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INVITED LECTURE

WORD PROBLEMS FOR VARIETIES OF ALGEBRAS (A SURVEY)

Siniša Crvenković

1. Introduction

In the algebraic sense, a *word* is a formal expression, or finite string of symbols, built up in a more or less transparent way from certain primitive symbols, called constants, and certain other symbols which represent algebraic operations. A *word problem* is the problem of deciding in a given context, whether or not two given words represent the same element of the algebra. For such a problem to have a definite sense, certain assumptions must be made. Typically, one is concerned with some specific variety of algebras, such as groups or associative rings or the like. Word problems range all the way from triviality to algorithmic unsolvability.

The origin of the field of word problems may be traced back to R. Dedekind who in 1900 described the free modular lattice on three generators. At the begining of the century Axel Thue had formulated the word problem for finitely presented semigroups—or, as one now says, Thue systems—and solved various special cases of the general problem.

But negative results, unsolvability results in algebra, were impossible before the notion of an *algorithmically unsolvable problem* was formulated. In 1935–1936 A. Church and, independently, A. M. Turing gave equivalent precise mathematical definitions of the intuitive notion of algorithm. *"Turing machines"* and *"Church's Thesis"*, led to Church's negative solution of the decision problem for first–order arithmetic; and, subsequently, to independent negative solutions by Church and Turing to Hilbert's *Entscheidungsproblem* for pure predicate logic. It seems that all unsolvability results in mathematics are, in final analysis, a translation of such classical results into a new setting.

In 1947 E. Post and A. A. Markov, independently, showed the word problem for semigroups unsolvable, constructing the bridge from logic to algebra.

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This result was the first unsolvability result outside the foundations of mathematics.

Perhaps the most celebrated result is the unsolvability of the word problem for groups obtained by P. S. Novikov in 1952.

This paper surveys, unifies, and extends a number of results on the word problems in the context of *universal algebra.* From our point of view, the theory of free spectra is also a part of algebra dealing with words. A model–theoretic argument is used to prove unsolvability of many word problems for varieties of universal algebras. Most of the presented results are small contributions of the authors to the great topic and appeard, or will be published, elswhere.

2. Definitions

In the sequel, $\mathcal{L}$ denotes a first order language which contains the symbol of identity $\approx$, and has no relation symbols. If t_1, t_2 are terms of the language $\mathcal{L}$, then $t_1 \approx t_2$ is called an *equation* or an *identity.* The set of all identities of $\mathcal{L}$ is denoted by $Eq(\mathcal{L})$. If θ is a set of formulas of $\mathcal{L}$, then by $Mod(\theta)$ we denote the class of all algebras $\mathcal{A}$ such that $\mathcal{A} \models \theta$.

If G is a set of new contant symbols ($\mathcal{L} \cap G = \emptyset$), then by $\mathcal{L}_G$ we denote $\mathcal{L} \cup G$. Usually, a symbol from G and its interpretation is denoted by the same letter. Let $\mathcal{A}$ be an algebra and $G \subseteq A$. Then by $\mathcal{A}_G$ we denote the algebra $(\mathcal{A}, x)_{x \in G}$. If R is a set of identities in $\mathcal{L}_G$ with no variables, then (G, R) is called a *presentation* in $\mathcal{L}_G$.

Definition 2.1. *Let θ be a set of identities of $\mathcal{L}$, $\mathcal{V} = Mod(\theta)$ and (G, R) a presentation in $\mathcal{L}_G$. For an algebra $\mathcal{A}$ in $\mathcal{L}$ we say that it is* presented by *(G, R) in $\mathcal{V}$ if the following hold:*

(i) *$\mathcal{A}$ is generated by G;*

(ii) *$\mathcal{A}_G \models \theta \cup R$;*

(iii) *For any identity e in $\mathcal{L}_G$, with no variables, we have $\theta \cup R \models e$ provided $\mathcal{A} \models e$.*

If an algebra $\mathcal{A}$ is presented by (G, R) in $\mathcal{V}$, then we put $\mathcal{A} = \mathcal{P}_{\mathcal{V}}(G, R)$. For an algebra $\mathcal{B}$ we say that it is *finitely presented* in $\mathcal{V}$ if there are finite sets G and R such that $\mathcal{B}$ is presented by (G, R) in $\mathcal{V}$. Note that the algebra presented by (G, R) in $\mathcal{V}$ is unique up to isomorphism.

Example 2.2. *Let (G, R) be a presentation in $\mathcal{L}_G$. Let θ be a set of identities of $\mathcal{L}$ and $\mathcal{V}$ the variety defined by the set $\theta \cup R$. Then the free algebra $\mathcal{F}_{\mathcal{V}}(\emptyset)$ of the variety $\mathcal{V}$ on the empty set of free generators is an algebra presented by (G, R) in $\mathcal{V} = Mod(\theta)$.*

Let θ be a set of identities of $\mathcal{L}$, $\mathcal{V} = Mod(\theta)$, and $\mathcal{A}$ the algebra finitely presented by (G, R) in $\mathcal{V}$. *The word problem for* $\mathcal{A} = \mathcal{P}_{\mathcal{V}}(G, R)$ *in* $\mathcal{V}$ asks if there is an algorithm to determine, for any identity e in $\mathcal{L}_G$ with no variables, whther or not $\mathcal{A} \models e$. If such an algorithm exists, the word problem is *solvable (decidable)*; otherwise it is *unsolvable (undecidable)*.

The following two options occur in the literature for what is meant by the solvability of the word problem for a variety $\mathcal{V}$:

(1) there is an algorithm which, given a finite presentation $\mathcal{P}_{\mathcal{V}}(G, R)$ solves the word problem for $\mathcal{P}_V(G, R)$ in $\mathcal{V}$;
(2) for each finite presentation $\mathcal{P}_V(G, R)$, there is an algorithm which solves the word problem for $\mathcal{P}_V(G, R)$ in $\mathcal{V}$.

We say that $\mathcal{V}$ has *uniformly solvable* word problem if (1) holds.

Varieties with uniformly solvable word problem include commutative semigroups and abelian groups, any finitely based locally finite or residually finite variety, and the variety of all algebras of a given finite type (see [21]).

Most of the examples which appear in the literature, of varieties with unsolvable word problem, provide a finite presentation for which the word problem is unsolvable. These include semigroups, groups and modular lattices.

In this paper we will apply the method of embedding to obtain several unsolvabilities of word problem.

3. Varieties with solvable but uniformly unsolvable word problems

Probably the first one who recognized the difference between the uniform solvability and solvability of the word problem was A. I. Mal'cev [27]. According to Benjamin Wells, A. Tarski was also interested in the existence of varieties with solvable but not uniformly solvable word problem.

An algebra $\mathcal{A}$ is *locally finite* if every finitely generated subalgebra is finite. A *variety* $\mathcal{V}$ *is lically finite* if every member of $\mathcal{V}$ is locally finite.

Let us recall some facts from mathematical logic. For an arbitrary first–order theory $\mathcal{K}$, we correlate with each symbol α of $\mathcal{K}$ a positive integer $\Gamma(\alpha)$, called the *Gödel number* of α. Thus, Γ is a one–one function from the set of symbols of $\mathcal{K}$, expressions of $\mathcal{K}$, and finite sequences of expressions of $\mathcal{K}$, into the set of positive integers.

A set of Gödel numbers is *recursive* if its characteristic function is a recursive function. Denote by $\mathcal{T}(x_1, x_2, \ldots, x_n)$ the set of all n–ary terms in the language of a variety $\mathcal{V}$. According to Church's Thesis, an algebra $\mathcal{A}$ finitely presented by (G, R) in $\mathcal{V}$ has a *solvable word problem* if the set

$$\{\Gamma(p \approx q) | \, p, q \in \mathcal{T}(x_1, x_2, \ldots, x_n),$$

$$p^{\mathcal{A}}(g_1, g_2, \dots, g_n) = q^{\mathcal{A}}(g_1, g_2, \dots, g_n),\ n \in N,\ g_1, g_2, \dots, g_n \in G\}$$

is recursive.

Proposition 3.1. *Let $\mathcal{B}$ be a finite algebra of a finite type. If $(c_1, c_2, \dots, c_n) \in B^n$ is a fixed n–tuple, then the set*

$$S = \{\Gamma(p \approx q) | p, q \in \mathcal{T}(x_1, x_2, \dots, x_n), p^{\mathcal{B}}(c_1, c_2, \dots, c_n) = q^{\mathcal{B}}(c_1, c_2, \dots, c_n)\}$$

is recursive.

Proof. The proof is straightforward. □

Proposition 3.2. *If a finitely presented algebra $\mathcal{A}$ is finite, then the word problem for $\mathcal{A}$ is solvable.*

Proof. Follows from the previous prposition □

Let e, e_1, e_2, ..., e_n (where $n \in N$) be identities of $\mathcal{L}$. Then the formula $e_1 \wedge e_2 \wedge \ldots \wedge e_n \rightarrow e$ is called a *quasi–identity.* The set of all quasi–identities of $\mathcal{L}$ is denoted by $Q(\mathcal{L})$. If $\mathcal{K}$ is a class of algebras in a language $\mathcal{L}$ then $Q(\mathcal{K}) = \{q \in Q(\mathcal{L}) | \mathcal{K} \models q\}$. *The problem of quasi–identities for a class $\mathcal{K}$* asks if the set $Q(\mathcal{K})$ is recursive (i.e. the set of Gödel numbers of the elements of $Q(\mathcal{K})$). If so, the problem of quasi–identities is *solvable*; otherwise it is *unsolvable.*

By the Church's Thesis, the problem of solvability (decidability) of the problem of quasi–identities for a class $\mathcal{K}$ is equivalent to the problem of the existence of an algorithm which, for every quasi–identity $q \in Q(\mathcal{L})$, decides whether or not $\mathcal{K} \models q$.

Remark 3.3. *Let θ be a set of formulas of $\mathcal{L}$ and $\mathcal{K} = Mod(\theta)$. Then we have*

$$Q(\mathcal{K}) = \{q \in Q(\mathcal{L}) | \theta \vdash q\}.$$

Therefore, the problem of quasi–identities for such a class $\mathcal{K}$ is solvable iff there exists an algorithm which, for any $q \in q(\mathcal{K})$, decides whether or not $\theta \vdash q$.

The following proposition is a part of the folklore.

Proposition 3.4. *Let θ be a set of identities in some language $\mathcal{L}$ and let $\mathcal{K} = Mod(\theta)$. Then $\mathcal{K}$ has uniformly solvable word problem iff the problem of quasi–identities for $\mathcal{K}$ is solvable.*

Similarly to the case of quasi–identities, if $\mathcal{K}$ is a class of algebras in a language $\mathcal{L}$, then $Eq(\mathcal{K}) = \{e \in Eq(\mathcal{L}) | \mathcal{K} \models e\}$. The set $Eq(\mathcal{K})$ is called the *equational theory* of the class $\mathcal{K}$. We say that equational theory of a class $\mathcal{K}$ is *decidable (solvable)* if the set $Eq(\mathcal{K})$ is recursive (i.e. the set of Gödel numbers of the elements of $Eq(\mathcal{K})$ is recursive).

Proposition 3.5. *Let $\mathcal{V}$ be a locally finite variety of a finite type. Then $\mathcal{V}$ has a solvable word problem.*

Proof. Follows from Proposition 3.2. □

As we all know, the set of all recursive functions is countable. For a class of algebras $\mathcal{K}$, let $H_{Eq(\mathcal{K})}$ denote the characteristic function of the set $\{\Gamma(p \approx q) | p \approx q \in Eq(\mathcal{K})\}$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two different varieties, then $H_{Eq(\mathcal{V}_1)} \neq H_{Eq(\mathcal{V}_2)}$. Therefore we have

Proposition 3.6. *If a class of varieties of the same type has uncountably many elements, then there is a variety from that class having undecidable equational theory.*

Corollary 3.7. *Let $\mathcal{V}$ be a locally finite variety of a finite type. If $\mathcal{V}$ has uncountably many subvarieties, then $\mathcal{V}$ has solvable but not uniformly solvable word problem.*

Proof. From Proposition 3.6. it follows that $\mathcal{V}$ has undecidable equational theory. This, of course, implies that the problem of $Q(\mathcal{V})$ is unsolvable which is equivalent with the uniform unsolvability of the word problem for $\mathcal{V}$. □

The student of A. Tarski, Benjamin Wells, in his Ph. D. thesis at the University of Berkeley (1982), presented the first examples of varieties having solvable but uniformly unsolvable word problem. This result appeared first in [36], later as Theorem 11.17. in [37], and recently as Theorem 1.7. in [38]. The last result is almost identical to our result even though they were obtained independently. Our construction is primarily based on an example appearing in the paper of Mekler, Nelson and Shelah [30].

Theorem 3.8 ([11]). *In a language of the type $(2, 0, 1, 1)$ there exists a variety having solvable word problem and undecidable equational theory. This variety is axiomatized by the following identities*

$$\begin{array}{ccc} x \cdot 0 \approx 0 & f(0) \approx 0 & h(h(x)) \approx h(x) \\ x^2 \approx 0 & f(f(x)) \approx f(x) & h(x)y \approx 0 \\ xy \approx yx & f(x \cdot y) \approx 0 & f(h(x)) \approx h(x) \\ x \cdot (y \cdot z) \approx (x \cdot y) \cdot z & h(0) \approx 0 & \end{array}$$

$$h^k(f(x_1)f(x_2) \cdots f(x_{\varphi(k)})) \approx 0,$$

where $\varphi(k)$ is a primitive recursive function such that $X = \{\varphi(k) | k \in N\}$ is a recursively enumerable nonrecursive set.

In [12] we proved the following

Theorem 3.9 ([12]). *In a language of the type* $(2,0,1,1)$ *there exists an infinite (isomorphic to* $\langle\omega,\leq\rangle$*) chain of varieties with solvable word problems and undecidable equational theories.*

The varieties from Theorem 3.9. are constructed in the following way. Let the variety defined in Theorem 3.8. be denoted by $\mathcal{V}_1$. Denote by ε_n, $n \geq 2$ the identity in $\{\cdot, f, h, 0\}$ of the form:

$$\varepsilon_n : f(x_1 f(x_2 \ldots f(x_n) \ldots)) \approx 0.$$

The variety whose set of definitional identities is same as the one for $\mathcal{V}_1$, with the exception of $f(x \cdot y) \approx 0$ being replaced by (ε_n), will be denoted by $\mathcal{V}_n$. Obviously,

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \ldots \subseteq \mathcal{V}_n \subseteq \ldots$$

It is easy to prove that all the inclusions are strict.

Definition 3.10. *Ternary discriminator on a set* A *is the function*

$$t_A(a,b,c) = \begin{cases} c, & \textit{for } a = b \\ a, & \textit{otherwise.} \end{cases}$$

Definition 3.11. *A variety* $\mathcal{V}$ *in a language* $\mathcal{L}$ *is said to be a* discriminator *variety if there exists a term in* $\mathcal{L}$ *inducing ternary discriminator on the universe of every subdirectly irreducible algebra in* $\mathcal{V}$.

Following the idea of Ross Willard we were able to prove

Theorem 3.12 ([12]). *There exists a recursively axiomatized discriminator variety in a finitary language with solvable word problem and undecidable equational theory.*

Discriminator varieties are only a part of wider class of so called *EDPC* varieties, arising in the algebraization of different logical systems.

Corollary 3.13. *There exists a recursively based EDPC variety in a finitary language having solvable word problem and undecidable equational theory.*

This result rules out the possibility of obtaining the converse of the following result, due to Blok and Pigozzi:

Theorem 3.14 ([2]). *Let* $\mathcal{V}$ *be an EDPC variety having decidable equational theory. Then* $\mathcal{V}$ *has solvable word problem.*

B. Wells proved that there is a variety of a finite type, with a base of not more than 350 000 axioms, having solvable but not uniformly solvable word problem. Mekler, Nelson and Shelah in [30] also presented a finitely based variety of a finite type having the same properties. However these examples seem to be too complicated and are not from any well known class of algebras. Also their varieties have decidable equational theories. The following problem is still open

Problem 3.15. *Is there a finitely based variety with solvable word problem having undecidable equational theory?*

4. Embedding

There are several undecidability proofs in the literature that use the result of Post and Markov on the existence of a finitely presented semigroup with unsolvable word problem. For example, in [26] we proved unsolvability of the word problem for the variety of relation algebras. We used the following result of Kogalovskii [25].

Proposition 4.1. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are classes of algebras such that*

(i) $\mathcal{K}_1 \subseteq \mathcal{K}_2$,

(ii) *every algebra from $\mathcal{K}_2$ is embeddable into an algebra from $\mathcal{K}_1$,*

then the theories of quasi-identities of $\mathcal{K}_1$ and $\mathcal{K}_2$ are the same.

Proof. See [27] and [25]. □

Corollary 4.2. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be varieties of algebras such that $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and every algebra from $\mathcal{K}_2$ is embeddable into an algebra from $\mathcal{K}_1$. Then, $\mathcal{K}_1$ and $\mathcal{K}_2$ have equivalent uniform word problems.*

Proof. Direct consequence of Proposition 4.1. □

So, if $\mathcal{K}_2$ is the class of all semigroups, and $\mathcal{K}$ is some class of algebras, such that some reduct $\mathcal{K}_1$ of $\mathcal{K}$ satisfies conditions of Corollary 4.2., then $\mathcal{K}$ has uniformly unsolvable word problem. But, this is not enough to obtain the result about the solvability of the word problem for $\mathcal{K}$. The following theorem gives something more than Corollary 4.2.

Theorem 4.3 ([5]). *Let $\mathcal{V}$ be a variety with an associative operation $*$ in its language. If every semigroup can be embedded into the $*$-reduct of some algebra from $\mathcal{V}$, then $\mathcal{V}$ has unsolvable word problem.*

The proof of Theorem 4.3. has been given in [5] and [26]. If we analyze this proof, we see that condition that $\mathcal{V}$ has to have semigroups in it is not necessary. The same goes for any variety with unsolvable word problem.

Theorem 4.3. enables us to obtain several undecidability results in a uniform way. For example, this theorem gives results on unsolvability of the word problem for some varieties which are obtained from the algebras of binary relations.

For an algebra $\mathcal{A} = (A, F)$ we say that it is an *algebra of binary relations* if $A = \mathcal{P}(S^2)$, for some set S, and F is a set of operations on binary relations.

Let $\mathcal{R}_F$ be a class of algebras of binary relations such that F contains the operation of relative multiplication of binary relations "$\circ$". Then the variety $HSP(\mathcal{R}_F)$ has unsolvable word problem. For example, we have unsolvability of the word problem for the following:

(a) variety generated by the class of all semigroups of binary relations ($F = \{\circ\}$);
(b) variety generated by the class of all involutive semigroups of binary relations ($F = \{\circ, ^{-1}\}$);
(c) (representable) relation algebras of Tarski ($F = \{\cup, \cap, \bar{}, \circ, ^{-1}, \Delta\}$);
(d) relation algebras of Jónsson ($F = \{\cap, \circ, ^{-1}, \Delta\}$);
(e) Kleene algebras ($F = \{\cup, \emptyset, \circ, ^{-1}, \Delta, ^{\text{rtc}}\}$), ($F = \{\cup, \emptyset, \circ, ^{-1}, \Delta\}$) and ($F = \{\cup, \emptyset, \circ, \Delta, ^{\text{rtc}}\}$);
(f) no special name (e.g. $F = \{\cup, \circ\}$, $F = \{\cap, \circ\}$).

Theorem 4.3. can easily be applied in the following cases, thus having unsolvable word problems

(g) rings ($F = \{+, \cdot, 0, 1\}$),
(h) involutive semigroups ($F = \{\cdot, ^{-1}\}$),
(i) semirings ($F = \{+, \cdot\}$),
(j) variety generated by Baer *–semigroups ($F = \{\cdot, ^*\}$),
(k) variety generated by the class of all simple semigroups,
(l) variety generated by the class of all bisimple semigroups,
(m) inverse semigroups ($F = \{\cdot, ^{-1}\}$),
(n) rings with involution ($F = \{+, \cdot, ^*\}$).

5. Partial algebras

Let A be a set and $B \subseteq A^n$. Then $f : B \to A$ is called a *partial operation* on A of type n. A *partial algebra* $\mathcal{A}$ is a pair (A, F), where A is a nonempty set and F is a collection of partial operations on A. In our considerations F will always be a finite set.

Let $\mathcal{A}$ be a partial algebra. Denote by $\Delta(\mathcal{A})$ the positive diagram of $\mathcal{A}$:

$$\Delta(\mathcal{A}) = \{f(a_1, a_2, \dots, a_n) = a \mid f \in F,\ a_1, a_2, \dots, a_n \in A,$$

$f(a_1, a_2, \dots, a_n)$ is defined and equals a in $\mathcal{A}$.}

Of course, if $\mathcal{A}$ is finite, then $\Delta(\mathcal{A})$ is finite.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are partial algebras. $\varphi : A \to B$ is called a *homomorphism* of $\mathcal{A}$ into $\mathcal{B}$ if, whenever $f(a_1, a_2, \dots, a_n)$ is defined, then so is $f(\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n))$ and

$$\varphi(f(a_1, a_2, \dots, a_n)) = f(\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n)).$$

A homorphism is an *isomorphism* if φ is a bijection.

Let $\mathcal{A} = (A, F)$ be a partial algebra and let $\emptyset \neq B \subseteq A$. Then

(i) $\mathcal{B}$ is a *subalgebra* of $\mathcal{A}$ if it is closed under all the operations in $\mathcal{A}$ i.e. if $b_1, b_2, \dots, b_n \in B$ and $f(b_1, b_2, \dots, b_n)$ is defined in $\mathcal{A}$, then $f(b_1, b_2, \dots, b_n) \in B$.

(ii) $\mathcal{B}$ is a *relative subalgebra* of $\mathcal{A}$ if for all $f \in F$ and all $b_1, b_2, \dots, b_n$, $b \in B$, we have:

$f(b_1, b_2, \dots, b_n)$ is defined and equals b iff $f(b_1, b_2, \dots, b_n)$ is defined in $\mathcal{A}$ and $f(b_1, b_2, \dots, b_n) = b$ in $\mathcal{A}$.

It is not dificult to give an example of a partial algebra $\mathcal{A}$ and a set $B \subseteq A$, such that B is the carrier of some relative subalebra of $\mathcal{A}$ but not the carrier of any subalgebra in $\mathcal{A}$.

Let $\mathcal{K}$ be a class of algebras, A nonempty set, and F a set of partial operations on A. Then $\mathcal{A} = (A, F)$ is a *partial $\mathcal{K}$-algebra* if (A, F) is a relative subalgebra of an algebra $\mathcal{B}$ in $\mathcal{K}$. For example, if $\mathcal{L}$ is the class of all lattices, then, a partial algebra $\mathcal{A}$ is a partial $\mathcal{L}$-algebra (or simply, partial lattice) if $\mathcal{A}$ is a relative subalgebra (or relative sublattice) of some lattices.

Definition 5.1. *Let $\mathcal{K}$ be a class of algebras and let $\mathcal{A}$ be a partial algebra. The algebra $F\mathcal{K}(\mathcal{A})$ is called the algebra freely generated by the partial algebra $\mathcal{A}$ over $\mathcal{K}$ if the following conditions are satisfied:*

(i) *$F\mathcal{K}(\mathcal{A})$ is generated by A' and there exists an isomorphism $\chi : \mathcal{A}' \to \mathcal{A}$ between $\mathcal{A}'$ and $\mathcal{A}$, where $\mathcal{A}'$ is a relative subalgebra of $F\mathcal{K}(\mathcal{A})$;*

(ii) *If φ is a homorphism of $\mathcal{A}$ into $\mathcal{C} \in \mathcal{K}$, then there exists a homorphism ψ of $F\mathcal{K}(\mathcal{A})$ into $\mathcal{C}$ such that ψ is an extension of $\chi\varphi$.*

It is not difficult to prove that $F\mathcal{K}(\mathcal{A})$ is unique up to isomorphism and, if $\mathcal{A}$ is an algebra from $\mathcal{K}$, then $F\mathcal{K}(\mathcal{A}) \cong \mathcal{A}$. Also, it is well known that if $\mathcal{K}$ is an equational class, then $F\mathcal{K}(\mathcal{A})$ exists if $\mathcal{A}$ is (isomorphic to) a relative subalgebra of an algebra $\mathcal{B}$ in $\mathcal{K}$. In other words, in the case of equational classes $\mathcal{K}$, $F\mathcal{K}(\mathcal{A})$ exists if $\mathcal{A}$ is a partial $\mathcal{K}$-algebra.

For example, if $\mathcal{A}$ is a partial lattice, then $F\mathcal{L}(\mathcal{A})$ always exists. It is well known (see [22]) these lattices (of the form $F\mathcal{L}(\mathcal{A})$) are the lattices that can be described by finitely many generators and finitely many relations.

Proposition 5.2 ([6]). *Let $\mathcal{K} = Mod(\Sigma)$ be a variety, $\mathcal{A}$ a partial algebra. Then,*

$$FK(\mathcal{A}) \cong \mathcal{P}_{\mathcal{K}}(A, \Delta(\mathcal{A})).$$

Proof. See [6]. □

Let $\mathcal{K}$ be a class of algebras in a language $\mathcal{L}$, and let $\mathcal{A}$ be a partial $\mathcal{K}$–algebra. The *problem of partial $\mathcal{K}$–algebra $\mathcal{A}$* asks if there is an algorithm to determine for any identity $p \approx q \in Eq(\mathcal{L} \cup G)$, with no variables, whether or not $FK(\mathcal{A}) \models p \approx q$.

The *problem of partial $\mathcal{K}$–algebras* asks if there is an uniform algorithm which for any finite partial $\mathcal{K}$–algebra $\mathcal{A}$, and any identity $p \approx q \in Eq(\mathcal{L} \cup G)$, with no variables, decides whether or not $FK(\mathcal{A}) \models p \approx q$.

Proposition 5.3 ([6]). *Let $\mathcal{K}$ be a variety in a language $\mathcal{L}$. If $\mathcal{K}$ has a uniformly solvable word problem, then the problem of partial $\mathcal{K}$–algebras is solvable too.*

Proof. Let $\mathcal{A}$ be a finite partial $\mathcal{K}$–algebra, $p \approx q \in Eq(\mathcal{L} \cup G)$, with no variables. Then, because of Proposition 5.2., $FK(\mathcal{A}) \cong \mathcal{P}_{\mathcal{K}}(A, \Delta(\mathcal{A}))$, so that

$$FK(\mathcal{A}) \models p \approx q \quad iff \quad \mathcal{P}_{\mathcal{K}}(A, \Delta(\mathcal{A})) \models p \approx q.$$

Hence, directly from the algorithm for the solution of the word problem, we obtain an algorithm for the solution of the problem of partial algebras. □

Denote by $|t|$ the length of a term t (i.e. the number of symbols in t). We can formulate two rules:

(α) If a set of identities I contains an identity of the form $p \approx q$, where p and q are terms $|p| = |q| = 1$, then we take out this identity from the set I and in all the other identities we replace the symbol q by p.

(β) If a set of identities I contains some identities of the form $t \approx t_1$, $t \approx t_2$, where $t_1 \neq t_2$, then from I we take out the identity $t \approx t_2$ and in all the other identities we replace the symbol t_2 by t_1.

Let I be a set of identities. Denote by $\alpha(I)$ the set of identities which appear from I, if the rule (α) is applied, and by $\beta(I)$ if the rule (β) is applied.

We say that the set of identities I is *α–pure* if $\alpha(I) = I$. Analogously, I is *β–pure* if $\beta(I) = I$. Obviously, if I is a finite set of identities, then there are natural numbers, m, n such that the set $\alpha^n(I)$ is α–pure and set $\beta^m(I)$ is β–pure.

Definition 5.4 ([6]). *Let $\mathcal{K}$ be a variety in a language $\mathcal{L}$ and (A, R) some finite presentation in $\mathcal{K}$. Then,*

(1) *If t is a term in $\mathcal{L}$, then by $Sub(t)$ we denote the set of all the subterms of t.*

(2) $Sub(R) = \bigcup\{Sub(t) \mid (\exists s)(s \approx t \in R \vee t \approx s \in R)\}$.
(3) $A' = \{C_\sigma \mid \sigma \in Sub(R)\} \cup A$.
(4) *Define the mapping* $\varphi : Sub(R) \rightarrow Eq(\mathcal{L} \cup A')$ *in the following way:*
  (i) *If* $|t| = 1$, *then* $\varphi(t)$ *is* $t \approx C_t$;
  (ii) *If* $t = f(t_1, t_2, \dots, t_n)$, *where* f *is an* n*–ary function symbol and* $t_1, t_2, \dots, t_n$ *are terms, then* $\varphi(t)$ *is* $t = f(C_{t_1}, C_{t_2}, \dots, C_{t_n}) \approx C_t$.
(5) *Define the set* R' *as*

$$R' = \varphi[Sub(R)] \cup \{C_p \approx C_q \mid |p| = 1 \text{ and } p \approx q \in R\} \cup$$

$$\cup \{f(C_{t_1}, C_{t_2}, \dots, C_{t_n}) \approx C_q \mid p = f(t_1, t_2, \dots, t_n) \text{ and } p \approx q \in R\},$$

where $\varphi[Sub(R)] = \{\varphi(t) \mid t \in Sub(R)\}$.

Note that if $t \in Sub(R)$ and $|t| = 1$, then $t \in A$ or t is a contant in $\mathcal{L}$ and the set R' is a set of identities, in the language $\mathcal{L} \cup A'$, with no variables.

Let $\mathcal{A} = (A, R)$ be a finite presentation in a variety $\mathcal{K}$. Let n be a finite natural number such that $\alpha^n(R')$ is α–pure and m be a natural number such that $\beta^m(\alpha^n(R'))$ is β–pure. Then let $R^* = \beta^m(\alpha^n(R'))$ and A^* be the set of all these symbols from $A' \cup const(\mathcal{L})$ which appear in the identities of R^*.

Theorem 5.5 ([6]). *Let* $\mathcal{A} = (A, F)$ *be a finite presentation in a variety* $\mathcal{K} = Mod(\Sigma)$, *in a language* $\mathcal{L}$, *and let* $\mathcal{A}^*$ *be a* $\mathcal{K}$*–partial algebra. Then, if the problem of the partial algebra* $\mathcal{A}^*$ *in* $\mathcal{K}$ *is solvable, the word problem for* $\mathcal{A} = (A, R)$ *in* $\mathcal{K}$ *is solvable, too.*

Proof. See [6]. □

6. Free spectra

Let $\mathcal{V}$ be a variety of type F. The cardinality of the free algebra over n generators ($n \geq 0$) in $\mathcal{V}$ is denoted by $f_n(\mathcal{V})$. The sequence of cardinal numbers

$$f(\mathcal{V}) = \langle f_n(\mathcal{V})\rangle_{n\geq 0} = \langle f_0(\mathcal{V}), f_1(\mathcal{V}), \dots, f_n(\mathcal{V}), \dots\rangle$$

is called the *free spectrum* of $\mathcal{V}$.

Let $\mathcal{A} = (A, F)$ be an algebra of type F. Every term of the type F in n variables $x_1, x_2, \dots, x_n$ ($n \geq 0$) defines an n–ary operation $t : A^n \rightarrow A$ in a natural way. These operations are called *n–ary term operations*. The number of differnt n–ary term operations on $\mathcal{A}$ is denoted by $s_n(\mathcal{A})$. If $\mathcal{A}$ generates the variety $\mathcal{V}$ then obviously $f_n(\mathcal{V}) = s_n(\mathcal{A})$ for all $n \geq 0$. The investigation of free spectra of specific varieties may have started with R. Dedekind [1900]. The *Dedekind problem*, the determination of the free spectrum of the variety $\mathcal{D}$ of distributive lattices, is still open.

In group theory, the famous Burnside problem asks whether $f_n(\mathcal{G}_m)$ is always finite, where $\mathcal{G}_m$ is the variety of groups of exponent m. This was solved in the negative by S. I. Adjan and P. S. Novikov. They proved that f_2 is infinite, for instance, for $m \geq 4381$. The choice of m was improved to $m \geq 115$ for odd exponents m.

A major problem of this field is to determine *what sequences can be represented as the free spectrum* of a variety.

If $\mathcal{V}$ is a variety and all $f_n(\mathcal{V})$ are finite, then $\mathcal{V}$ is a locally finite variety. In what follows we are going to consider only locally finite varieties.

Is there a variety $\mathcal{V}$ having $f_0(\mathcal{V}) = 0$, $f_1(\mathcal{V}) = 10$ and $f_2(\mathcal{V}) = 18$? To answer this question we use the concept of s_n*-sequence* (or p_n*-sequence* in the literature). Denote

$$s(\mathcal{A}) = \langle s_0(\mathcal{A}), s_1(\mathcal{A}), \ldots, s_n(\mathcal{A}), \ldots \rangle.$$

For a nontrivial variety $\mathcal{V}$, we define s_n-sequence of $\mathcal{V}$ as the s_n-sequence of $\mathcal{F}_{\mathcal{V}}(\omega)$, the free algebra on ω generators in $\mathcal{V}$. $s_0(\mathcal{A})$ is the number of unary contant term operations and $s_1(\mathcal{A}) \geq 1$.

The following two formulas connect the free spectrum and the s_n-sequence for an algebra $\mathcal{A}$:

(C1)

$$f_n(\mathcal{A}) = \sum_{k=0}^{n} \binom{n}{k} s_k(\mathcal{A});$$

(C2)

$$s_n(\mathcal{A}) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_k(\mathcal{A}).$$

Back to the variety $\mathcal{V}$ with $f_0(\mathcal{V}) = 0$, $f_1(\mathcal{V}) = 10$ and $f_2(\mathcal{V}) = 18$. By formula (C2), we have $s_2 = f_0 - 2f_1 + f_2 = -2$, a contradiction. So a necessary condition for the representability of a sequence as a free spectrum of an algebra is that the corresponding s_n-sequence be nonnegative.

Let $\mathbf{S}$ be a semilattice with more than one element. Using formula (C1) we see that $f_n(\mathbf{S}) = 2^n - 1$, for all $n \geq 0$. Let $\mathcal{S}$ be the variety of semilattices then

$$s(\mathcal{S}) = \langle 0, 1, 1, \ldots, 1, \ldots \rangle.$$

Let us see some examples which explain the flavor of the field.

Proposition 6.1 ([15]). *Let $\mathcal{A}$ be an idempotent groupoid with $s_3(\mathcal{A}) < 6$. Then $\mathcal{A}$ is equivalent to a semilattice, a diagonal semigroup, a groupoid with $s_n(\mathcal{A}) = n$, or a distributive Stainer quasigroup.*

Proof. See [15]. □

Proposition 6.2. *If $\mathcal{A}$ has two commutative binary term operations, then*

(1) $s_3(\mathcal{A}) \geq 9$ *([17]);*

(2) $s_n(\mathcal{A}) \geq 3 + n!$ *for all* $n > 3$.

Proof. See [17]. □

The following result of J. Dudek seems especially attractive:

Proposition 6.3. *Let d_n be the n-th Dedekind's number, that is, $d_n = |\mathcal{F}_{\mathcal{D}}(n)|$, where $\mathcal{D}$ is the variety of ditributive lattices. For a variety $\mathcal{V}$, $f_n(\mathcal{V}) = d_n$ holds, for all $n \geq 0$, iff $\mathcal{V}$ is equvivalent to $\mathcal{D}$.*

Proof. See [20]. □

Among other things, J. Dudek proved the following

Theorem 6.4. *Let $(A, +, \cdot)$ be an idempotent commutative algebra of the type $(2, 2)$ such that $+$ and $\cdot$ are distinct. Then*

(i) $(A, +, \cdot)$ *is a distributive lattice iff* $s_3((A, +, \cdot)) = 9$.

(ii) *If $(A, + \cdot)$ is a bisemilattice, then $(A, + \cdot)$ is a lattice iff* $s_2((A, + \cdot)) = 2$.

There is no bisemilattice $(A, + \cdot)$ for which $s_2((A, + \cdot)) = 3$.
$(A, +, \cdot)$ is a nondistributive modular lattice iff $s_3((A, + \cdot)) = 19$.

Proof. See [16], [17]. □

In a joint paper with J. Dudek we investigated so called *rectangular* groupoids

Definition 6.5. *A groupoid $(G, \cdot)$ is called rectangular (right) if it satisfies the following laws*

$$x^2 \approx x,$$

$$(xy)z \approx xz.$$

Proposition 6.6 ([13]). *For any rectangular groupoid $(G, \cdot)$, being not a semigroup, we have*

$$s_n((G, \cdot)) \geq n^2, \quad \textit{for } n \geq 3.$$

Proof. See [13]. □

This estimation is the best posible because we have

Theorem 6.7 ([13]). *Let $(G, \cdot)$ be a rectangular groupoid. Then the following conditions are equivalent*

(i) *$(G, \cdot)$ is not a semigroup and satisfies*

$$x(y(zu)) \approx x(z(yu));$$

(ii) $s_n((G,\cdot)) = n^2$, *for all* n;
(iii) $s_4((G,\cdot)) = 16$.

Proof. See [13]. □

In [9] we proved the following and therefore, solving the Problem 25. in [23].

Theorem 6.8 ([9]). *Let* $\mathcal{V}$ *be a variety of semigroups. Then* $s_n(\mathcal{V}) = n^2$, *for all* $n \geq 0$, *iff* $\mathcal{V}$ *is the variety of normal bands.*

Proof. See [9]. □

From Theorem 6.8. and by duing some technical calculations we were able to prove

Theorem 6.9 ([8]). *Let* $\mathcal{G}$ *be a groupoid. Then* $S_n(\mathcal{G}) = n^2$, *for all* $n \geq 0$, *iff one of the following conditions hold*

(i) $\mathcal{G}$ *generates the variety of normal bands;*
(ii) $\mathcal{G}$ *is not a semigroup and satisfies*

$$\begin{aligned} xx &\approx x \\ x(yz) &\approx xz \\ ((xy)z)u &\approx ((xz)y)u; \end{aligned}$$

(iii) $\mathcal{G}$ *is not a semigroup and satisfies*

$$\begin{aligned} xx &\approx x \\ (xy)z &\approx xz \\ x(y(zu)) &\approx x(z(yu)). \end{aligned}$$

Proof. See [8]. □

E. Marczewski formulated in [28] the problem of representability of s_n-sequences by algebras. He and his colleagues in Wroclaw considered many associated problems.

If one considers semigroups, the following problem can be formulated.

Problem 6.10. *Characterize* s_n*-sequences for the class of semigroups.*

We may start with the representability of sequences

$$s_\alpha = < 0, \alpha, \alpha, ..., \alpha, ... > \quad \alpha \in N$$

in the variety of semigroups.

Proposition 6.11. *If a semigroup* $\mathbf{S}$ *has* $s_0(\mathbf{S}) = 0$ *and* $s_1(\mathbf{S}) = \alpha$, $\alpha > 0$, *then the following hold:*

(i) $x, x^2, ..., x^\alpha$ *are different essentially unary term operations;*
(ii) $\mathbf{S}$ *satisfies* $x^{\alpha+1} \approx x^\beta$ *for some* $\beta \in \{1, 2, ..., \alpha\}$;

(iii) $\mathbf{S}$ *satisfies* $x^\gamma \approx x^\delta$ *iff* $\gamma, \delta \geq \beta$ *and* $\gamma \equiv \delta$ *(mod* $\alpha + 1 - \beta$*);*

(iv) *If* p, q *are two terms having lenghts* l_p, l_q *such that* $l_p \neq l_q$ *and* $l_p, l_q \leq \alpha$, *then* $p \neq g$. *Specially, all the terms* $xy, xy^2, ..., xy^{\alpha-1}$ *are different.*

(v) *If a semigroup* $\mathbf{S}$ *has an essentially* n*–ary term operation, then the term* $x_1 x_2 \cdots x_n$ *induces essentially* n*–ary term operation.*

Proof. Follows immediately. □

Proposition 6.12. *If a semigroup* $\mathbf{S}$ *satisfies*

$$xy \approx yx, \; xy^2 \approx x^2 y, \; x^{\alpha+1} \approx x^\beta, \; (\alpha \geq \beta \geq 0),$$

then every nontrivial n*–ary term operation is equal to one of the following*

$$x_1 x_2 \cdots x_{n-1} x_n, \quad x_1 x_2 \cdots x_{n-1} x_n^2, \quad \ldots, \quad x_1 x_2 \cdots x_{n-1} x_n^\alpha.$$

Proof. Straightforward. □

It is easy to demonstrate that a semigroup has $< 0, 1, 1, ..., 1, ... >$ as the s_n sequence iff it is nontrivial semilattice. For the case $< 0, 2, 2, ..., 2, ... >$ we have the following.

Proposition 6.13. *A semigroup* $\mathbf{S}$ *has the* s_n*–sequence* $< 0, 2, 2, ..., 2, ... >$ *iff* $\mathbf{S}$ *generates the variety of semigroups determined by the identities*

$$x^3 \approx x^2, \quad xy \approx yx, \quad xy^2 \approx x^2 y.$$

Proof. $(\rightarrow)$. Let $\mathbf{S}$ be a semigroup having $< 0, 2, 2, ..., 2, ... >$ as the s_n–sequence. If $\mathbf{S}$ satisfies $x^3 \approx x$, then, because of $xy^2 = x^2 y \Rightarrow x^5 = x^4$, it follows that xy, xy^2, $x^2 y$ are three different essentially binary term operations. Hence, $\mathbf{S}$ satisfies $x^3 \approx x^2$. If $\mathbf{S}$ is a non commutative semigroup, then xy, yx are only essentially binary term operations of $\mathbf{S}$. The term xyz is essentially 3–ary (Proposition 6.12.) so that from $s_3(\mathbf{S}) = 2$ it follows that $\mathbf{S}$ satisfies $xyz = zxy = yzx$ which implies $xy^2 = y^2 x = yxy$. But then $x^2 y^2 = y^2 x^2$ which is a contradiction since $\mathbf{S}$ does not have a commutative binary term operation. Therefore, $\mathbf{S}$ is a commutative semigroup. $s_2(\mathbf{S}) = 2$ implies that both of essentially binary term operation are commutative. Specially, $xy^2 = x^2 y$. Therefore $\mathbf{S}$ belongs to the variety given by

$$x^3 \approx x^2, \quad xy \approx yx, \quad \text{and} \quad xy^2 \approx x^2 y.$$

If $\mathcal{A}$ is an arbitrary semigroup from the variety above, then Proposition 6.13. implies that every essentially n–ary term operation is equal to $x_1 \cdots x_{n-1} x_n$ or $x_1 \cdots x_{n-1} x_n^2$. Hence, $s_2(\mathcal{A}) \leq 2$. Since $s_n(\mathbf{S}) = 2$ for all $n \geq 1$, it follows that $\mathbf{S}$ generates the variety.

$(\leftarrow)$. It is sufficient to prove that the free semigroup $\mathbf{F}$ in the variety $x^3 \approx x^2$, $xy \approx yx$, $xy^2 \approx x^2 y$ over an infinite set of generators has

$< 0,2,2,...,2,... >$ as the s_n–sequence. It was demonstrated above that $s_n(\mathbf{F}) \leq 2$ for all $n \geq 1$. Obviously, both of terms $x_1 \cdots x_{n-1}\ x_n$ and $x_1 \cdots x_{n-1}x_n^2$ induce essentially n–ary term operation in $\mathbf{F}$. $\mathbf{F}$ satisfies $x_1 \cdots x_{n-1}x_n \approx x_1 \cdots x_{n-1}x_n^2$ iff this identity can be deduced from the defining identities. However, we can only apply $xy \approx yx$ to $x_1 \cdots x_{n-1}x_n$ and hence obtain a permutation of it. Therefore, $s_n(\mathbf{F}) = 2$ for all $n \geq 1$ and it is obvious that $s_0(\mathbf{F}) = 0$. □

Having done some more calculations we will be able to prove the following.

Theorem 6.14. (i) *For $\alpha \geq 3$ the sequence $< 0, \alpha, \alpha, ...\alpha, ... >$ is not representable in the class of all semigroups.*

(ii) *The sequence $< 0,1,1,...,1,... >$ is the s_n–sequence for a semigroup* $\mathbf{S}$ *iff* $\mathbf{S}$ *is a nontrivial semilattice.*

(iii) *The sequence $< 0,2,2,...,2,... >$ is the s_n–sequence for a semigroup* $\mathbf{S}$ *iff* $\mathbf{S}$ *generates the variety determined by the identities*

$$x^3 \approx x^2,\quad xy \approx yx,\quad xy^2 \approx x^2y.$$

Proof. Follows from the considerations above.

A variety $\mathcal{V}$ is *log–linear* if it is locally finite and there exists a constant $c > 0$ such that $\log f_n(\mathcal{V}) \leq cn$ for all $n > 1$. Obviously $\mathcal{V}$ is log–linear iff the free spectrum of $\mathcal{V}$ has subexponential rate of growth, i.e. iff there exist constants $a, c > 0$ such that $f_n(\mathcal{V}) \leq ac^n$ for all $n \geq 0$.

In [10] we gave a solution of the following problem of Grätzer and Kisielewicz.

Problem 6.15 ([23], Problem 29). *Characterize log–linear varieties of semigroups. Is there any algebraic property of semigroups equivalent to (or following form) log–linearity?*

Theorem 6.16 ([10]). *For any semigroup variety $\mathcal{V}$ the following conditions are equivalent:*

(i) *$\mathcal{V}$ is log–linear;*

(ii) *$\mathcal{V}$ satisfies the identities*

$$x^{\alpha+1} \approx x^\beta$$
$$x_1x_2 \cdots x_m \approx x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(m)},$$

for some $\alpha \geq \beta > 0$, $m > 1$, and some non–trivial permutation σ of the set $\{1, ..., m\}$;

(iii) *$\mathcal{V}$ satisfies the identities*

$$x^{\alpha+1} \approx x^\beta$$
$$x_1x_2 \cdots x_{i-1}x_ix_{i+1}x_{i+2} \cdots x_m \approx x_1x_2 \cdots x_{i-1}x_{i+1}x_ix_{i+2} \cdots x_m,$$

for some $\alpha \geq \beta > 0$, $m > i \geq 1$.

Proof. See [10]. □

Corollary 6.17. *Let* $\mathbf{S} = (S, \cdot)$ *be an arbitrary finite semigroup and let* n *be a natural number such that* $S^{n-1} = S^n$. $\mathbf{S}$ *generates a log–linear variety iff* $cabd = cbad$ *holds for all* $a, b \in S$ *and all* $c, d \in S^n$.

It was proved in [31] that every semigroup satisfying condition (iii) of Theorem 6.16. has a finite basis for its identities, so that every log–linear semigroup variety is finitely based. Moreover, every subvariety of a log–linear semigroup variety is log–linear, and therefore finitely based. So we have:

Corollary 6.18. *Every log–linear variety of semigroups is a hereditarily finitely based.*

However, log–linearity is not necessary condition for a semigroup variety to be finitely based, even if the variety is locally finite. The variety $\mathcal{V}$ defined by the identity $xyzx \approx x^2$ was shown in [33] to be hereditarily finitely based. On the other hand, it is easy to check that $n! \leq f_n(\mathcal{V}) \leq (n+1)^n$ for all $n \geq 1$, so that $\mathcal{V}$ is locally finite but not log–linear.

7. Decidability

Let Σ be a fixed set of identities of a given similarity type. The *elementary theory* based on Σ is the set of sentences of first–order logic which are logical consequences of Σ. An elementary, quasi–identies, equational theory is deciable iff it is a recursive set of sentences. The connections between these concepts are given in the diagram below. This diagram refers to any fixed set Σ of equations.

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\text{Decidable elementary theory}} \\
\downarrow & & \downarrow \\
\text{Decidable theory of quasi–identities} & \Longleftrightarrow & \text{Uniformly solvable word problem} \\
\downarrow & & \downarrow \\
\text{Decidable equational theory} & & \text{Solvable word problem}
\end{array}
$$

In general, none of the implications above can be reversed.

It is well known ([24]) that in the case of the variety of relation algebras of Tarski every quasi–identity is equivalent to some identity. Since, in the case of relation algebras, there is an algorithm to construct, for every quasi–identity, the equivalent identity, the problem of quasi–identities is equivalent to the problem of decidability of the equational theory. It was mentioned, as a consequence of Theorem 4.3., that the word problem is unsolvable for the class of relation algebras of Tarski. Therefore, the theory of quasi–identities of relation algebras is unsolvable. Hence, we obtain, as a consequence, the well known theorem of Tarski.

Theorem 7.1. *The equational theory of the class of relation algebras is undecidable.*

Starting from the result on unsolvability of the word problem for rings, we can prove that some varieties of modules have undecidable equational theory. The main reason for that is that every ring $\mathcal{R} = (R, +, \cdot, 0)$ can be considered as an $\mathcal{R}$ module $\mathcal{M} = (R, +, \cdot, 0, (f_r)_{r\in R})$, where $f_r(x) = r \cdot x$ for every $x \in R$. Then, to every equality between two words in $\mathcal{R}$, corresponds an identity in $\mathcal{M}$, and from the unsolvability of the word problem for $\mathcal{R}$ we can prove the undecidability of equational theory for $\mathcal{M}$ (and HSP($\mathcal{M}$)).

The same idea, with some additional ones, can be applied for the class of dynamic algebras.

There are several algebraic structures which correspond to some notions from computer science. Such are Kleene and dynamic algebras. We consider Kleene algebras which are obtained from the so–called Kleene relation algebras (without inversion). Kleene relation algebra, with some base $\mathbf{U}$, is an algebra having the set of all binary relations on the set $\mathbf{U}$ as the carrier, and the fundamental operation are set–theoretical union, composition, and reflexive–transitive closure. Kleene algebra is an algebra that belongs to the variety generated by all Kleene relation algebras.

Because of the relationship between Kleene relation algebras and regular languages, it follows the the equational theory of Kleene algebras is decidable.

We proved in [5] that the word problem for the class of all Kleene algebras is unsolvable.

Dynamic algebras are algebraic counterparts of propositional dynamic logic. Roughly speaking, dynamic logic is a classical propositional logic with some modal operators $\langle x\rangle$ associated with the elements x of a Kleene algebra. We can say that the corresponding algebraic structure, dynamic algebras, are Boolean algebras with normal unary operators which are indexed by the elements of a Kleene algebra. Although the equational theory of Kleene algebras is decidable, we proved in [7] that there are infinitely many finitely generated varieties of dynamic algebras having undecidable equational theories.

Definition 7.2. *Let* $\mathcal{K} = (K, \bigvee, ;, \star)$ *be a Kleene algebra. An algebra* $\mathcal{D} = (B, \cdot, -, F_a(a \in K))$ *is a* dynamic $\mathcal{K}$ algebra *if it satisfies the following conditions:*

(1) $(B, \cdot, -)$ *is a Boolean algebra,*
(2) $F_a(0) \approx 0$,

(3) $F_a(x+y) \approx F_a(x) + F_a(y)$,
(4) $F_{a \bigvee b}(x) \approx F_a(x) + F_b(x)$,
(5) $F_{a;b} \approx F_a F_b(x)$,
(6) $x + F_a F_{a^*}(x) \leq F_{a^*}(x)$,
(7) $F_{a^*}(x) \leq x + F_{a^*}(-x \cdot F_a(x))$,

for all $a, b \in K$, $x, y \in B$.

The definition above is from the paper of B. Jónson [24].

Let $\mathbf{S}$ be a semigroup with an identity. By $\mathcal{T}(\mathbf{S})$ we denote the so-called *semigroup of left translations* of $\mathbf{S}$.

Definition 7.3. *Let* $\mathbf{S}$ *be a semigroup with an identity. By* $\Psi(\mathbf{S})$ *we denote the subalgebra of the Kleene relation algebra* $\mathcal{K}(S)$ *generated by the set* $\mathcal{T}(\mathbf{S})$. *We define the dynamic set algebra* $\mathcal{D}(\mathbf{S})$ *to be* $(\mathcal{P}S, \bigcap, -, F_a(a \in \Psi(\mathbf{S})))$.

Definition 7.4. *The* semigroup of Cejtin *is the semigroup* $\boldsymbol{C}$ *presented by* $(G(\boldsymbol{C}), R(\boldsymbol{C}))$, *where*

$G(\mathbf{C}) = \{a, b, c, d, e\}$,
$R(\mathbf{C}) = \{ac = ca, ad = da, bc = cb, bd = db, abac = adacc, eca = ac, edb = be\}$

It is well known that the semigroup of Cejtin has unsolvable word problem.

Proposition 7.5. *There is a sequence* $\mathbf{C}_0\ \mathbf{C}_1, \ldots, \mathbf{C}_n, \ldots$ *of finitely presented semigroups such taht*

(a) *all semigroups* $\mathbf{C}_i$, $(i \in N)$*have unsolvable word problems;*
(b) $HSP(\mathcal{D}(\mathbf{C}_i)) \neq HSP(\mathcal{D}(\mathbf{C}_j))$ *for all* $i \neq j$, $i, j \in N$.

Proof. See [7]. □

Theorem 7.6 ([7]). *There are infinitely many finitely generated varieties of dynamic algebras, with countably many operations, having undecidable equational theories. All these varieties are generated by representable dynamic algebras.*

Proof. See [7]. □

Corollary 7.7. *There are infinitely many finitely generated varieties od dynamic algebras with countably many operations, having uniformly unsolvable word problems.*

Theorem 7.6. does not give any information on the word problem of dynamic algebras. Therefore we can formulate

Problem 7.8. *Is the word problem for all the varieties of dynamic algebras solvable?*

Also, the following is still open

Problem 7.9. *Is there a finitely based variety of dynamic algebras having undecidable equational theory?*

Problem 7.10. *Is there finite dynamic algebra which is not finitely based?*

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Institute of Mathematics, University of Novi Sad, 21000 Novi Sad, Trg D. Obradovića 4, Serbia, Yugoslavia

E-mail: sima@unsim.ns.ac.yu