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INVITED LECTURE

THE SECOND LARGEST EIGENVALUE OF A GRAPH (A SURVEY)

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ABSTRACT. This is a survey paper on the second largest eigenvalue λ_2 of the adjacency matrix of a graph. Among the topics presented are the graphs with small λ_2, bounds for λ_2, algebraic connectivity, graphs with good expanding properties (such as Ramanujan graphs), rapidly mixing Markov chains etc. Applications to computer science are mentioned. Recent results of the authors are included.

0. Introduction

Let G be a graph on vertices $1, 2, \dots, n$. The adjacency matrix of G is the matrix $A = [a_{ij}]_1^n$, where $a_{ij} = 1$ if vertices i and j are adjacent and $a_{ij} = 0$ otherwise. Since A is symmetric its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are real. Assuming that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, we also say that $\lambda_i (= \lambda_i(G))$ is the i-th eigenvalue of G ($i = 1, 2, \dots, n$). In particular, $\lambda_2(G)$ is the second largest eigenvalue of a graph G.

For general theory of graph spectra see monographs [26] and [27].

Concerning particular eigenvalues the following eigenvalues have been studied in some detail:

1° the largest eigenvalue;
2° the second largest eigenvalue;
3° the smallest positive eigenvalue;
4° the largest negative eigenvalue;
5° the second smallest eigenvalue;
6° the smallest eigenvalue.

For a survey on the largest eigenvalue of a graph see the paper [27] by D. Cvetković and P. Rowlinson (see also [26], the third edition, pp. 381–392). Concerning the smallest eigenvalue, particular attention has been paid to graphs with the smallest eigenvalue -2 (see [26], the third edition, pp. 378–381).

Graphs with small second largest eigenvalue have interesting structural properties. The second largest eigenvalue (in modulus) of a regular graph turned out to be an important graph invariant. This paper provides a survey of research on such graphs and on the second largest eigenvalue in general. The starting point for writing this survey was a shorter survey on the same subject given on pp. 392–394 of the third edition of [26].

1. Graphs with small λ_2

It is an elementary fact (see, for example, [26], p. 163) that for non-trivial connected graphs $\lambda_2(K_n) = -1$ $(n \geq 2)$, $\lambda_2(K_{n_1,n_2,\dots,n_k}) = 0$ $(\max(n_1, n_2, \dots, n_k) \geq 2)$ and $\lambda_2(G) > 0$ for other graphs G.

A graph property $\mathcal{P}$ is called *hereditary* if the following implication holds for any graph G: if G has property $\mathcal{P}$, then any induced subgraph of G also possesses property $\mathcal{P}$. (In this paper, when we say that "a graph G *contains* a graph H" we mean that G contains H as an induced subgraph). A graph H is *forbidden* for a property $\mathcal{P}$ if it does not have property $\mathcal{P}$. If a graph G contains (as an induced subgraph) the forbidden graph H (for a property $\mathcal{P}$), then G does not have property $\mathcal{P}$. Then H is called a *forbidden subgraph.* A forbidden subgraph H is called *minimal* if all vertex deleted subgraphs $H - i$ have property $\mathcal{P}$. Graphs having property $\mathcal{P}$ can be characterized by a collection (possibly infinite) of minimal forbidden subgraphs for property $\mathcal{P}$.

For any real a and any integer i the property expressed by the inequality $\lambda_i(G) \leq a$ is a hereditary property. This conclusion follows from the *interlacing theorem* (cf., e.g., [26], p. 19) which says that $\lambda_i(H) \leq \lambda_i(G)$ for any induced subgraph H of G.

The hereditary property of the form $\lambda_2(G) \leq a$, and in principal, the second largest eigenvalue of a graph, has been studied in some detail for the first time by L. Howes [50] and [51] in early seventies. The following characterization is taken from [51]:

Theorem 1. *Let $\mathcal{G}$ be an infinite set of graphs, then the following statements about $\mathcal{G}$ are equivalent:*

- 1^o *There exists a real number a such that $\lambda_2(G) \leq a$ for every $G \in \mathcal{G}$.*
- 2^o *There exists a positive integer s such that for each $G \in \mathcal{G}$ the graphs $(K_s \cup K_1) \bigtriangledown K_s$, $(sK_1 \cup K_{1,s}) \bigtriangledown K_1$, $(K_{s-1} \cup sK_1) \bigtriangledown K_1$, $K_s \cup K_{1,s}$, $2K_{1,s}$, $2K_s$ and the graphs on Fig. 1 (each obtained from two copies of $K_{1,s}$ by adding extra edges) are not subgraphs of G.*

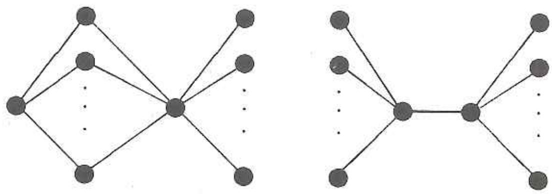


Fig. 1.

Here $\bigtriangledown$ denotes the join of two graphs, while $\cup$ refers to union of two disjoint graphs. Notice that $G_1 \bigtriangledown G_2 = \overline{\overline{G_1} \cup \overline{G_2}}$

In the rest of this section, we shall focuss our attentoin the following values for a: $a = \frac{1}{3}$, $a = \sqrt{2} - 1$, $a = (\sqrt{5} - 1)/2$, $a = 1$ and $a = 2$.

1.1 The golden section bound

There are several results in which the (upper) bound for λ_2 does not exceed the golden section $(\sqrt{5} - 1)/2$.

It is proved in 1993 by D. Cao and Y. Hong [17] that the second largest eigenvalue of a graph G on n vertices is between 0 and $\frac{1}{3}$ if and only if $G = (n - 3)K_1 \bigtriangledown (K_1 \cup K_2)$. The problem of characterizing graphs G with $\frac{1}{3} < \lambda_2(G) < (\sqrt{5} - 1)/2$ was also posed in [17]. Graphs G with $\lambda_2(G) < \sqrt{2} - 1$ are determined by M. Petrović [75]. An independent characterization of graphs with $\lambda_2 \leq \sqrt{2} - 1$ has been given by J. Li in [56]; in addition, all minimal forbidden subgraphs for the property $\lambda_2 \leq \sqrt{2} - 1$ are given there. It is proved by S. Simić [85] that the set of minimal forbidden subgraphs for the property $\lambda_2(G) < (\sqrt{5} - 1)/2$ is finite. The structure of graphs G with $\lambda_2(G) \leq (\sqrt{5} - 1)/2$ has been studied by D. Cvetković and S. Simić [29]. A part of results has been announced in [28].

We shall introduce the notation $\sigma = (\sqrt{5} - 1)/2 \approx 0.618033989$. Obviously, we have $\sigma^2 + \sigma - 1 = 0$.

Graphs having property $\lambda_2(G) \leq \sigma$ (σ-property) will be called *σ-graphs.* For convenience graphs G for which $\lambda_2(G) < \sigma$, $\lambda_2(G) = \sigma$, $\lambda_2(G) > \sigma$ will be called *σ^--graphs, σ^0-graphs, σ^+-graphs*, respectively.

The next proposition, taken from [95] (see also [9]), enables the definition of a class of graphs to which every σ^--graph belongs.

Proposition 2. *If $\overline{G}$ is a connected graph and if G has no isolated vertices, then G contains an induced subgraph equal to $2K_2$ or P_4.*

Assume now G is a σ^--graph. If $\overline{G}$ is a connected graph, then G must have at least one isolated vertex (otherwise G contains $2K_2(= E)$ or P_4 as an induced subgraph, and hence is not a σ^--graph). On the other hand, if $\overline{G}$

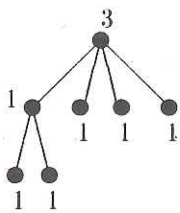
is a disconnected graph, then G itself is a join of at least two graphs. Since the σ-property is hereditary, it follows that G belongs to a class of graphs (here, as in [85], denoted by $\mathcal{C}$) which is defined as the smallest family of graphs that contains K_1 and is closed under adding isolated vertices (i.e., if $G \in \mathcal{C}$, then $G \cup K_1 \in \mathcal{C}$) and taking joins (i.e., if $G_1, G_2 \in \mathcal{C}$, then $G_1 \bigtriangledown G_2 \in \mathcal{C}$). An alternative way to describe graphs from the class $\mathcal{C}$ is in terms of minimal forbidden induced subgraphs. Actually, $\mathcal{C}$ is a class of graphs having no induced subgraphs equal to E $(= 2K_2)$ or P $(= P_4)$.

Clearly, any σ^--graph belongs to $\mathcal{C}$, but not vice versa.

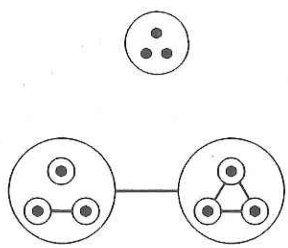
The class $\mathcal{C}$ has been introduced and studied in [85]. Weighted rooted trees (with weights assigned to vertices) were used also in [85] in representing graphs from the class $\mathcal{C}$.

To any graph G from $\mathcal{C}$ we associate a weighted rooted tree T_G (also called an *expression tree* of G) in the following way:
if $H = (H_1 \bigtriangledown \ldots \bigtriangledown H_m) \cup nK_1$ is any subexpression of a graph G (i.e. a graph obtained by using the above rules), then a subtree T_H with a root v corresponds to H; $n(= w(v))$ is a weight of v, whereas for each i $(i = 1, \ldots, m)$ there is a vertex v_i (a son of v) representing a root of H_i.

Example. If $G = (((((K_1 \bigtriangledown K_1) \cup K_1) \bigtriangledown K_1) \bigtriangledown K_1) \bigtriangledown K_1) \cup 3K_1$, then the corresponding expression tree is depicted in Fig. 2(a). In Fig. 2(b) we represent the same graph as a set diagram (a line between two circumscribed sets of vertices denotes that each vertex inside one set is adjacent to any vertex inside the other set).



(a)



(b)

Fig. 2.

It turned out that the set of σ^--graphs falls into a finite number of structural types. These types are given in Fig. 3 by the corresponding expressioned trees.

It has been proved along the same lines in [85] that the set of minimal forbidden subgraphs for the σ^--property is finite. They all belong to $\mathcal{C}$ except for E and P_4. The whole list of these forbidden subgraphs will be described in a forth-comming paper [30].

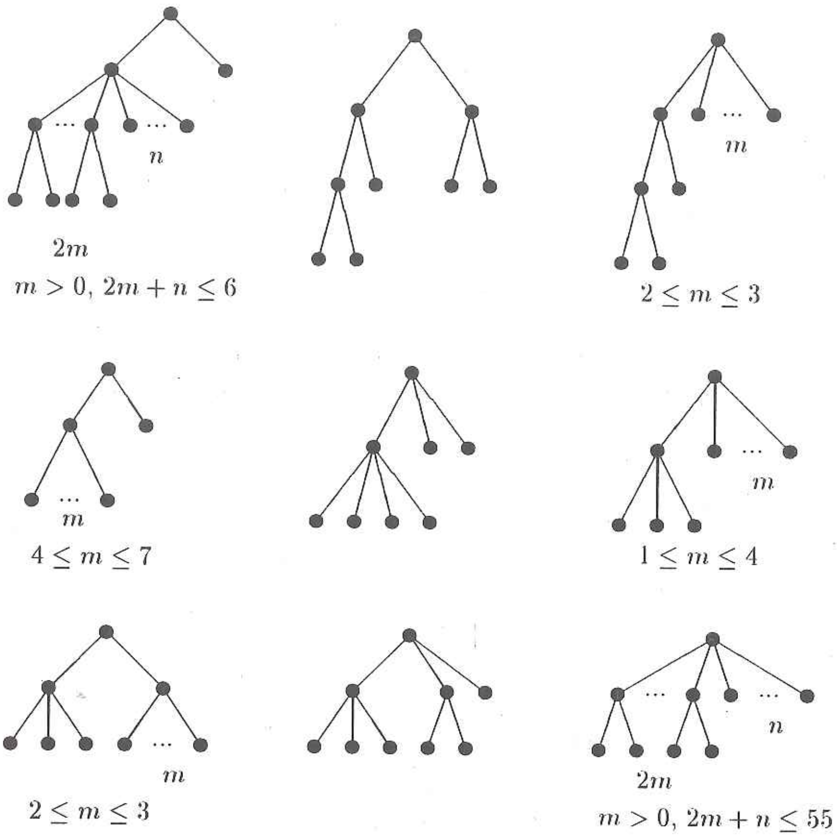


Fig. 3.

We present now main results of [29].

Theorem 3. *A σ-graph has at most one non-trivial component G for which one of the following holds:*

1^o G is a complete multipartite graph;
2^o G is an induced subgraph of C_5;
3^o G contains a triangle.

Before proceeding to describe σ-graphs mentioned in 3^o we introduce some notation.

Let G be a σ-graph with the vertex set V. Let T be a triangle in G induced by the vertices x, y, z. Next, let $A(G,T) = A$, $B(G,T) = B$, $C(G,T) = C$ be the sets of vertices outside T which are adjacent to exactly one, two, three vertices from T, respectively. Also, let G_A, G_B, G_C be the component, containing T, of the subgraph of G induced by the vertex set $V - B - C$, $V - A - C$, $V - A - B$, respectively.

Let $d(u,T)$ denote the distance of the vertex u from the triangle T, i.e. the length of the shortest path between u and a vertex from T.

σ-graphs containing triangles are now described in more detail in terms of induced subgraphs G_A, G_B, G_C.

Theorem 4. *Let G be a connected σ-graph which contains a triangle. For any triangle T of G the following holds for subgraphs G_A, G_B, G_C :*

1^o *G_A is an induced subgraph of one of the graphs from Fig. 4.*

2^o *For G_B one of the following holds:*

- i) *G_B is an induced subgraph of graphs from Fig. 5;*
- ii) *$G_B = P_4 \bigtriangledown (H \cup K_1)$ for some σ-graph H;*
- iii) *$G_B = H_1 \bigtriangledown H_2 \bigtriangledown H_3$ for some σ-graphs H_1, H_2, H_3.*

3^o *For G_C one of the following holds:*

- i) *G_C is an induced subgraph of $(K_3 \cup K_1) \bigtriangledown H$ for some σ-graph H;*
- ii) *G_C is obtained from $K_n \bigtriangledown K_3 \bigtriangledown H$ by adding a pendant edge to each vertex of K_n, where $n \geq 2$ and H is a σ-graph containing no induced subgraphs isomorphic to some of graphs $K_3 \cup K_1$, $K_2 \cup 3K_1$, $K_{1,2} \cup 2K_1$, $K_{2,4} \cup K_1$, $K_{3,3} \cup K_1$.*

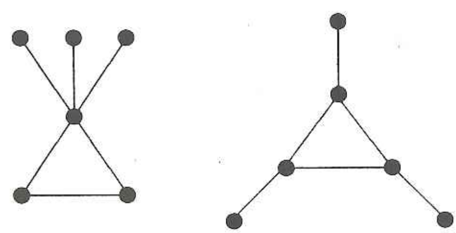


Fig. 4.

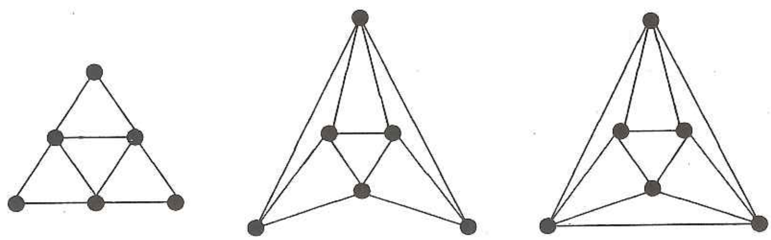
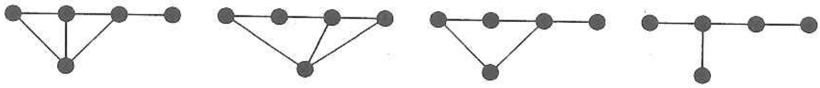


Fig. 5.

It is also proved in [29] that the set of minimal forbidden subgraphs for the $\sigma-$property is finite. The next theorem (taken from [29]) provides more details.

Theorem 5. *If H is a minimal forbidden (induced) subgraph for the $\sigma-$property, then:*

1° H is one of the graphs $E(=2K_2)$, F_1, F_2, F_3, F_4 (see Fig. 6), or
2° H belongs to the class $\mathcal{C}$.



F_1 F_2 F_3 F_4

Fig. 6.

All minimal forbidden subgraphs for $\sigma-$property are not yet known. On the other hand, more can be said if we require that both, the graph and its complement are $\sigma-$graphs. Then, due to S. Simić [86], there are exactly 27 minimal forbidden subgraphs for this property. Here we rather give explicitely (following[86]) all graphs with the property in question.

Theorem 6. *G and $\overline{G}$ are both $\sigma-$graphs, if and only if either of them is one of the following graphs:*

$K_m \cup nK_1$ $(m, n \geq 0)$, $K_{2,1,1} \cup mK_1$, $K_{2,1} \cup mK_1$ $(m \geq 0)$,
$K_{3,1} \cup mK_1$ $(m \leq 3)$, $K_{2,1,1,1} \cup mK_1$ $(m \leq 2)$,
$((K_{2,1,1} \cup K_1) \bigtriangledown K_1) \cup K_1$, $((K_{2,1} \cup 2K_1) \bigtriangledown K_1) \cup K_1$,
$((K_{2,1} \cup K_1) \bigtriangledown K_1) \cup K_1$, $(K_{m,1} \cup K_1) \bigtriangledown K_n$ $(m \geq 2,\ n \geq 0)$,
$(K_{2,1,1} \cup K_1) \bigtriangledown K_m$, $(K_{2,1} \cup 2K_1) \bigtriangledown K_m$ $(m \leq 2)$,
$(K_{3,1} \cup 2K_1) \bigtriangledown K_1$, $(K_{2,1,1,1} \cup K_1) \bigtriangledown K_1$, $(((K_{2,1} \cup K_1) \bigtriangledown K_1) \cup K_1) \bigtriangledown K_1$.

1.2 Bounds equal to 1 and 2

Graphs with $\lambda_2(G) \leq 1$ have been studied in 1982 by D. Cvetković [24]. It turned out that some of these graphs are the complements of the graphs whose least eigenvalue is greater than or equal to -2. More precisely, $\lambda_n(\overline{G}) > -2$ implies $\lambda_2(G) < 1$. If $\lambda_n(\overline{G}) = -2$, then $\lambda_2(G) \leq 1$ equality holding if and only if the eigenvalue -2 of $\overline{G}$ is either non-simple or non-main (all eigenvectors are orthogonal to the vector $(1, 1, \ldots, 1)$). For other graphs G with $\lambda_2(G) \leq 1$, the complement $\overline{G}$ has exactly one eigenvalue smaller than -2. However, $\lambda_n(\overline{G}) < -2$ and $\lambda_{n-1}(\overline{G}) \geq -2$ does not imply $\lambda_2(G) \leq 1$. These results are derived by the well-known Courant-Weyl inequalities for eigenvalues of matrices. For further details see the original paper or monograph [25] (p. 11, where [Cve5] is wrongly given as [Cve7]).

A representation of graphs with $\lambda_2(G) = 1$ in the Lorentz space is given in 1983 by A. Neumaier and J. J. Seidel [72].

Bipartite graphs G with $\lambda_2(G) \leq 1$ have been characterized in 1991 by M. Petrović [74]. Three families of graphs and four particular graphs with $\lambda_2(G) \leq 1$ are constructed. It is proved that a connected bipartite graphs have the property $\lambda_2(G) \leq 1$ if and only if it is an induced subgraph of the mentioned graphs.

In particular, trees with the second largest eigenvalue less than 1 were treated by A. Neumaier [70]. More generaly, an algorithm for deciding if the second largest eigenvalue of any tree is less than some bound was also proposed by A. Neumaier.

The exact characterization of graphs with second largest eigenvalue around 1 still remains an interesting open question in spectral graph theory.

Graphs with $\lambda_2 \leq 2$ are called *reflexive* graphs [72]. Some classes of reflexive graphs are studied in [72]. In particular, trees with $\lambda_2 = 2$ are called *hyperbolic* [60]. All hyperbolic trees are known [60], [70] and [72].

2. Bounds for λ_2

Upper and lower estimates for the second largest eigenvalue of a graph under various restrictions were studied in literature (but not as extensively as for the largest eigenvalue).

The most general result concerns the connected graphs with prescribed number of vertices. According to D. Powers, for a connected graph G on n vertices the following holds

$$-1 \leq \lambda_2(G) \leq \lfloor \frac{n}{2} \rfloor - 1.$$

The upper bound is achieved, for n odd ($n = 2s+1$), if G is a graph consisting of two cliques of size s (graphs equal to K_s) bridged by a path of length 2; for n even this bound is only asymptotically sharp (see [78], or [79]; see also [48]). The lower bound is achieved if and only if G is a complete graph (see Section 1). It is interesting to note that the above (upper) estimate is proved by making use of the following more general estimate of the second largest eigenvalue in terms of the largest eigenvalue of some parts of a graph. Namely, due to D. Powers we have:

$$\lambda_2(G) \leq \max_{(G_1,G_2)} \min\{\lambda_1(G_1), \lambda_1(G_2)\},$$

where G_1 and G_2 denote the subgraphs induced by vertex sets of some bisection of (the vertex set of) a (connected) graph G. The key argument for proving this was based on partitioning the vertices of G according to sign pattern of the eigenvector corresponding to the second largest eigenvalue (see [77] for details).

If G is a connected graph on n vertices and m edges, then, due to R.C. Brigham and R.D. Dutton [13], the following inequality holds:

$$\lambda_2(G) \leq \sqrt{\frac{m(n-2)}{n}}.$$

In particular, this estimate in not too good for trees. If we assume that G is not a tree, then some refinements are possible, as shown in [79]. Then the result is expressed in terms of the estimates for the largest eigenvalue of a connected graph with a fixed number of edges (but not vertices). The latter problem is completely solved by P. Rowlinson [81] to within the graphs which realize the bounds. More precisely, as remarked in [79], then

$$\Lambda_1(\lfloor\frac{m}{2}\rfloor - 1) \leq max\{\lambda_2(G)\} \leq \Lambda_1(\lfloor\frac{m-1}{2}\rfloor),$$

where $\Lambda_1(m)$ is the maximum for the largest eigenvalue of a connected graph with m edges); thus the estimate is very tight.

In particular, for triangle-free (and bipartite) graphs some further estimates are obtained in [13].

Much better estimates for trees are known. If T is a tree with $n \geq 3$ vertices, then

$$0 \leq \lambda_2(T) \leq \sqrt{\lfloor\frac{n-2}{2}\rfloor}.$$

The upper bound was obtained by Y. Hong [47]. It is the best possible for n (≥ 3) odd (then it coincides with the bound of A. Neumaier [70] $\lambda_2(T) \leq \sqrt{\frac{n-3}{2}}$ which holds only for n odd). As remarked by D. Powers [78], with more careful analysis one can get:

$$\sqrt{\lfloor\frac{n-1}{2}\rfloor - 1} \leq max\{\lambda_2(T)\} \leq \sqrt{\lfloor\frac{n-2}{2}\rfloor},$$

i.e. the bound for n even is asimptotically sharp. The lower bound is clear from the above (it is achived for a tree isomorphic to a star, i.e. for $T = K_{1,n-1}$). Otherwise, if $T \neq K_{1,n-1}$, then $\lambda_2(T) < 1$ only for $T = S^2_{n-2}$ (here S^2_{n-2} is the graph obtained from a star with $n-2$ arms by subdividing one arm). Also then $\lambda_2(T) = \sqrt{\frac{n-1-\sqrt{(n-3)^2+4}}{2}}$. Thus if $T \neq K_{1,n-1}, S^2_{n-2}$, then $\lambda_2(T) \geq 1$.

Star-like trees are trees homeomorphic to a star ($K_{1,s}$ for some $s \geq 3$). The second largest eigenvalue of star-like trees (with a fixed number of vertices and fixed number of arms) were studied by F. K. Bell and S. K. Simić (see [87]). We only mention here that for fixed $s \geq 4$, the trees with minimum and maximum second largest eigenvalue (on fixed number of vertices) are those as intuitively expected (i.e., those having the length of all arms as equal as possible in the former case, and those having the length of all arms but one equal to 1 in the latter case). If $s = 3$, then some interesting phenomena do occur (for detail see [87]).

Results on regular graph are given in the next section.

3. Regular graphs

There are two main reasons why regular graphs deserve special interest in this context. The first is that the largest eigenvalue of a regular graph of degree d is equal to d, so then the second largest eigenvalue becomes the dominant feature in many asspects (in particular, in spectral orderings). The second is that regular graphs allow a simple connection between the eigenvalues (of the adjacency matrix) and the eigenvalues of some other matrices associated with graphs, in particular, with the eigenvalues of the graph Laplacian (see below).

3.1 λ_2 and spectral ordering of regular graphs

The role of the second largest eigenvalue in ordering cubic graphs has been observed in 1976 by F.C. Bussemaker, S. Čobeljić, D. Cvetković and J.J. Sedel [16] (see also [26], pp. 268–269). The 621 connected cubic graphs with not more than 14 vertices, together with eigenvalues and many other data, are displayed. The sequence of eigenvalues is given in non-increasing order for each graph, and for a fixed number of vertices the graphs are ordered lexicographicaly with respect to their sequences of eigenvalues. Since the largest eigenvalue λ_1 is equal to 3 in cubic graphs, the second largest eigenvalue λ_2 determines roughly the ordering of graphs. Decreasing λ_2 shows graphs of more "round" shape (smaller diameter, higher connectivity and girth).

A partial theoretic explanation of these empirical observations was offered in 1978 by D. Cvetković [23].

Theorem 1. *Let G be a $d-$regular graph on n vertices. Let x be any vertex of G and let δ be the average vertex degree of the subgraph induced by the vertices not adjacent to x. Then we have:*

$$\delta \leq d\frac{\lambda_2^2 + \lambda_2(n-d)}{\lambda_2(n-1)+d}.$$

The same inequality (see also [25], p. 71) was derived in [6] by quite different method.

In further we shall offer some other theoretic support to these (empirical) observations (see Section 3.3).

3.2 Algebraic connectivity

For a graph G on n vertices, let $d_1, d_2, \ldots, d_n$ denote the corresponding vertex degrees. The matrix $L = D - A$ with $D = [d_i \delta_{ij}]_1^n$ (δ_{ij} the Kronecker symbol) is called the Laplacian of G. The graph Laplacian is positive semi-definite and the second smallest eigenvalue of L (here denoted by $\alpha(= \alpha(G))$) is called the *algebraic connectivity* of G. It was introduced in 1973 by M. Fiedler [35].

The algebraic connectivity α of a graph (in regular case) can be expressed in terms of the second largest eigenvalue. If G is a d-regular graph, then $\alpha = d - \lambda_2$. (Thus the algebraic connectivity increases as the second largest eigenvalue becomes smaller).

Definition 1. An $(n, d, \epsilon)-$enlarger is a d-regular graph G on n vertices with $\alpha(G) \geq \epsilon$.

The significance of enlargers lies, among others, in the fact that they enable an explicit construction of graphs with good expansion properties (such as expanders). One such construction of expanders is obtained by N. Alon and V.D. Milman [6]. For this aim we need the following definition:

Definition 2. Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$. The extended double cover of G is a bipartite graph $H = (X, Y, F)$ with $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ where $F = \{x_i y_j : i = j \text{ or } v_i v_j \in E\}$.

Remark. Actually, an extended double cover is a NEPS (Non-Complete Extended P-Sum, see [26], pp. 65-66) of G and K_2 in the basis $\{(0,1),(1,1)\}$ (see, also [25], p. 60).

Now the following theorem from [6] offers an explicit construction of an expander (see Definition 2 from Section 6).

Theorem 2. *Let $G = (V, E)$ be an $(n, d, \epsilon)-$enlarger and let H be its extended double cover. Then H is a strong $(n, d+1, \delta)-$expander for*

$$\delta = \frac{4\epsilon}{d + 4\epsilon}.$$

The next theorem of N. Alon [3] points that good enlargers are in fact good magnifiers (see Definition 3 from Section 6).

Theorem 3. *Every* (n, d, ϵ)*−enlarger is an* (n, d, δ)*-magnifier, where*

$$\delta = \frac{2\epsilon}{d + 2\epsilon}.$$

It is interesting to note that the converse also holds, i.e. that every magnifier is an enlarger with some appropriate parameters (see [3], for further details).

Remark. Generally, the fact that the algebraic connectivity is relevant to expansion property of a graph (see Definition 1 from Section 6) can be also justified by the following relation (cf. [42], Lemma 5.7) given below. Namely, for any graph $G = (V, E)$ we have:

$$(\forall X \subset V)(|X| \leq \frac{1}{2} \Rightarrow |\partial X| \geq \frac{|X||V \setminus X|}{|V|}\alpha(G),$$

where $\partial X = \{y : xy \in E, x \in X\}$. A similar result is due to R.M. Tanner [92].

More information on expansion property of graphs, and other related graphs can be found in Section 6.

The work on algebraic connectivity and graph Laplacian for graphs in general (in particular, for non-regular graphs) will not be reported in this paper. For more information see papers by M. Fiedler [36], [37], [38] and also [44], [45]. Much information on graph Laplacians can be found in the book [26] and in expository papers [43], [61], [62], [66], [67].

3.3 Second largest eigenvalue in modulus

The second largest eigenvalue (in modulus) of a regular graph turned out to be an important graph invariant since it has relations with various graph invariants (such as diameter and *covering number* etc.) and graph properties (including expanding properties and convergence properties of simple random walks).

Let G be a d-regular graph, and let $\Lambda(= \Lambda(G)) = max\{|\lambda_i| : |\lambda_i| \neq d\}$. Notice that for bipartite graphs we have $\Lambda(G) = \lambda_2(G)$ (due to symmetry of the spectrum with respect to the origin).

Let G be a connected d-regular graph on n vertices. According to N. Alon and V.D. Milman [6] we have the following bound:

$$\text{(1)} \qquad diam(G) \leq 2\lceil\sqrt{\frac{2d}{d - \Lambda}}\log_2 n\rceil.$$

This bound was improved by several authors, in several directions. Interesting improvements are given by B. Mohar in [65], but expressed in terms of the second smallest, and the largest eigenvalue of Laplacian matrix of any (not necessarily regular) graph. Also the following lower bound, $diam(G) \geq \frac{4}{n\alpha}$, valid for any (connected) graph on n vertices, can be found in [65]. Bound (1) is also improved by F.R.K. Chung [22]. For regular graphs this bounds reads:

$$diam(G) \leq \lceil \frac{\log(n-1)}{\log(\frac{d}{\Lambda})} \rceil. \tag{2}$$

For this bound it was observed in [31] that it is indeed the upper bound for covering index of a graph (i.e. it is the smallest integer c such that any pair of not necessarily distinct vertices is connected by a walk of length exactly c). Let $cover(G)$ denote the covering number of a connected graph G. As proved in [31], for any (connected) graph we have $diam(G) \leq cover(G)$ (it is also true that $cover(G) \leq diam(G) + s$ if every vertex of G is in some closed walk of odd length, at most $2s+1$; if G is a bipartite graph, then $cover(G) = \infty$). By convenient distinction between diameter and covering index we have: if G is a d-regular connected graph on n vertices and t some positive number then:

(i) if $d^m/n > \lambda^m(1-\frac{1}{n})$, then $cover(G) \leq m$ (by F.R.K. Chung, restatement of (2));

(ii) if $d^{m-1}(d+t)/n > \lambda^{m-1}|\lambda+t|(1-\frac{1}{n})$ ($\lambda \in \{\lambda_2, \ldots, \lambda_n\}$ and $t > 0$), then $diam(G) \leq m$ (by C. Delorme and P. Sóle [31]).

According to P. Sarnak [82] (see also [80]) the following estimate holds:

$$diam(G) \leq \frac{\operatorname{arccosh}(n-1)}{\operatorname{arccosh}(\frac{d}{\Lambda})}, \tag{3}$$

for any d-regular graph G on n vertices. By considering separately non-bipartite, and bipartite case, some further refinements of (3) are obtained by G. Quenell in [80]:

$$diam(G) \leq \begin{cases} \dfrac{\operatorname{arccosh}(n-1)}{\operatorname{arccosh}\left(\frac{d}{\Lambda}\right)} + 1 & G \text{ non-bipartite,} \\ \dfrac{\operatorname{arccos}(n/2-1)}{\operatorname{arccosh}\left(\frac{d}{\Lambda}\right)} + 2 & G \text{ bipartite.} \end{cases}$$

The inequality (3) can be further refined, by introducing the *injectivity radius* r of G into consideration. According to G. Quenell (see [80] for the definition of r) it holds:

$$(4) \qquad diam(G) \leq \frac{\operatorname{arccosh}(\frac{n}{d(d-1)^{r-1}})}{\operatorname{arccosh}(\frac{d}{\Lambda})} + 2r + 1.$$

As also remarked in [80], the estimate (3) is better than (4) provided $\Lambda \geq 2\sqrt{d-1}$ (in other words, see below, (4) is better only for Ramanujan graphs).

Finally, let us mention that the inequality (3) has been generalized to the case of biregular graphs and regular directed graphs [31]. The authors also discuss connections to finite non-abelian simple groups, primitive association schemes, primitivity exponent of the adjacency matrix, covering radius of a linear code and Cayley graphs.

The relationship between the second largest eigenvalue in moduli of a graph and girth, was investigated by P. Solé [90]. For the graphs with small diameter we have:

$$\Lambda(G) \geq \begin{cases} \sqrt{d} & g(G) \geq 4 \\ \sqrt{2d-1} & g(G) \geq 6 \end{cases} \;; G \text{ is non-bipartite},$$

$$\Lambda(G) \geq \begin{cases} \sqrt{d(n-2d)/(n-2)} & g(G) \geq 6 \\ \sigma\sqrt{d-1} & g(G) \geq 8 \end{cases} \;; G \text{ is bipartite}.$$

(Here, as in Section 1, σ denotes the golden section.) For the graphs with larger diameter we have:

$$\Lambda(G) \geq \begin{cases} 2\sqrt{d-1}\cos\frac{\pi}{s+1} & G \text{ is non-bipartite}, \\ 2\sqrt{d-1}\cos\frac{2\pi}{s+1} & G \text{ is bipartite}, \end{cases}$$

where $s = \lfloor \frac{g(G)-1}{2} \rfloor$.

From the above (upper) bounds for diameter it generally follows that the diameter is expected to be smaller as λ_2 (or Λ is smaller). Thus, by these inequalities, we have at least partial explanations for the shape of cubic graphs.

We now turn to important class of graphs in this context, so called Ramanujan graphs:

Definition 3. Let G be a (conected) $d-$regular graph. Then G is called a Ramanujan graph if $\Lambda(G) \leq 2\sqrt{d-1}$.

Remark. The importance of the number $2\sqrt{d-1}$ in the above definition lies in the following lower bound due to N. Alon and R. Boppana (cf. [58];

see Proposition 4.2). Suppose $G_{d,n}$ is a d-regular (connected) graph on n vertices (d being fixed). Then for any sequence of such graphs we have:

$$\liminf_{n\to\infty} \Lambda(G_{n,d}) \geq 2\sqrt{d-1}.$$

Thus if one wants graphs with as small λ_2 as possible, the above number serves as the lower limit of what can be done. More information this kind of results can be found, for example, in [73].

Since the second largest eigenvalue is small in Ramanujan graphs, they are also good enlagers (see Definition 1 from above), and hence good magnifiers.

An infinite family of Ramanujan graphs have been constructed, for the first time, by A. Lubotzky, R. Phillips and P. Sarnak in 1988. These graphs were realised as Cayley graphs of some groups (such as, for example, group $PGL(2, F_q)$) relative to some symmetric subset (or, alternatively, as quotients of a quaternion group); see [58] for details. In particular, cubic Ramanujan graphs are treated in [20]. It is remarkable that the diameter of Ramanujan graphs cannot be too large (besides the bounds for particular Ramanujan graphs from [58], see the result of A. Nilli, given below). The girth of Ramanujan graphs is investigated in [10].

The following result of A. Nilli [73] explains some effects on λ_2 when, in fact, diameter increases. Let G be a d-regular graph, and suppose that G contains two edges the distance between which is at least $2k+2$ (the distance between two edges is the length of shortest path whose terminal vertices are the vertices of edges in question). Then we have:

$$\lambda_2(G) \geq 2\sqrt{d-1}\left(1 - \frac{1}{k+1}\right) + \frac{1}{k+1}.$$

4. Rapidly mixing Markov chains

The second largest eigenvalue of graphs is of some interest in the theory of *rapidly mixing Markov chains*.

Consider a Markov chain on a finite *state space* $S_n = \{1, 2, \dots, n\}$ with *transition matrix* $P = [p_{ij}]_1^n$. Thus for any ordered pair i, j of states the quantity p_{ij} is the *transition probability* from state i to state j and is independent in the time t. The matrix P is non-negative and *stochastic*, i.e. its row sums are all equal to 1. Let π_i ($i = 1, 2, \dots, n$) be a probability distribution over S_n and suppose that $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S_n$. Then P is said to be *reversible* w.r.t. probability distribution π_i and the Markov chain is *ergodic* with the stationary distribution π_i.

As is well known, P has real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_1 = 1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n > -1$. The rate of convergence to π_i is governed by the second largest eigenvalue in absolute value, i.e. by $\max(\lambda_2, |\lambda_n|)$. One can show that the influence of λ_n can be neglected so that the really important quantity is λ_2. A reversible Markov chain is called *rapidly mixing* if λ_2 is sufficiently small.

It is useful to identify an ergodic reversible Markov chain with a weighted undirected graph G (possibly containing loops) as follows. The vertex set is the state space S_n of the chain. If $p_{ij} \neq 0$, there is an edge in G between vertices i and j with the weight $q_{ij} = \pi_i p_{ij} = \pi_j p_{ji}$. The eigenvalues of G (i.e. of the weight matrix $Q = [q_{ij}]_1^n$) are equal to the eigenvalues of P. In this way we see that the theory of graph spectra is relevant to the problem considered. There are two immediate consequences of the above facts. Firstly, one can use the theory of graph spectra to evaluate or estimate λ_2 in Markov chains, in particular to find upper bounds for λ_2. Secondly, one can use known graphs with small λ_2 to construct rapidly mixing Markov chains.

Detailed elaboration of above ideas can be found in papers [2], [32], [33], [53], [88] and [89], just to mention a few among several papers by the same authors (D. J. Aldous, P. Diaconis, M. Jerrum, A. Sinclair). Note that rapidly mixing Markov chains are important parts in stochastic algorithms for enumeration of large combinatorial sets.

5. Miscelaneous

In this section we briefly mention other results concerning λ_2.

Let us define

$$\mu_2(G) = \liminf_{d \to \infty} \{\lambda_2(H) : G \subset H, d(H) > d\}.$$

A.J. Hoffman [46] proved the following result.

Theorem 1. *Let G be a graph with n vertices and with adjacency matrix A. Let Γ be the set of all $(0,1)$ matrices C with n rows and at least two columns such that every row sum of C is positive, and if C has more than two columns, no column can be deleted without destroying the property that C has positive row sums. Then*

$$\mu_2(G) = min_{C \in \Gamma} \lambda_1(A - C(J-I)^{-1}C^T).$$

It was proved by M. Doob [34] that the set of all second-largest eigenvalues is dense in the interval $(\sqrt{2+\sqrt{5}}, \infty)$. The same set has infinitely many accumulation points, but is nowhere dense in the interval $(-\infty, -1+\sqrt{2}]$. These points are described in some detail by J. Li [56].

It is proved by C. Licata and D.L. Powers [57] that the Platonic solids are *self-reproducing* in the following specific sense. We consider an eigenvalue λ (in this case $\lambda = \lambda_2$) of the graph of the solid P considered, and the corresponding eigenspace $\mathcal{E}(\lambda)$ which is of dimension k. The convex hull of a basis of $\mathcal{E}(\lambda)$ is a polytope Q. If Q is isomorphic with P, then P is called self-reproducing. It is also proved in [57] that some other polyhedra are self-reproducing.

Spectra of weighted adjacency matrices have been used by Y. C. de Verdiére to introduce a new important graph invariant in [94]. For a connected graph G we introduce the class $\mathcal{A}_G$ of matrices $A = [a_{ij}]$ for which $a_{ij} > 0$ if i and j are adjacent and $a_{ij} = 0$ otherwise. Let $\mu_1, \mu_2, \dots, \mu_m$ ($\mu_1 > \mu_2 > \cdots > \mu_m$) be distinct eigenvalues of A with multiplicities $k_1 = 1, k_2, \dots, k_m$, respectively. Let $\mu(G) = \max k_2$, where maximum is taken over the class $\mathcal{A}_G$. For example, $\mu(K_n) = n-1$ and $\mu(K_{3,3}) = 4$. It is proved that G is planar if and only if $\mu(G) \leq 3$. It is conjectured that $\mu(G) \geq \chi(G) - 1$, where $\chi(G)$ is the chromatic number of G. The validity of this conjecture would imply the four colour theorem!

Various inequalities involving the isoperimetric number and the spectrum of graphs are provided by B. Mohar [63] and [64].

Second largest eigenvalue in random graphs is studied in [15], [39] and [40].

It is interesting to note that expanding properties in infinite graphs are related to the spectral radius of the graph [11].

6. Some applications

The topic concerning the second largest eigenvalue has many theoretical and practical applications. Its major interest stems from the fact that it is significantly related to various types of expansion (and concentration) properties of graphs. These properties, in turn, are of great practical and theoretical interest in many brances of mathematics and/or computer science (such as extremal graph theory (see, e.g., [8]), graph pebbling (see, e.g., [55]), computational complexity (see, e.g., [52]), parallel sorting algorithms (to be treated below), etc.) as well as other branches of science (like electrical engineering; some detals in connection with various networks are also included below).

We shall not attempt within this paper to go into details. Rather, we shall try to gain the importance of the topic toward various applications. The key idea is that many spectral parameters (invariants of graphs) are important link to structural properties (such as various expansion properties).

Informally, a graph has a "good" *expanding property* if each (its) vertex subset has a large neighbourhood. For bipartite graphs, more precisely, we have:

Definition 1. Let $G = (U, V, E)$ be a bipartite graph with $|U| = |V| = n$. Then G is an (n, α, β)- expanding $(0 < \alpha \leq \beta \leq n)$ if the following condition holds:

$$(\forall X \subseteq U)(|X| \geq \alpha \Rightarrow |\partial X| \geq \beta).$$

Here, for the sake of completness, we recall that $\partial X = \{y : d(y, X) = 1\}$, where d stands for the usual metric on a graph.

Bipartite graphs having good expanding properties are known as *expanders*. One of the most general definition reads as follows:

Definition 2. Let $G = (U, V, E)$ be a bipartite graph with $|U| = |V| = n$, and $|E| \leq dn$. Then G is an (n, d, δ, α)- expander $(\alpha \leq 1)$ if the follwing condition holds:

$$(\forall X \subseteq U)(|X| \leq \alpha n \Rightarrow |\partial X| \geq (1 + \delta(1 - \frac{|X|}{n}))|X|).$$

In particular, if $\alpha = \frac{1}{2}$, then G is called an (n, d, δ)-expaders and if $\alpha = 1$, then G is called a strong (n, d, δ)-expader.

In the above definition d and δ are regarded as *density* and *extension*, respectively. Notice also that the expression $(1 + \delta(1 - \frac{|X|}{n}))$ is larger as $|X|$ is smaller, which supports the fact that small subsets X, more likely, have large neighbourhood.

For non-bipartite graphs, the above definition has to be modified (since the verices are generally not distingused according to colour classes, or viewed as "input - output parts" of some system). According to [3], the non-bipartite analogon of expanders are *magnifiers*. The cooresponding definition (most frequently refering to regular graphs) reads as follows:

Definition 3. Let $G = (V, E)$ be a graph on n vertices, and maximal vertex degree d. Then G is an (n, d, δ)-magnifier if the following condition holds:

$$(\forall X \subseteq U)(|X| \leq \frac{1}{2}n \Rightarrow |\partial X| \geq \delta|X|.$$

Some examples of (good) expanders and magnifiers we have encountered in Section 3. To provide some hints on applications, we need some further definitions.

We first define two classes of graphs (having special connectivity properties and possibly small number of edges) which can be viewed as communication networks: concentrators (defined by M.S. Pinsker [76] in 1973) and superconcentrators (defined by L.G. Valiant [93] in 1975). There is an extensive literature on applications of these graphs in communication problems (a good source of references can be found, e.g., in [22]; see also [91] on construction of low complexity error-correcting codes).

Definition 4. An $(n, m)-$concentrator is a graph with n input vertices and m output vertices, $n \geq m$, having the property that, for any set of r $(\leq m)$ inputs, there exists a flow (a set of vertex-disjoint paths) that join the given inputs to some set of r outputs.

With a slight modification, we get the definition of superconcentrators.

Definition 5. An $n-$superconcentrator is a graph with n input vertices and n output vertices having the property that, for any set of r $(\leq n)$ inputs and any set of r outputs, there exists a flow that join the given inputs to given outputs.

Remark. Besides these two classes of graphs, which were firstly used in construction of various switching networks, there are many others of similar kind: for example, nonblocking networks where the partial correspondence between inputs and outputs by disjoint paths can be always extended without disturbing existing paths (see, e.g., [21] for more precise definition).

It is also worth mentioning that superconcentrators can be constructed from concentrators, but also from expanders (see, e.g., [41] and [83]). Superconcentrators, among others, are used in construction of parallel sorting networks [1].

As is well known from literature, expanding graphs within some properly choosen classes do exist. Moreover, by probabilistic arguments, one can show, with relative ease, that within many such classes almost every graph posseesses the desired property (see, for example, [12]). On the other hand, if one needs some of these graphs, there is no efficient algorithm, for a randomly choosen graph, to decide if it indeed satisfies the required properties (for example, it is known that the problem of checking if a given graph is an $(n, d, 0)-$expander is coNP-complete). So explicit constructions are desirable (but, as a rule, are very complicated). The first breakthrough was given by G.A. Margulis [59] (but without explicit estimate on expansion magnitude; only non-zero estimate is proved to exist). By a slight modification of the previous construction, O. Gabber and Z. Galil [41] have provided the estimate explicitely. Another important construction is due to N. Alon and V.D. Milman [5] (based on theory of group representations or harmonic analysis). For further constructions see [7], [4] (where finite geometries are used - points and hyperplains are the vertices of bipartite graph), [54] (bipartite graphs are obtained from affine transformations), [22] (graphs represented as k-sum are used), etc. On the other hand, it is worth noting that explicit constructions are in many circumstances poor substitute for probabilistic ones, since giving graphs with worse expanding properies then probabilistic ones.

Besides the particular graphs with good expanding properties, very frequently the (infinite) families of such graphs are more preferable.

In the rest, we give some details on sorting in rounds.

Suppose we are given n elements $x_1, \dots, x_n$ with linear order unknown to us. Our task is to determine this linear order by as few probes as possible. Each probe (or question) is a binary comparision (say, is $x_i > x_j$?). The (information) theoretic bound is, clearly, $\log_2 n!$ ($\sim n \log_2 n$). The sorting in rounds is organized as follows: In the first round we ask m_1 ($\leq m$) simultaneous questions. Having the answers, we deduce all implications and ask, in the second round, another $m_2(\leq m)$ questions, deduce their implications, and so on. After r rounds, we need to have the unknown order. The need for such algorithms arises in structural modeling.

The sorting described above is in fact parallel sorting. Here m is a number of processors (also called the width of algorithm); r is a (parallel) time requred by algorithm (also called the depth of algorithm). The object is to minimize the size of the algorithm (equal to the number of comparisions), here denoted by $f_r(n)$.

It is known, for example that $f_1(n) = \binom{n}{2}$; $f_2(n) = O(n^{\frac{3}{2}} \log n)$ (probabilistic bound) and $f_2(n) = O(n^{\frac{7}{4}})$ (explicit construction by expanders).

Here the idea of using expanders is based on the fact that after each round enough comparisions are avoided due to good expanding properties of partial graph so far grown.

For more details see [4], [1] and [12].

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