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INVITED LECTURE

THEORY OF GREATEST DECOMPOSITIONS OF SEMIGROUPS (A SURVEY)

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Dedicated to Professor L. N. Shevrin on his 60th birthday

CONTENTS

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Introduction

As known, one of the best methods used in studying of structure of semigroups, as well as other algebras, is the *decomposition method*. The main idea of this method is to decompose a semigroup into components, possibly of simpler structure, to study the components in details and to establish mutual relationships between the components within the entire semigroup. We differentiate two general kinds of decompositions: *external decompositions*, where we include decompositions into a direct product and related concepts, and *internal decompositions*, by which we mean decompositions by equivalence relations. In this paper our attention will be aimed only to internal decompositions, which will be here called simply *decompositions*.

By a *kind of decompositions* we will mean a mapping $\mathfrak{T} : S \mapsto \mathfrak{T}_S$ by which to any semigroup S we associate a subset $\mathfrak{T}_S$, possibly empty, of the partition lattice Part(S) of S. But it is often of interest to consider such kinds of decompositions which can be applied on any semigroup, i.e. such that $\mathfrak{T}$ is nonempty subset of Part(S), for any semigroup S. For example, many kinds of decompositions have the property that for any semigroup S, $\mathfrak{T}_S$ contains the zero of Part(S), i.e. the one-component partition $\{S\}$. For that reason we define a *type of decompositions*, or a *decomposition type*, as a mapping $\mathfrak{T} : S \mapsto \mathfrak{T}_S$ by which to any semigroup S we associate a subset $\mathfrak{T}_S$ of the partition lattice Part(S) of S, containing its zero. In other words, a decomposition type $\mathfrak{T}$ is a collection of sets $\mathfrak{T}_S$ indexed by the set of all semigroups, and it is defined if for any semigroup S we define what are the elements of $\mathfrak{T}_S$. Of course, any type $\mathfrak{T}$ of decompositions induces a mapping $\mathfrak{T}' : S \mapsto \mathfrak{T}'_S$ by which to any semigroup S we associate a subset $\mathfrak{T}'_S$ of the lattice $\mathcal{E}(S)$ of equivalence relations on S, containing the universal relation on S, called a *type of equivalences*, and vice versa. For a given type $\mathfrak{T}$ of decompositions and a semigroup S, the elements of $\mathfrak{T}_S$ will be called $\mathfrak{T}$-*decompositions* of S, and related equivalence relations will be called $\mathfrak{T}$-equivalences on S, and S will be called $\mathfrak{T}$-*indecomposable* if the one-component partition $\{S\}$ is the unique $\mathfrak{T}$-decomposition of S, i.e. if the universal relation is the unique $\mathfrak{T}$-equivalence on S.

Consider a decomposition type $\mathfrak{T}$ and a semigroup S. Since $\mathfrak{T}_S$ is a subset of the lattice Part(S), then $\mathfrak{T}_S$ is a poset with respect to usual ordering of partitions, from where several very important questions follow:

(1) Does $\mathcal{T}_S$ have a greatest element?
(2) Is $\mathcal{T}_S$ a complete lattice?
(3) Does $\mathcal{T}_S$ a complete sublattice of the partition lattice on S?

Such problems have been treated first by T. Tamura and N. Kimura [112],

1954, and [113], 1955. After that, they have been considered by many authors. The aim of this paper is to make a survey of main ideas, concepts and results concerning greatest decompositions of semigroups of various types. We will talk about the mostly important decomposition types and the results concerning these.

We know that one of the most important algebraic theorems is the famous *Birkhoff's representation theorem*, proved by G. Birkhoff in [3], 1944, which says that any algebra can be decomposed into a subdirect product of subdirectly irreducible algebras. Of course, in Theory of semigroups similar theorems are also very important. A decomposition type $\mathfrak{T}$ will be called *atomic* if there exists the greatest $\mathfrak{T}$-decomposition and their components are $\mathfrak{T}$-indecomposable. But only four atomic types of decompositions of semigroups are known: *semilattice decompositions*, whose atomicity has been proved by T. Tamura [110], 1956, *ordinal decompositions*, whose atomicity has been proved by E. S. Lyapin [62], 1960, $\cup$-*decompositions*, whose atomicity has been proved by L. N. Shevrin [96], 1965, and *orthogonal decompositions*, whose atomicity has been established by S. Bogdanović and M. Ćirić in [10], 1995. In this paper these decomposition types will take an outstanding place.

This paper is divided into five chapters.

In the first chapter we introduce notions and notations that will be used in the further text, we give a classification of decomposition types and define the types that will considered in this paper, and we also present several general results concerning decompositions by congruences.

Because of the great importance and enormous quantity of the results concerning semilattice decompositions of semigroups, these results will be separated from the ones concerning band decompositions and they will be presented in Chapter 2.

Chapter 3 is devoted to the remaining significant types of band decompositions. Namely, in this chapter we make a survey of the results on matrix and normal band decompositions of semigroups.

In Chapter 4 we consider decompositions of semigroups with zero: orthogonal decompositions, decompositions into a left, right and matrix sum of semigroups, and quasi-semilattice decompositions.

Finally, in Chapter 5 we talk about yet other types of decompositions: $\cup$-decompositions, ordinal decompositions, I-matrix decompositions and semilattice-matrix decompositions.

1. Preliminaries

This chapter is divided into three sections. In Section 1.1 we introduce notions and notations that will be used in the further text. In Section 1.2 we make a classification of decompositions and we single out the most important decomposition types, which will be treated later. Finally, in Section 1.3 we consider decompositions by congruence relations and we present several general results concerning these decompositions.

1.1. Basic notions and notations

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. Further, $S = S^0$ means that S is a semigroup with zero 0, and $S \neq S^0$ means that S is a semigroup without zero. If $S = S^0$, we will write 0 instead $\{0\}$, and if A is a subset of S, then $A^\bullet = A - 0$, $A^0 = A \cup 0$ and $A' = (S - A)^0$. If A is a subset of a semigroup S, then $\sqrt{A} = \{x \in S \mid (\exists n \in \mathbf{Z}^+)\, x^n \in A\}$.

For a binary relation ξ on a set A, ξ^∞ will denote the transitive closure of ξ, ξ^{-1} will denote the relation defined by $a\,\xi^{-1}\,b \Leftrightarrow b\,\xi\,a$, and for $a \in A$, $a\xi = \{x \in A \mid a\,\xi\,x\}$ and $\xi a = \{x \in A \mid x\,\xi\,a\}$. By a *quasi-order* we mean a reflexive and transitive binary relation. If ξ is a quasi-order on a set A, then the relation $\widetilde{\xi}$ defined by $\widetilde{\xi} = \xi \cap \xi^{-1}$ is an equivalence relation called the *natural equivalence* of ξ. A relation ξ on a semigroup $S = S^0$ is called *left 0-restricted* if $0\xi = 0$. A *right 0-restricted* relation on S is defined dually, and a relation ξ on $S = S^0$ will be called *0-restricted* if it is both left and right 0-restricted, i.e. if $0\xi = \xi 0 = 0$. We say that a relation ξ on a semigroup S satisfies the *common multiple property*, briefly the *cm-property*, if for all $a, b, c \in S$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$. Similarly, for a relation ξ on a semigroup $S = S^0$ we say that ξ satisfies the *0-common multiple property*, briefly the *0-cm-property*, if for all $a, b, c \in S$, $ab \neq 0$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$.

Let K be a subset of a lattice L (not necessary complete). If K contains the meet of any its nonempty subset having the meet in L, then K is called a *complete meet-subsemilattice* of L. A *complete join-subsemilattice* is defined dually. If K is both complete meet-subsemilattice and complete join-subsemilattice of L, then it is called a *complete sublattice* of L. If L is a lattice with unity, then any sublattice of L containing its unity is called a *1-sublattice* of L. Dually we define a *0-sublattice* of a lattice with zero, and we define a sublattice of a lattice L with zero and unity to be a *0,1-sublattice* if it is both 0-sublattice and 1-sublattice of L. If any element of L is the meet of some nonempty subset of K, then K is called *meet-dense* in L.

An element a of a lattice L with the zero 0 is an *atom* of L if $a > 0$ and

there exists no $x \in L$ such that $a > x > 0$. A complete Boolean algebra B is *atomic* if every element of B is the join of some set of atoms of B. If L is a distributive lattice with zero and unity, then the set $\mathfrak{B}(L)$ of all elements of L having a complement in L is a Boolean algebra and it is called the *greatest Boolean subalgebra* of L.

For a nonempty set A, $\mathcal{P}(A)$ will denote the *lattice of subsets* of A. Let A be a nonempty set and let L be a sublattice of $\mathcal{P}(A)$ containing its unity and having the property that any nonempty intersection of elements of L is also in A. Then for any $a \in A$ there exists the smallest element of L containing a (it is the intersection of all elements of L containing a), which will be called the *principal element* of L generated by a.

A subset A of a semigroup S is called *completely semiprime* if for $x \in S$, $x^2 \in A$ implies $x \in A$, *completely prime* if for $x, y \in S$, $xy \in A$ implies the either $x \in A$ or $y \in A$, *left consistent* if for $x, y \in S$, $xy \in A$ implies $x \in A$, *right consistent* if for $x, y \in S$, $xy \in A$ implies $y \in A$, and it is *consistent* if it is both left and right consistent. Clearly, the empty set has any of these properties and the sets of completely semiprime, completely prime, left consistent, right consistent and consistent subsets are complete sublattices of $\mathcal{P}(S)$. A consistent subsemigroup of a semigroup S will be called a *filter* of S. The empty set will be also defined to be a filter. By $\mathcal{F}(S)$ we denote the *lattice of filters* of S, which is a complete meet-subsemilattice of $\mathcal{P}(S)$, and therefore a complete lattice, but it is not necessary a sublattice of $\mathcal{P}(S)$. It is well known that a subset A of a semigroup S is a filter of S if and only its complement is either empty or a completely prime ideal of S. The principal element $\mathcal{F}(S)$, called the *principal filter*, generated by $a \in S$ will be denoted by $N(a)$.

In studying of semigroups with zero we use some similar notions. Namely, a subset A of a semigroup $S = S^0$ is called *left 0-consistent* if $A^\bullet$ is left consistent, *right 0-consistent* if $A^\bullet$ is right consistent, and it is *0-consistent* if $A^\bullet$ is consistent. Similarly, an equivalence relation θ on $S = S^0$ will be called *left 0-consistent* if for $x, y \in S$, $xy \neq 0$ implies $xy\,\theta\,x$, *right 0-consistent* if for $x, y \in S$, $xy \neq 0$ implies $xy\,\theta\,y$, and *0-consistent* if it is both left and right 0-consistent.

Let S be a semigroup. By $\mathcal{I}d(S)$ we denote the *lattice of ideals* of S. This lattice is a sublattice of $\mathcal{P}(S)$, it is also a complete join-subsemilattice of $\mathcal{P}(S)$, but it is not necessary a complete meet-subsemilattice, since the empty set is not included in $\mathcal{I}d(S)$. The principal element of $\mathcal{I}d(S)$, called the *principal ideal*, generated by $a \in S$ will be denoted by $J(a)$.Further, $\mathcal{LI}d(S)$ will denote the *lattice of left ideals* of a semigroup S defined in the following way: if $S = S^0$, then $\mathcal{LI}d(S)$ consists of all left ideals of S, and if S has no zero, then $\mathcal{LI}d(S)$ consists of the empty set and all left

ideals of S. The *lattice of right ideals* of S, in notation $\mathcal{RI}d(S)$, is defined dually. Lattices $\mathcal{LI}d(S)$ and $\mathcal{RI}d(S)$ are complete sublattices of $\mathcal{P}(S)$. The principal element of $\mathcal{LI}d(S)$, called the *principal left ideal*, generated by $a \in S$ will be denoted by $L(a)$. The *principal right ideal* generated by $a \in S$, defined dually, will be denoted by $R(a)$. By $\mathcal{I}d^{\mathrm{cs}}(S)$ we denote the *lattice of completely semprime ideals* of S, which is a complete 1-subsemilattice of $\mathcal{I}d(S)$. The principal element of $\mathcal{I}d^{\mathrm{cs}}(S)$, called the *principal radical*, generated by $a \in S$ will be denoted by $\Sigma(a)$. By $\mathcal{RI}d^{\mathrm{lc}}(S)$ and $\mathcal{LI}d^{\mathrm{rc}}(S)$ we denote the lattice of left consistent right ideals and the lattice of right consistent left ideals of S, which are complete sublattices of $\mathcal{RI}d(S)$ and $\mathcal{LI}d(S)$, respectively.

For a nonempty subset A of a semigroup S define the relations P_A, R_A and L_A by:

$$\begin{aligned} a\,P_A\,b &\Leftrightarrow (\forall x, y \in S)(xay \in A \Leftrightarrow xby \in A),\\ a\,R_A\,b &\Leftrightarrow (\forall y \in S)(ay \in A \Leftrightarrow by \in A),\\ a\,L_A\,b &\Leftrightarrow (\forall x \in S)(xa \in A \Leftrightarrow xb \in A). \end{aligned}$$

Then P_A is a congruence on S called the *principal congruence* on S defined by A, R_A is a right congruence called the *principal right congruence* on S defined by A, and L_A is a left congruence called the *principal left congruence* on S defined by A. If $\mathcal{A}$ is a nonempty family of subsets of S, then $P(\mathcal{A})$ will denote the congruence which is the intersection of all principal congruence on S defined by elements from $\mathcal{A}$.

Let A be a nonempty set and let $X \in \mathcal{P}(A)$. The relation Θ_X on A defined by

$$a\,\Theta_X\,b \quad \Leftrightarrow \quad a, b \in X \text{ or } a, b \in A - X \qquad (a, b \in A),$$

is an equivalence relation on A whose classes are precisely the nonempty sets among the sets X and $A - X$. Clearly, when $X = \varnothing$ or $X = A$, then Θ_X is the universal relation on A. Also, for any $X \in \mathcal{P}(A)$, $\Theta_X = \Theta_{A_X}$. Further, for a nonempty subset $\mathcal{A}$ of $\mathcal{P}(A)$, $\Theta(\mathcal{A})$ will denote the equivalence relation on A defined by:

$$\Theta(\mathcal{A}) = \bigcap_{X \in \mathcal{A}} \Theta_X.$$

If $\mathcal{A}$ is a complete meet-subsemilattice of $\mathcal{P}(A)$, and it contains the unity of $\mathcal{P}(A)$, then $\Theta(\mathcal{A})$ can be alternatively defined by:

$$a\,\Theta(\mathcal{A})\,b \Leftrightarrow \mathcal{A}(a) = \mathcal{A}(b) \qquad (a, b \in S),$$

Using this theorem, M. Petrich [71] characterized normal band congruences and the smallest normal band congruence on a semigroup by the following two theorems:

Theorem 3.20. (M. Petrich [71]) *A relation θ on a semigroup S is a normal band congruence on S if and only if $\theta = \Phi(\mathcal{A})$, for some nonempty subset $\mathcal{A}$ of $\mathcal{X}$, where $\mathcal{X}$ denotes the set of all left normal complexes and all right normal complexes of S.*

Theorem 3.21. (M. Petrich [71]) *The smallest normal band congruence on a semigroup S equals the relation $\theta = \Phi(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all left normal complexes and all right normal complexes of S.*

In Theorem 3.20, $\mathcal{X}$ cannot be replaced by the set of all normal complexes, but this can be done in Theorem 3.21:

Theorem 3.22. (M. Petrich [71]) *The smallest normal band congruence on a semigroup S equals the relation $\theta = \Phi(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all normal complexes of S.*

4. Decompositions of semigroups with zero

The first known type of decompositions of semigroups with zero have been orthogonal decompositions, called also 0-direct unions, which have been first defined and studied by E. S. Lyapin in [60] and [61], 1950, and Š. Schwarz [90], 1951. After that, they have been studied by many authors, mainly as orthogonal sums of completely 0-simple semigroups. General study of orthogonal decompositions of semigroups with zero has done by S. Bogdanović and M. Ćirić in [10], 1995, and [13]. The results obtained there will be a topic of Section 4.1. Among these results we emphasize Theorem 4.8 on atomicity of orthogonal decompositions.

Decompositions of a semigroup with zero into a left, right and matrix sum of semigroups have been first defined and studied by S. Bogdanović and M. Ćirić in [13]. The results concerning these decompositions obtained in this paper will be presented in Section 4.2. We also give Theorem 4.21 which establish connections between the decompositions into a left, right and matrix sum, and orthogonal decompositions inside the lattice of 0-decompositions of a semigroup with zero. Note also that some decompositions of semigroups with zero, similar to decomposition into a matrix sum, have been considered by O. Steinfeld in [105].

Quasi-semilattice decompositions of a semigroup with zero, which are carried by partially ordered sets, appeared recently in the paper of M. Ćirić

and S. Bogdanović [26]. These decompositions will be considered in Section 4.3.

Note finally that decompositions into a left, right and matrix sum of semigroups, and quasi-semilattice decompositions of semigroups with zero are generalizations (or analogues) of left zero band, right zero band, matrix and semilattice decompositions, respectively, as showed by Theorems 4.22 and 4.28. Orthogonal sums have no such analogue.

4.1. Orthogonal decompositions

In studying of orthogonal decompositions of semigroups with zero, S. Bogdanović and M. Ćirić [10], 1995, has started from the notion of 0-consistent ideal. They defined a *0-consistent* ideal of a semigroup $S = S^0$ as an ideal A having the property that $A^\bullet$ is a consistent subset of S. They denoted by $\mathcal{I}d^{0c}(S)$ the set of all 0-consistent ideals of a semigroup $S = S^0$ and they proved the following theorem:

Theorem 4.1. (S. Bogdanović and M. Ćirić [10]) *The poset $\mathcal{I}d^{0c}(S)$ of all 0-consistent ideals of a semigroup $S = S^0$ is a complete atomic Boolean algebra and $\mathcal{I}d^{0c}(S) = \mathfrak{B}(\mathcal{I}d(S))$.*

Furthermore, any complete atomic Boolean algebra is isomorphic to the Boolean algebra of 0-consistent ideals of some semigroup with zero.

Using this theorem, S. Bogdanović and M. Ćirić [10] obtained the following theorem concerning orthogonal decompositions:

Theorem 4.2. (S. Bogdanović and M. Ćirić [10]) *Any semigroup $S = S^0$ has a greatest orthogonal decomposition and its summands are all the atoms in $\mathcal{I}d^{0c}(S)$.*

Another approach to orthogonal decompositions, through certain equivalence relations, has done by S. Bogdanović and M. Ćirić in [13]. A 0-restricted and 0-consistent equivalence relation on a semigroup $S = S^0$ will be called an *orthogonal equivalence*. This name will be justified by the role of these equivalences in orthogonal decompositions, which will be demonstrated in Theorem 4.4. Namely, the authors proved in [13] the following two theorems:

Theorem 4.3. (S. Bogdanović and M. Ćirić [13]) *The poset of orthogonal equivalences on a semigroup $S = S^0$ is a complete sublattice of the lattice $\mathcal{E}(S)$.*

Theorem 4.4. (S. Bogdanović and M. Ćirić [13]) *The poset of orthogonal decompositions of a semigroup $S = S^0$ is a complete lattice and it is dually isomorphic to the lattice of orthogonal equivalences on S.*

Note that the sumands in an orthogonal decomposition of a semigroup $S = S^0$ are precisely the nonzero classes of the related orthogonal equivalence, with the zero adjoined, and vice versa.

By Theorems 4.3 and 4.4 we deduce the following:

Theorem 4.5. *The lattice of orthogonal decompositions of a semigroup $S = S^0$ is a complete sublattice of the partition lattice of S.*

The lattice of orthogonal decompositions has been also characterized by some Boolean subalgebras of $\mathcal{I}d^{0c}(S)$ as follows:

Theorem 4.6. (S. Bogdanović and M. Ćirić [13]) *The lattice of orthogonal decompositions of a semigroup $S = S^0$ is isomorphic to the lattice of complete Boolean subalgebras of $\mathcal{I}d^{0c}(S)$.*

Note that any complete Boolean subalgebra of $\mathcal{I}d^{0c}(S)$ is atomic and its atoms are precisely the summands in the related orthogonal decomposition of S, and vice versa.

To describe the smallest orthogonal equivalence on a semigroup with zero, S. Bogdanović and M. Ćirić in [10] defined the relation $\sim$ on a semigroup $S = S^0$ by

$$a \sim b \quad \Leftrightarrow \quad J(a) \cap J(b) \neq 0, \quad \text{for } a, b \in S^\bullet, \qquad\qquad 0 \sim 0,$$

and they proved the following:

Theorem 4.7. (S. Bogdanović and M. Ćirić [10]) *The smallest orthogonal equivalence on a semigroup $S = S^0$ equals the relation $\sim^\infty$.*

Note also that the lattice of orthogonal equivalences on S is the principal dual ideal of the lattice $\mathcal{E}^\bullet(S)$ of 0-restricted equivalence relations on S, generated by $\sim^\infty$. Since $\mathcal{E}^\bullet(S)$ is the principal ideal of $\mathcal{E}(S)$, generated by the equivalence relation χ on S determined by the partition $\{S^\bullet, 0\}$, then the lattice of orthogonal equivalences on S is precisely the (closed) interval $[\sim^\infty, \chi]$ of $\mathcal{E}(S)$.

The main theorem of the theory of orthogonal decompositions of semigroups with zero is the theorem on atomicity of orthogonal decompositions, proved by S. Bogdanović and M. Ćirić in [10], 1995. This is the following theorem:

Theorem 4.8. (S. Bogdanović and M. Ćirić [10]) *The summands of the greatest orthogonal decomposition of a semigroup $S = S^0$ are orthogonally indecomposable semigroups.*

S. Bogdanović and M. Ćirić in [13] also observed that orthogonal decompositions of a semigroup $S = S^0$ are closely connected with direct decompositions of the lattice of ideals of S. This connection is demonstrated by the following three theorems:

Theorem 4.9. *The lattice $\mathcal{I}d(S)$ of ideals of a semigroup $S = S^0$ is a direct product of lattices L_α, $\alpha \in Y$, if and only if S is an orthogonal sum of semigroups S_α, $\alpha \in Y$, and $L_\alpha \cong \mathcal{I}d(S_\alpha)$, for any $\alpha \in Y$.*

Theorem 4.10. (S. Bogdanović and M. Ćirić [13]) *The lattice $\mathcal{I}d(S)$ of ideals of a semigroup $S = S^0$ is directly indecomposable if and only if S is orthogonally indecomposable.*

Theorem 4.11. (S. Bogdanović and M. Ćirić [13]) *If S_α, $\alpha \in Y$, are summands of the greatest orthogonal decomposition of a semigroup $S = S^0$, then the lattice $\mathcal{I}d(S)$ can be decomposed into a direct product of lattices $\mathcal{I}d(S_\alpha)$, $\alpha \in Y$, which are directly indecomposable.*

4.2. Decompositions into a left, right and matrix sum

In studying of decompositions of semigroups with zero into a left sum of semigroups, S. Bogdanović and M. Ćirić in [13] used the methods similar to the ones used in studying of orthogonal decompositions. At first, they considered equivalence relations on a semigroup with zero which we call here left sum equivalences. Namely, a 0-restricted, left 0-consistent equivalence relation on a semigroup $S = S^0$ will be called an *left sum equivalence*. *Right sum equivalences* on S are defined dually. These names will be explained by the role of these equivalences in decompositions of S into a left sum and a right sum of semigroups, respectively, as demonstrated in Theorem 4.13. But, first we give the following theorem:

Theorem 4.12. (S. Bogdanović and M. Ćirić [13]) *The poset of left sum equivalences on a semigroup $S = S^0$ is a complete sublattice of the lattice $\mathcal{E}(S)$.*

Theorem 4.13. (S. Bogdanović and M. Ćirić [13]) *The poset of decompositions of a semigroup $S = S^0$ into a left sum of semigroups is a complete lattice and it is dually isomorphic to the lattice of left sum equivalences on S.*

As in orthogonal decompositions, the sumands in a decomposition of a semigroup $S = S^0$ into a left sum are the nonzero classes of the related left sum equivalence, with the zero adjoined, and vice versa.

By Theorems 4.12 and 4.13 we obtain the following:

Theorem 4.14. *The lattice of decompositions of a semigroup $S = S^0$ into a left sum of semigroups is a complete sublattice of the partition lattice of S.*

To characterize the smallest left sum equivalence on a semigroup, the authors used the relation $\overset{r}{\sim}$ defined by G. Lallement and M. Petrich [59], 1966, on a semigroup $S = S^0$ by:

$$a \overset{r}{\sim} b \quad \Leftrightarrow \quad R(a) \cap R(b) \neq 0, \quad \text{for } a, b \in S^\bullet, \qquad\qquad 0 \overset{r}{\sim} 0.$$

The relation $\overset{\ell}{\sim}$ on S is defined dually. Using the above relation, S. Bogdanović and M. Ćirić [13] characterized the smallest left sum equivalence as follows:

Theorem 4.15. (S. Bogdanović and M. Ćirić [13]) *The smallest left sum equivalence on a semigroup $S = S^0$ equals the relation $\overset{r}{\sim}{}^{\infty}$.*

As in orthogonal equivalences, the set of left sum equivalences on a semigroup $S = S^0$ equals the interval $[\overset{r}{\sim}{}^{\infty}, \chi]$ of the lattice $\mathcal{E}(S)$.

Instead of 0-consistent ideals, used in studying of orthogonal decompositions, in studying of decompositions of a semigroup with zero into a left sum of semigroups, S. Bogdanović and M. Ćirić used in [13] the notion of the left 0-consistent right ideal. Namely, they defined a right ideal A of a semigroup $S = S^0$ to be *left 0-consistent* if $A^\bullet$ is a left consistent subset of S. The set of all left 0-consistent ideals of a semigroup they denoted by $\mathcal{RI}d^{\mathrm{l0c}}(S)$ and they proved the following two theorems:

Theorem 4.16. (S. Bogdanović and M. Ćirić [13]) *The poset $\mathcal{RI}d^{\mathrm{l0c}}(S)$ of all left 0-consistent right ideals of a semigroup $S = S^0$ is a complete atomic Boolean algebra and $\mathcal{RI}d^{\mathrm{l0c}}(S) = \mathfrak{B}(\mathcal{RI}d(S))$.*

Theorem 4.17. (S. Bogdanović and M. Ćirić [13]) *The lattice of decompositions of a semigroup $S = S^0$ into a left sum of semigroup is isomorphic to the lattice of complete Boolean subalgebras of $\mathcal{RI}d^{\mathrm{l0c}}(S)$.*

As in orthogonal decompositions, the summands in a decomposition of a semigroup $S = S^0$ into a left sum of semigroups are precisely the atoms in the related complete Boollean subalgebra of $\mathcal{RI}d^{\mathrm{l0c}}(S)$, which is atomic, and vice versa.

As S. Bogdanović and M. Ćirić in [13] observed, the previous two theorems can be applied to direct decompositions of the lattice of right ideals of a semigroup with zero:

Theorem 4.18. *The lattice $\mathcal{RI}d(S)$ of right ideals of a semigroup $S = S^0$ is a direct product of lattices L_α, $\alpha \in Y$, if and only if S is a left sum of semigroups S_α, $\alpha \in Y$, and $L_\alpha \cong \mathcal{RI}d(S_\alpha)$, for any $\alpha \in Y$.*

Theorem 4.19. (S. Bogdanović and M. Ćirić [13]) *If S_α, $\alpha \in Y$, are the summands of the greatest decomposition of a semigroup $S = S^0$ into a left sum of semigroups, then the lattice $\mathcal{LI}d(S)$ can be decomposed into a direct product of its intervals $[0, S_\alpha]$, $\alpha \in Y$, which are directly indecomposable.*

Note that the interval $[0, S_\alpha]$ in Theorem 4.19 cannot be replaced by the lattice $\mathcal{RI}d(S_\alpha)$, in contrast to Theorem 4.11.

In order to characterize decompositions of a semigroup with zero into a matrix sum of semigroups, S. Bogdanović and M. Ćirić consider in [13] equivalence relations that are the intersection of a left sum equivalence and a right sum equivalence, which will be called here *matrix sum equivalences*, and they proved the following theorems:

Theorem 4.20. (S. Bogdanović and M. Ćirić [13]) *The poset of matrix sum equivalences on a semigroup $S = S^0$ is a complete lattice.*

Theorem 4.21. (S. Bogdanović and M. Ćirić [13]) *The poset of decompositions of a semigroup $S = S^0$ into a matrix sum of semigroups is a complete lattice and it is dually isomorphic to the lattice of matrix sum equivalences on S.*

Note that the sumands in a decomposition of a semigroup $S = S^0$ into a matrix sum are exactly the nonzero classes of the related matrix sum equivalence, with the zero adjoined, and vice versa.

Note also that the previous two theorems give a connection between the decompositions into a left sum, decompositions into a right sum and decompositions into a matrix sum. The authors in [13] established a similar connection between the decompositions into a left sum, decompositions into a right sum and orthogonal decompositions. This connection is given by the following theorem:

Theorem 4.22. (S. Bogdanović and M. Ćirić [13]) *The join in $\mathcal{E}(S)$ of any left sum equivalence and any right sum equivalence on a semigroup $S = S^0$ is an orthogonal equivalence on S.*

Especially, the join of $\overset{r}{\sim}{}^{\infty}$ and $\overset{\ell}{\sim}{}^{\infty}$ equals $\sim^{\infty}$.

The above quoted results can be summarized by the following theorem:

Theorem 4.23. *In the partition lattice of a semigroup $S = S^0$, the meet of any decomposition of S into a left sum and any decomposition of S into a right sum of semigroups is an orthogonal decomposition, and its join is a decomposition into a matrix sum of semigroups.*

Especially, the meet of the greatest decomposition of S into a left sum and the greatest decomposition of S into a right sum of semigroups is the greatest

orthogonal decomposition of S, and its join is the greatest decomposition of S into a matrix sum of semigroups.

Note finally that decompositions of a semigroup with zero into a left sum, right sum and matrix sum of semigroups can be treated as generalizations of left zero band, right zero band and matrix decompositions, respectively. This follows by the following theorem:

Theorem 4.24. *The lattice of left zero band (right zero band, matrix) decompositions of a semigroup S is isomorphic to the lattice of decompositions into a left (right, matrix) sum of semigroups of a semigroup T arising from S by adjoining the zero.*

4.3. Quasi-semilattice decompositions

Studying of quasi-semilattice decompositions of semigroups with zero began in the paper of M. Ćirić and S. Bogdanović [20], 1994. In this paper, some notions which appears in studying of semilattice decompositions of semigroups the authors modified for semigroups with zero. Namely, the authors defined a *0-positive* quasi-order on a semigroup $S = S^0$ as a quasi-order ξ having the property that for $a, b \in S$, $ab \neq 0$ implies $a\,\xi\,ab$ and $b\,\xi\,ab$, they defined a quasi-order ξ on S to satisfy the *0-cm-property* if for all $a, b, c \in S$, $ab \neq 0$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$, and they proved the following theorem:

Theorem 4.25. (M. Ćirić and S. Bogdanović [20]) *The poset of left 0-restricted positive quasi-orders on a semigroup $S = S^0$ satisfying the 0-cm-property and the poset of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property are complete lattices and they are isomorphic.*

Further, M. Ćirić and S. Bogdanović defined in [20] a *completely 0-semiprime* ideal of a semigroup $S = S^0$ as an ideal A of S having the property that $A^\bullet$ is a completely semiprime subset of S. Similarly, A is said to be *completely 0-prime* if $A^\bullet$ is a completely prime subset. The set of all completely 0-semiprime ideals of S, denoted by $\mathcal{I}d^{\mathbf{c0s}}(S)$ is clearly a complete lattice. A sublattice L of $\mathcal{I}d^{\mathbf{c0s}}(S)$ is defined to satisfy the *c-0-pi-property* if the set of completely 0-prime ideals from L is meet-dense in L, i.e. if any element of L can be written as the intersection of some family of completely 0-prime ideals from L. Using these notions, M. Ćirić and S. Bogdanović [20] proved the following theorem:

Theorem 4.26. (M. Ćirić and S. Bogdanović [20]) *For a semigroup $S = S^0$, the poset of complete 0,1-sublattices of the lattice $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi-property is a complete lattice and it is dually isomorphic to the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property.*

The investigation of quasi-semilattice decompositions of semigroups with zero M. Ćirić and Bogdanović continued in [26], where they proved the following three theorems that characterize the lattice of quasi-semilattice decompositions of a semigroup with zero:

Theorem 4.27. (M. Ćirić and S. Bogdanović [20]) *The poset of quasi-semilattice decompositions of a semigroup $S = S^0$ is a complete lattice and it is dually isomorphic to the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property.*

Theorem 4.28. (M. Ćirić and S. Bogdanović [26]) *The lattice of quasi-semilattice decompositions of a semigroup $S = S^0$ it is dually isomorphic to the lattice of left 0-restricted positive quasi-orders on S satisfying the 0-cm-property.*

Theorem 4.29. (M. Ćirić and S. Bogdanović [26]) *The lattice of quasi-semilattice decompositions of a semigroup $S = S^0$ is isomorphic to the lattice of complete 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi-property.*

We finish this chapter by the theorem which give a connection between quasi-semilattice decompositions of semigroups with zero and semilattice decompositions. Note that this connection is incorporated in the name of quasi-semilattice decompositions.

Theorem 4.30. (M. Ćirić and S. Bogdanović [26]) *The lattice of semilattice decompositions of a semigroup S is isomorphic to the lattice of quasi-semilattice decompositions of the semigroup T arising from S by adjoining the zero.*

5. Yet other decompositions

In this paper we talk about yet other types of decompositions having the greatest one.

The topic of Section 5.1 will be $\cup$-decompositions, introduced and first studied by L. N. Shevrin [93], 1961, as a powerful tool in studying of lattices of subsemigroups of a semigroup. We quote the theorem considering the properties of the poset of $\cup$-decompositions, the theorem on atomicity of these decompositions, as Theorem 5.3, and also three theorems on application of $\cup$-decompositions to direct decompositions of the lattice of subsemigroups of a semigroup. For informations on other applications of $\cup$-decompositions, and related $\cup$-band decompositions, in studying of the lattice of subsemigroups of a semigroup the reader is referred to the books

of L. N. Shevrin and A. Ya. Ovsyanikov [102], 1990, and [103], 1991, their survey article [101], 1983, and the book of M. Petrich [73], 1977. Note that L. N. Shevrin used the names "strong decomposition" and "strong band decomposition" for these decompositions. But, because the notion "strong band of semigroups" has been also used for other concepts of the semigroup theory, here we use the names used also in the book of M. Petrich [73], 1977.

Ordinal decompositions, treated in Section 5.2, came out from studying of linearly ordered groups in the papers of F. Klein-Barmen [55] and [56], 1942, and [57], 1948, and A. M. Kaufman [51] and [52], 1949. They have been introduced by A. M. Kaufman [51], 1949, where they have been called successively-annihilating sums (bands) of semigroups. General study of these decompositions has done by E. S. Lyapin in his book [62], 1960, where he showed that the poset of ordinal decompositions of any semigroup is a complete sublattice of the partition lattice of this semigroup, and proved the theorem on atomicity of ordinal decompositions, given here as Theorem 5.8. Here we also present the results of M. Ćirić and S. Bogdanović that characterize lattices of ordinal decompositions of semigroups. For more applications on applications of ordinal decompositions see the books: E. S. Lyapin [62], 1960, M. Petrich [73], 1977, and L. N. Shevrin and A. Ya. Ovsyanikov [102], 1990, and [103], 1991.

I-matrix decompositions have arisen in the paper of G. Lallement and M. Petrich [59], 1966, as a generalization of matrix decompositions. The very nice results obtained in this paper will be presented in Section 5.3. For some applications of such decompositions see the papers of J. Fountain and M. Petrich [43], 1986, and [44], 1989.

The last section of this chapter is devoted to semilattice-matrix decompositions of semigroups. These decompositions have been first studied by A. H. Clifford [29], 1941, who proved that unions of groups (completely regular semigroups) are semilattices of completely simple semigroups, which are in fact semilattices of matrices of groups. After that, semilattice-matrix decompositions have been studied by many authors, for example by P. Chu, Y, Guo and X. Ren [28], 1989, L. N. Shevrin [100], 1994, S. Bogdanović and M. Ćirić [11], 1995, and [15], and other. By the well-known theorem of D. McLean [64], 1954, and A. H. Clifford [30], 1954, on the decomposition of a band into a semilattice of rectangular bands, semilattice-matrix decompositions can be treated as generalizations of band decompositions, and in many papers these decompositions have been used to make preparations for band decompositions. Here we present some general properties of these decompositions discovered by M. Ćirić and S. Bogdanović in [27].

5.1. $\bigcup$-decompositions

We said in the introduction of this paper that general study of $\cup$-decompositions has been done by L. N. Shevrin in [96], 1965. There he has obtained the result that can be formulated in the following way:

Theorem 5.1. (L. N. Shevrin [96]) *The poset of $\cup$-decompositions of a semigroup S is a principal ideal of the partition lattice of S.*

In the same paper L. N. Shevrin considered also $\cup$-band decompositions and some their special types. Namely, for any subvariety $\mathcal{V}$ of the variety of bands a *$\cup$-$\mathcal{V}$-band decomposition* of a semigroup S is defined as a decomposition which is both $\cup$-decomposition and $\mathcal{V}$-band decomposition. By Theorems 5.1 and 1.6 the following theorem follows:

Theorem 5.2. *For any subvariety $\mathcal{V}$ of the variety of bands, the poset of $\cup$-$\mathcal{V}$-band decompositions of a semigroup S is a principal ideal of the partition lattice of S.*

L. N. Shevrin [96] also proved the theorem on atomicity of $\cup$-decompositions, which is given below.

Theorem 5.3. (L. N. Shevrin [96]) *The components of the greatest $\cup$-decomposition of a semigroup S are $\cup$-indecomposable.*

Among the numerous applications of $\cup$-decompositions in studying of lattices of subsemigroups of a semigroup we emphasize the application to decompositions of these lattices into a direct product, which is demonstrated by the following three theorems:

Theorem 5.4. (L. N. Shevrin [94]) *The lattice $\mathrm{Sub}(S)$ of subsemigroups of a semigroup S is a direct product of lattices L_α, $\alpha \in Y$, if and only if S has a $\cup$-decomposition into subsemigroups S_α, $\alpha \in Y$, and $\mathrm{Sub}(S_\alpha) \cong L_\alpha$, for any $\alpha \in Y$.*

Theorem 5.5. (L. N. Shevrin [96]) *The lattice $\mathrm{Sub}(S)$ of subsemigroups of a semigroup S is directly indecomposable if and only if S is $\cup$-indecomposable.*

Theorem 5.6. (L. N. Shevrin [96]) *If S_α, $\alpha \in Y$, are the components of the greatest $\cup$-decomposition of a semigroup S, then the lattice $\mathrm{Sub}(S)$ of subsemigroups of S can be decomposed into a direct product of lattices $\mathrm{Sub}(S_\alpha)$, $\alpha \in Y$, which are directly indecomposable.*

We advise the reader to compare the previous three theorems with Theorems 4.9–4.11, concerning direct decompositions of the lattice of ideals of a

semigroup with zero, Theorems 4.18 and 4.19, concerning direct decompositions of the lattice of right ideals of a semigroup with zero, and Theorems 3.15 and 3.16, concerning direct decompositions of the lattice of right ideals of a semigroup without zero.

5.2. Ordinal decompositions

General study of ordinal decompositions has been made by E. S. Lyapin in his book [62] from 1960. There he showed the following property of the poset of ordinal decompositions:

Theorem 5.7. (E. S. Lyapin [62]) *The poset of ordinal decompositions of a semigroup S is a complete sublattice of the partition lattice of S.*

E. S. Lyapin [62] also proved the very important theorem on atomicity of ordinal decompositions, whose another proof has been given by M. Ćirić and S. Bogdanović in [25].

Theorem 5.8. (E. S. Lyapin [62]) *The components of the greatest ordinal decomposition of a semigroup S are ordinally indecomposable.*

To characterize the lattice of ordinal decompositions, M. Ćirić and S. Bogdanović [25] have used the next theorem, obtained in their earlier paper [23], which gives a characterization of the poset of chain decompositions of a semigroup through completely prime ideals.

Theorem 5.9. (M. Ćirić and S. Bogdanović [23]) *The poset of chain decompositions of a semigroup S is isomorphic to the poset of complete 1-sublattices of $\mathcal{I}d^{\mathrm{cs}}(S)$ consisting of completely prime ideals of S.*

Note that another characterization of the poset of chain decompositions can be given by filters as follows:

Theorem 5.10. (S. Bogdanović and M. Ćirić [12]) *The poset of chain decompositions of a semigroup S is isomorphic to the poset of complete 0,1-sublattices of $\mathcal{P}(S)$ consisting of filters of S.*

M. Ćirić and S. Bogdanović [25] defined a *strongly prime* ideal of a semigroup S as an ideal P of S having the property that for all $x, y \in S$, $xy = p \in P$ implies that either $x = p$ or $y = p$ or $x, y \in P$, and they proved that the set of all strongly prime ideals of a semigroup S, denoted by $\mathcal{I}d^{\mathrm{sp}}(S)$, is a complete 1-sublattice of the lattice $\mathcal{I}d(S)$ of ideals of S. Moreover, they gave the following characterization of the lattice of ordinal decompositions of a semigroup:

Theorem 5.11. (M. Ćirić and S. Bogdanović [25]) *The lattice of ordinal decompositions of a semigroup S is isomorphic to the lattice of complete 1-sublattices of $\mathcal{I}d^{\mathrm{sp}}(S)$.*

5.3. I-matrix decompositions

If θ is a congruence on a semigroup S and S/θ is a rectangular 0-band, then θ is said to be an *I-matrix congruence*, where I is an ideal of S which is the θ-class that is the zero of S/θ. The corresponding decomposition is an *I-matrix decomposition* of S, and I is called a *matrix ideal* of S. G. Lallement and M. Petrich [59] defined a *quasi-completely prime* ideal of a semigroup S as an ideal I satisfying the condition that for all $a, b, c \in S$, $abc \in I$ implies that either $ab \in I$ or $bc \in I$, and they proved the following theorem:

Theorem 5.12. (G. Lallement and M. Petrich [59]) *An ideal I of a semigroup S is a matrix ideal if and only if it is prime and quasi-completely prime.*

To characterize I-matrix congruences, G. Lallement and M. Petrich [59] introduced the following notions: if A is a nonempty subset of a semigroup S, then an equivalence relation θ on S is called a *left A-equivalence* if the following conditions hold:

(1) A is a θ-class of S;
(2) θ is a left congruence;
(3) for all $x, y \in S$, $xy \notin A$ implies $xy\,\theta\,x$.

A *right A-equivalence* is defined dually. Necessary and sufficient conditions for existence of a left A-equivalence and a right A-equivalence on a semigroup have been determined by the following theorem:

Theorem 5.13. (G. Lallement and M. Petrich [59]) *Let A be a subset of a semigroup S. Then there exists a left A-equivalence and a right A-equivalence if and only if A is a quasi-completely prime ideal of S.*

The following theorem has been also proved in [59]:

Theorem 5.14. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the poset of left I-equivalences on S is a complete sublattice of the lattice of left congruences on S.*

G. Lallement and M. Petrich [59] characterized the smallest I-equivalence in three ways. At first, they defined a *left I-complex* of a semigroup S as a nonempty subset A of S having the following properties:

(1) $A \cap I = \varnothing$;

(2) A is a left consistent subset of S;
(3) $A \cup I$ is a right ideal of S.

A *right I-complex* has been defined dually. For an element $a \in S - I$, let $C(a)$ they denoted the smallest left I-complex of S containing a, i.e. the intersection of all left I-complexes of S containing a, called the *principal left I-complex* of S generated by a, and they proved the following

Theorem 5.15. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the relation θ on S defined by:*

$$a \,\theta\, b \quad \Leftrightarrow \quad a, b \in I \text{ or } C(a) = C(b) \qquad (a, b \in S),$$

equals the smallest left I-equivalence on S.

The second and third characterization of the smallest left I-equivalence on a semigroup have been given by the following two theorems:

Theorem 5.16. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the relation θ on S defined by*

$$a \,\theta\, b \quad \Leftrightarrow \quad (\forall x \in S)\,(axa \in I \quad \Leftrightarrow \quad bxb \in I) \qquad (a, b \in S),$$

equals the smallest I-matrix congruence on S.

Theorem 5.17. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the smallest left I-equivalence equals the principal left congruence L_I.*

Following the ideas used by M. Petrich in studying of matrix decompositions, G. Lallement and M. Petrich proved in [59] the next theorem, similar to Theorem 3.5.

Theorem 5.18. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the intersection of a left I-equivalence and a right I-equivalence is a I-matrix congruence on S.*

Conversely, any I-matrix congruence on S can be written uniquely as the intersection of a left I-equivalence and a right I-equivalence on S.

Using the previous theorem and Theorem 5.14, G. Lallement and M. Petrich proved also in [59] the following two theorems:

Theorem 5.19. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the poset of I-matrix congruences on S is a complete sublattice of* Con (S).

Theorem 5.20. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S and let θ denote the smallest I-matrix congruence on S. Then $\theta = R_I \cap L_I = R_K = L_K$, where $K = \{x \in S \mid x^2 \notin I\}$.*

5.4. Semilattice-matrix decompositions

Let a semigroup S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for any $\alpha \in Y$, let S_α be a left zero band (right zero band, matrix) of semigroups S_i, $i \in I_\alpha$. M. Ćirić and S. Bogdanović [27] called the partition $\{S_i \mid i \in I\}$, where $I = \bigcup_{\alpha \in Y} I_\alpha$, a *semilattice-left (semilattice-right, semilattice-matrix) decomposition* of S, or briefly *s-l-(s-r-, s-m-)decomposition*. If θ denotes the equivalence relation determined by this partition and if ϱ denotes the semilattice congruence determined by the partition $\{S_\alpha \mid \alpha \in Y\}$, then θ is called a *semilattice-left (semilattice-right, semilattice-matrix) equivalence* on S *carried by* ϱ, or briefly *s-l-(s-r-, s-m-)equivalence*, and ϱ is called a *carrier* of θ. Clearly, an equivalence relation θ on a semigroup S contained in a semilattice-congruence ϱ on S is a s-l-(s-r-, s-m-)equivalence carried by ϱ if and only if for all $a, b \in S$, $a \varrho b$ implies $ab \theta a$ ($a \varrho b$ implies $ab \theta b$, $a \varrho b$ implies $aba \theta a$).

M. Ćirić and S. Bogdanović studied in [27] some general properties of s-l-, s-r- and s-m-equivalences and their carriers, and they proved the next four theorems. Note that Theorem 5.24 is similar to Theorems 3.5, 3.19 and 5.18.

Theorem 5.21. (M. Ćirić and S. Bogdanović [27]) *The set of s-l-(s-r-, s-m-) equivalences on a semigroup S carried by a semilattice congruence ϱ on S is a closed interval of $\mathcal{E}(S)$.*

Theorem 5.22. (M. Ćirić and S. Bogdanović [27]) *The set of carriers of a s-l-(s-r-, s-m-)equivalence θ on a semigroup S is a convex subset, with the smallest element, of the lattice of semilattice congruences on S.*

Theorem 5.23. (M. Ćirić and S. Bogdanović [27]) *The poset of all a s-l-(s-r-, s-m-)equivalences on a semigroup S is a complete lattice.*

Theorem 5.24. (M. Ćirić and S. Bogdanović [27]) *The intersection of a s-l-equivalence and a s-r-equivalence on a semigroup S is a s-m-equivalence.*

Conversely, any s-m-equivalence can be written, uniquely up to a carrier, as the intersection of a s-l-equivalence and a s-r-equivalence.

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