d(p^k) = k+1$ (9) holds in this case for any $c > 0$, and in the case of $a(n)$ it holds with $c = 1/2$, since $P(k) \le e^{A\sqrt{k}}$ (see [5]). A simple proof that

$$\limsup_{n\to\infty} \frac{\log f(n) \log\log n}{\log n} = \max_{k\ge 1}(f(2^k))^{1/k} \tag{10}$$

if $f(n) \in F$ was given by P. Shiu [6]. We shall be interested here in the maximal order of $f(f(n))$ when $f(n) \in F$. Lack of information about the arithmetic structure of $g(k)$ makes this, in general, quite a difficult problem. Even in the relatively simple case of $d(n)$ the existing bounds (2) and (3) are of a different order of magnitude. We shall prove an upper bound result, contained in

Theorem 1. *If $f(n) \in F$ and c is given by* (9), *then*

$$\log f(f(n)) \ll (\log n)^{11+2c-c^2} (\log_2 n)^{(c^2-3)/2}. \tag{11}$$

Proof. We shall prove first that

$$\omega(f(n)) \ll (\log n)^{(c+1)/2} (\log_2 n)^{-(c+3)/2}, \tag{12}$$

which seems to be of independent interest. Let the a_j's denote the distinct exponents in the canonical decomposition of $n (n > 1)$. Since

$$\omega(mn) \le \omega(m) + \omega(n),\ \omega(n^k) = \omega(n),\ \omega(n) \ll \frac{\log n}{\log\log n},$$

we have, for suitaqble integers $\beta_i \geq 1$,

$$\omega(f(n)) = \omega\left(\prod_{a_i<S} g^{\beta_i}(a_i) \prod_{a_i\geq S} G^{\beta_i}(a_i)\right) \leq \sum_{a_i<S} \omega(g(a_i)) + \sum_{a_i\geq S} \omega(g(a_i))$$

$$\ll \frac{S^c}{\log S} \sum_{a_i<S} 1 + \sum a_i \geq S \frac{a_i^c}{\log a_i}$$

$$\ll \frac{S^{c+1}}{\log S} + \sum_{j=0}^{O(\log\log n)} \sum_{2^j S \leq a_i \leq 2^{j+1} S} \frac{a_i^c}{\log a_i} + \sum_{a_i \geq (\log n)^{(3+c)/4}} \frac{a_i^c}{\log a_i}$$

$$\ll \frac{S^{c+1}}{\log S} + \sum_{j=0}^{O(\log\log n)} \frac{2^{jc} S^c}{\log S} Q(2^j S, n) + (\log n)^c Q((\log n)^{(3+c)/4}, n)$$

$$\ll \frac{S^{c+1}}{\log S} + \sum_{j=0}^{O(\log\log n)} \frac{2^{jc} S^c}{\log S} \cdot \frac{\log n}{2^j S \log\log n} + (logn)^{(3+c)/4}$$

$$\ll \frac{S^{c+1}}{\log S} + \frac{S^{c-1}\log n}{\log S \log\log n} + (logn)^{(3+c)/4},$$

where summation is over j such that $2^{j+1} S \geq (\log n)^{(3c+1)/4}$, and where we used (7). Now the choice

$$S = \left(\frac{\log n}{\log\log n}\right)^{1/2}$$

gives (12), since $0 < c < 1$. To obtain (11) from (12) note that, if (6) holds (the exponents now do not have to be distinct), then by Hölder's inequality and (7) it follows that ($\Omega(n)$ is the number of all prime divisors of n)

$$\log f(n) \leq A \sum_{i=1}^{r} a_i^c \leq A(\Omega(n))^c(\omega(n))^{1-c}. \tag{13}$$

In (13) we replace n by $f(n)$, use (12),(10) and the fact that $\Omega(n) \leq \log n / \log 2$ for all $n \geq 1$. We obtain

$$\log f(f(n)) \ll \left(\frac{\log n}{\log\log n}\right)^c ((\log n)^{(c+1)/2} (\log\log n)^{-(c+1)/2})^{1-c},$$

which gives then (11).This ends the proof of Theorem 1.

We recall that $a(n) \in F$ with $c = 1/2$, so that (12) and (11) yield $B = 7/4$ and $C = 11/8$ in (8), as already mentioned.

It follows from (10) that (11) gives a non-trivial upper bound for $log f(f(n))$. However, Theorem 1 certainly does not resolve the problem of the maximal order of $log f(f(n))$, whose solution requires additional information on the function $g(k)$ in (9). To see that $f(f(n))$ may assume both large and very small values infinitely often if $f(n) \in F$, we present the following two examples.

Example 1. Let $f_1(n) \in F$ with $f_1(p^k) = g_1(k)$, $g_1(1) = g_1(2) = 2$ $g_1(k) = [e^{k^c}]$ for $k \geq 3$ and a fixed c such that $0 < c < 1$, where $[x]$ denotes the integer part of x. Then if $n = (p_1 p_2 \cdots p_K)^2$ $(K \to \infty)$ we have

$$f_1(n) = 2^K,\ f_1(f_1(n)) = [e^{K^C}],\ \log n = 2\theta(p_K) \sim 2K \log K.$$

Thus for infinitely many n we have

$$\log f_1(f_1(n)) \gg \left(\frac{\log n}{\log\log n}\right)^c. \tag{14}$$

By construction the constant c in (14) is, for $f_1(n)$, the same as the one appearing in (9). If we compare the bounds in (11) and (14) for $log f_1(f_1(n))$ it is hard to tell which one lies closer to the true order of magnitude of $log f_1(f_1(n))$. Although $g_1(k)$ in this example is of simple form, its arithmetic structure is obscure, and for this reason the problem is a hard one.

Example 2. Let $f_2(n) \in F$ with

$$f_2(p^k) = \begin{cases} 1 & k \neq 2 \\ 2 & k = 2. \end{cases}$$

In the previous example the function $f_1(f_1(n))$ exhibited large values, but in this case we clearly have

$$\liminf_{n\to\infty} f_2(f_2(n)) = 1,\ \limsup_{n\to\infty} f_2(f_2(n)) = 2,$$

since $f_2(f_2(n))$ equals either 1 or 2. Here, at least, the problem of the maximal order of $f_2(f_2(n))$ is solved. Note, however, that $f_2(n)$ itself takes large values, since by (10) one has

$$\limsup_{n\to\infty} \frac{\log f_2(n) \log\log n}{\log n} = \frac{\log 2}{2}.$$

Related to the functions $\omega(n)$, $\Omega(n)$ is the function $Q(n)$, which for $n > 1$ we define as the number of *distinct* exponents a_j in the canonical decompositon (6) of n, and for convenience we set $Q(1) = 1$. Note that the function $Q(n)$ is neither multiplicative nor additive. We shall determine the maximal and average order of $Q(n)$ and $Q(Q(n))$. The results on the maximal order are contained in

Theorem 2. *For $n \geq n_o$ we have*

$$Q(n) \leq 2\left(\frac{\log n}{\log_2 n}\right)^{1/2}\left(1 + O\left(\frac{\log_3 n}{\log_2 n}\right)\right), \tag{15}$$

and equality holds in (15) *for infinitely many n. We also have*

$$Q(Q(n)) \leq \left(\frac{2\log_2 n}{\log_3 n}\right)^{1/2}\left(1 + O\left(\frac{\log_4 n}{\log_3 n}\right)\right), \tag{16}$$

and equality holds in (16) *for infinitely many n.*

Proof. Take

$$n = p_1^1 p_2^2 \cdots p_K^K, \quad K \to \infty. \tag{17}$$

Then

$$K = \omega(n) = Q(n), \quad Q(Q(n)) = Q(K). \tag{18}$$

But from (17) we have

$$\log n = \sum_{j \leq K} j \log p_j = \sum_{j \leq K} j(\log j + O(\log_2 j)) \tag{19}$$

$$= \frac{1}{2}K^2 \log K + O(K^2 \log_2 K),$$

which gives

$$Q(n) = 2\left(\frac{\log n}{\log_2 n}\right)^{1/2}\left(1 + O\left(\frac{\log_3 n}{\log_2 n}\right)\right)$$

for n given by (17), that is, for infinitely many n. From (18) we have

$$Q(Q(n)) = Q(K) = 2\left(\frac{\log K}{\log_2 K}\right)^{1/2}\left(1 + O\left(\frac{\log_3 K}{\log_4 K}\right)\right)$$

for infinitely many K of the form

$$K = p_1^1 p_2^2 \cdots p_r^r, \; r \to \infty. \tag{20}$$

But from (19) it follows that

$$\log K = \frac{1}{2}\log_2 n + O(\log_3 n), \; \log\log K = \log_3 n + O(1).$$

Inserting those values in the expression for $Q(Q(n))$ it follows that equality holds in (16) if n is given by (17) and $K$by (20).

To obtain an upper bound in (15) note that if $(j_1, j_2, \ldots, j_Q)$ is any permutation of $(1, 2, \ldots, Q)$ and $1 \le a_1 < \cdots < a_Q$ $(Q = Q(n))$ are the distinct exponents in the canonical decomposition of n, then

$$a_i \ge i \qquad (i = 1, \ldots, Q).$$

Thus we have, for some permutation $(j_1, j_2, \ldots, j_Q)$ of $(1, 2, \ldots, Q)$,

$$\begin{aligned}\log n &\ge \sum_{i=1}^{Q} a_i \log p_{j_i} \ge \sum_{i=1}^{Q} a_i \log p_{Q-i+1} \ge \sum_{i=1}^{Q} i \log p_{Q-i+1} \\ &= \sum_{i=1}^{Q} (Q - i + 1) \log p_i = \sum_{i=1}^{Q} (Q - i + 1)(\log i + O(\log_2 i)) \\ &= Q \sum_{i=1}^{Q} \log i - \sum_{i=1}^{Q} i \log i + O(Q^2 \log_2 Q) = \frac{1}{2} Q^2 (\log Q + O(\log_2 Q)).\end{aligned}$$

The above expression is similar to (19) and easily implies the upper bound in (15). Since the right-hand side of (15) is an increasing function of n for $n \ge n_1$ we have

$$Q(Q(n)) \le 2 \left(\frac{\log Q(n))}{\log_2 Q(n)} \right)^{1/2} \left(1 + O\left(\frac{\log_3 Q(n)}{\log_2 Q(n)} \right) \right),$$

and if apply (15) to the right-hand side of the last inequality, we obtain (16). This completes the proof of Theorem 2.

To investigate the average order of $Q(n)$ and $Q(Q(n))$ we shall use the approach developed by G. Tenenbaum and the author [4]. Therein an s-function $f(n)$ was defined as an arithmetic function for which $f(n) = f(s(n))$, where $s(n)$ denotes the squarefull part of n (s is called squarefull if $s = 1$ or if $p^2 \mid s$ whenever $p \mid s$, p a prime). Thus $a(n)$ and $\Omega(n) - \omega(n)$ are both s-functions, the former being multiplicative and the atter additive. Now $Q(n)$ is neither multiplicative nor additive, but it turns out that it is "nearly" an s-function. Every n can be uniquely written as $n = qs$, $(q, s) = 1$, where $q = q(n)$ is squarefree (meaning that it is either 1 or a product of distinct primes) and $s = s(n)$ is squarefull. But then

$$Q(n) = \begin{cases} 1 + Q(s(n)) & \text{if } q(n) > 1, \\ Q(s(n)) & \text{if } q(n) = 1. \end{cases} \tag{21}$$

Therefore

$$\sum_{n\le x} Q(n) = \sum_{s\le x}(1+Q(s)) \sum_{1<n\le x/s,(q,s)=1} 1 + \sum_{s\le x} Q(s).$$

We evaluate the sum over q by (1.4) and (1.5) of [4], noting that $\sum_{s\le x} 1 \ll \sqrt{x}$. We obtain

$$(22)\quad \sum_{n\le x} Q(n) = \sum_{s\le x}(1+Q(s))\times$$
$$\times\left\{\frac{6x}{\pi^2 s}\prod_{p|s}(1+p^{-1})^{-1} + O(B(s)s^{-1/2}x^{1/2}\log x\right\}$$

with

$$B(n) = \prod_{p|n}(1+p^{-1/2}.$$

To estimate the error term in (22) we use (15) and

$$\sum_{s\le x} B(s)s^{-1/2} \le \prod_{p\le x}(1+B(p)\sum_{m=2}^{\infty} p^{-m/2}) \ll \log x.$$

In a similar way we may evaluate the summatory function of $Q(Q(n))$. The expression will be similar to (22), only instead of $1+Q(s)$ we shall have $Q(1+Q(s))$. We obtain

Theorem 3. *We have*

$$\sum_{n\le x} Q(n) = Dx + O(x^{1/2}\log^{5/2} x(\log_2 x)^{-1/2}),$$

$$D = \frac{6}{\pi^2}\sum_{s=1}^{\infty}\frac{1+Q(s)}{s}\prod_{p|s}(1+p^{-1})^{-1},$$

$$\sum_{n\le x} Q(Q(n)) = Ex + O(x^{1/2}\log^2 x(\log_2 x)^{1/2}(\log_3 x)^{-1/2}),$$

$$E = \frac{6}{\pi^2}\sum_{s=1}^{\infty}\frac{Q(1+Q(s))}{s}\prod_{p|s}(1+p^{-1})^{-1}.$$

It may be noted that by similar arguments one also obtains

$$(23)\quad \sum_{n\le x, Q(n)=k} 1 = d_k x + O(x^{1/2}\log^2 x),$$

where the so-called "local density" d_k is given by

$$d_k = \frac{6}{\pi^2} \prod_{s=1, Q(s)=k-1}^{\infty} \frac{1}{s} \prod_{p|s} (1+p^{-1})^{-1} \qquad (k \geq 2),$$

and $d_1 = 6\pi^{-2}$ (since $Q(n) = 1$ if n is a power of a squarefree number). The error term in (23) is uniform in k, and each $d_k > 0$, since for any given $k > 1$ the equation $Q(s) = k - 1$ has a solution in s, namely

$$s = p_1^2 p_2^3 \cdots p_{k-1}^k.$$

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