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COMPUTING PSEUDOINVERSES USING MINORS OF AN ARBITRARY MATRIX

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ABSTRACT. *In this paper we establish a general determinantal representation of generalized inverses in terms of minors of an arbitrary matrix of an adequate order. Then we obtain a general algorithm for exact computation of different classes of pseudoinverses: Moore-Penrose inverse, group inverse, left, right inverses and Radić's and Stojaković's inverse. In this way, this paper is a generalization of an earlier paper [12], where an algorithm for computing of the Moore–Penrose inverse, Radić's and Stojaković's inverse is described. We also give some examples which illustrate our results.*

1. Introduction

Let $\mathbb{C}_r^{m\times n}$ be the set of $m \times n$ complex matrices whose rank is r. Conjugate, transpose and conjugate-transpose matrix of A will be denoted by $\overline{A}$, A^T and A^* respectively. Submatrix and minor of A containing rows $\alpha_1, \dots, \alpha_t$ and columns $\beta_1, \dots, \beta_t$ will be denoted by $A\begin{bmatrix}\alpha_1\dots\alpha_t\\ \beta_1\dots\beta_t\end{bmatrix}$ and $A\begin{pmatrix}\alpha_1\dots\alpha_t\\ \beta_1\dots\beta_t\end{pmatrix}$ respectively, and the *algebraic complement* corresponding to the element a_{ji} is defined by

$$A_{ij}\begin{pmatrix}\alpha_1 \dots \alpha_{p-1} & i & \alpha_{p+1} \dots \alpha_t\\ \beta_1 \dots \beta_{q-1} & j & \beta_{q+1} \dots \beta_t\end{pmatrix} = (-1)^{p+q} A\begin{pmatrix}\alpha_1 \dots \alpha_{p-1} & \alpha_{p+1} \dots \alpha_t\\ \beta_1 \dots \beta_{q-1} & \beta_{q+1} \dots \beta_t\end{pmatrix}.$$

For any matrix $A \in \mathbb{C}^{m\times n}$, consider the following equations in X:

(1) $AXA = A$ (2) $XAX = X$ (3) $(AX)^* = AX$ (4) $(XA)^* = XA$

and if $m = n$, also

$$(5) \qquad AX = XA\,.$$

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For a subset $\mathcal{S}$ of $\{1,2,3,4,5\}$, the set of matrices G obeying the conditions represented in $\mathcal{S}$ will be denoted by $A\{\mathcal{S}\}$. A matrix $G \in A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$. In particular, for any $A \in \mathbb{C}^{m\times n}$, the set $A\{1,2,3,4\}$ consists of a single element, the *Moore-Penrose inverse* of A, denoted by $A^\dagger$ [9]. In the case $m = n$, the *group inverse*, denoted as $A^\#$, of A is the unique $\{1,2,5\}$ inverse, and exists if and only if $ind(A) = \min\{k : \; k > 0 \text{ and } rank(A^{k+1}) = rank(A^k)\} = 1$.

The starting point of the investigations of this paper is the determinantal representation of *Moore-Penrose inverse*, studied in [1], [2], [3], [4], [8]. The main result of these papers is:

Theorem 1.1. *Element $a_{ij}^\dagger$ lying on the i-row and j-column of the Moore-Penrose pseudoinverse of a given matrix $A \in \mathbb{C}_r^{m\times n}$ is given by*

$$a_{ij}^\dagger = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<\alpha_r\le m}} \overline{A}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\ldots<\delta_r\le n\\ 1\le\gamma_1<\ldots<\gamma_r\le m}} \overline{A}\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}}, \quad \begin{pmatrix}1\le i\le n\\ 1\le j\le m\end{pmatrix}.$$

Determinantal representation of the *Group inverse* of a singular n by n matrix is introduced in [7]:

Theorem 1.2. *The group inverse $A^\# = \left(a_{ij}^\#\right)$ of $A \in \mathbb{C}_r^{n\times n}$ has the following determinantal representation:*

$$a_{ij}^\# = \frac{\displaystyle\sum_{\substack{1\le\alpha_1<\ldots<\alpha_r\le n\\ 1\le\beta_1<\ldots<\beta_r\le n}} A^T\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\gamma_1<\ldots<\gamma_r\le n\\ 1\le\delta_1<\ldots<\delta_r\le n}} A^T\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}}.$$

For the sake of completeness, in the following definition we unify the definitions of generalized inverses introduced by M. Radić [10], [11], M. Stojaković [13] and V.N. Joshi [5].

Definition 1.1. *Let i, j be integers, $1 \le i \le n$, $1 \le j \le m$. Then the (i,j)-th entry of Radić's, Stojaković's and Joshi's generalized inverse $A \in \mathbb{C}_r^{m\times n}$ is defined by*

$$a_{ij}^\epsilon = \frac{\displaystyle\sum_{\substack{1\le j_1<\ldots<j<\ldots<j_r\le n\\ 1\le i_1<\ldots<i<\ldots<i_r\le m}} \epsilon^{(i_1+\ldots+i_r)+(j_1+\ldots+j_r)} A_{ji}\begin{pmatrix}i_1 & \ldots & j & \ldots & i_r\\ j_1 & \ldots & i & \ldots & j_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\alpha_1<\ldots<\alpha_r\le m\\ 1\le\beta_1<\ldots<\beta_r\le n}} \epsilon^{(\alpha_1+\ldots+\alpha_r)+(\beta_1+\ldots+\beta_r)} A\begin{pmatrix}\alpha_1 & \ldots & \alpha_r\\ \beta_1 & \ldots & \beta_r\end{pmatrix}}, \quad \epsilon \in \{-1,1\}.$$

For $\epsilon = 1$, we get Stojaković's definition, and for $\epsilon = -1$, we get Radić's definition.

Now, we describe the main results of the paper. First we define a general determinantal representation for the *Moore-Penrose, group inverse* , and the class of *left* and *right* generalized inverses. Later we describe algorithms for exact computation of generalized inverses based on the introduced determinantal representation. Finally, we give several examples which illustrate presented theory and algorithms.

2. General determinantal representation

According to Theorem 1.1, Theorem 1.2 and Definition 1.1, we define a general determinantal representation which includes the determinantal representations of the *Moore-Penrose* pseudoinverse and the *group* inverse. Also, this determinantal representation represents the class of *left* and *right* inverses for full-rank matrices and generalized inverses introduced by M. Stojaković, M. Radić and V.N. Joshi.

Theorem 2.1. *For $A \in \mathbb{C}_r^{m\times n}$ determinantal representation of an (i,j)-element of an arbitrary left and right inverse, the Moore-Penrose pseudoinverse, the group inverse, Radić's and Stojaković's inverse is*

$$(2.1)\qquad g_{ij} = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\ldots<\beta_t\le n\\ 1\le\alpha_1<\ldots<\alpha_t\le m}} \overline{R}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_t\\ \beta_1 & \ldots & i & \ldots & \beta_t\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_t\\ \beta_1 & \ldots & i & \ldots & \beta_t\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\ldots<\delta_t\le n\\ 1\le\gamma_1<\ldots<\gamma_t\le m}} \overline{R}\begin{pmatrix}\gamma_1 & \ldots & \gamma_t\\ \delta_1 & \ldots & \delta_t\end{pmatrix} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_t\\ \delta_1 & \ldots & \delta_t\end{pmatrix}},$$

where $R \in \mathbb{C}_r^{m\times n}$ and $t = r_c(A) \le r \le \min\{m,n\}$ is the greatest integer which ensures $DET_{(R,t)}(A) \neq 0$.

For the briefness sake, we denote the numerator of the expression (2.1) by $A_{ij}^{(R,t)}$ and call it the generalized algebraic complement corresponding to element a_{ij}. The denominator is shortly denoted by $DET_{(R,t)}(A)$, and it is called the generalizd determinant.

Proof. Consider the following cases:

1. Suppose that $t = m \le n$. Using the Laplace's development for the square minors $A\begin{pmatrix}1 & \ldots & m\\ j_1 & \ldots & j_m\end{pmatrix}$, we get

$$DET_{(R,m)}(A) = \sum_{j_1<\ldots<j_m} \overline{R}\begin{pmatrix} 1 & \ldots & m \\ j_1 & \ldots & j_m \end{pmatrix} \left[\sum_{k=1}^{r} a_{ij_k} A_{ij_k} \begin{pmatrix} 1 & \ldots & m \\ j_1 & \ldots & j_m \end{pmatrix}\right] =$$

$$= \sum_{l=1}^{n} a_{il} \left[\sum_{j_1<\ldots<j_m} \overline{R}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & m \\ j_1 & \ldots & l & \ldots & j_m \end{pmatrix} A_{il} \begin{pmatrix} 1 & \ldots & \ldots & \ldots & m \\ j_1 & \ldots & l & \ldots & j_m \end{pmatrix}\right] = \sum_{l=1}^{n} a_{il} A_{li}^{(R,m)}.$$

For two integers $p \neq q$, $1 \leq p,q \leq m$, substituting in the minors of A the q-th row by the p-th row, and using

$$DET_{(R,m)}(A) = \sum_{j_1<\ldots<j_m} \overline{R}\begin{pmatrix} 1 & \ldots & m \\ j_1 & \ldots & j_m \end{pmatrix} A \begin{pmatrix} 1 & \ldots & m \\ j_1 & \ldots & j_m \end{pmatrix} = 0,$$

The relation $\sum_{l=1}^{n} a_{pl} A_{lq}^{(R,m)} = 0$ can be proved in the same way. Hence, $g_{ij} = \delta_{ij} DET_{(R,m)}(A)$, and consequently $A \cdot A_{(R,m)}^{-1} = I_m$, for arbitrary R. It means that $A_{(R,m)}^{-1}$ represents the class of *right inverses* of the full-rank matrix A.

On the other hand, it can be proved that $A_{(R,n)}^{-1}$, in the case $t = n \leq m$, represents the class of *left inverses* of A. Now, it is obvious that (2.1) represents the general determinantal representation of *right/left inverses* of a full rank matrix A.

2. For $R = A$, we obtain determinantal representation of $A^{\dagger}$, presented in Theorem 1.1. In this case, $r_c(A) = r$, which represents the known result in [4].

3. If $m = n$, $ind(A) = 1$ and $R = A^*$ the determinantal representation of the *group inverse* is obtained (Theorem 1.2).

4. If $r = r_c(A)$ and a matrix R satisfies condition

(1) $\overline{R}\begin{pmatrix} i_1 & \ldots & i_r \\ j_1 & \ldots & j_r \end{pmatrix} = K \cdot \epsilon^{(i_1+\ldots+i_r)+(j_1+\ldots+j_r)}$, where $K \in \mathbb{C}$, $\epsilon \in \{-1, 1\}$, for all combinations $1 \leq i_1 < \ldots < i_r \leq m$; $1 \leq j_1 < \ldots < j_r \leq n$,

then, in the case $\epsilon = 1$, the inverse $A_{(R,r)}^{-1}$ is equal to the Stojaković's inverse and reduces to the Radić's inverse in the case $\epsilon = -1$ (Definition 1.1).

5. If A is regular square matrix, then (2.1) reduces into the well known inversion of regular square matrices, for an arbitrary regular matrix R of an adequate size. □

Note that the partial cases 4. and 2. are studied in [12].

3. Algorithms

In this section we give a high-level description of the algorithms for computing generalized inverses. Theoretical base of these algorithms is contained in Theorem 2.1.

In all presented algorithms complex and rational numbers are represented by an adequate union in programming language C, called the internal form of numbers. The internal form of a given matrix A is the two-dimensional array or the binary tree of the internal forms of the elements of A. Addition, subtraction, multiplication and division of complex or rational numbers in the internal forms are denoted by $\oplus$ $\ominus$ $\otimes$ $\oslash$, respectively (so called makrooperations).

Various implementation details about the generating combinations are presented in [6].

Here presented procedures receive the following global parameters:

◇ S: the actual value of $DET_{(R,t)}(A)$.

◇ $p(1:n)$, $q(1:n)$: The sequences representing combinations of rows and columns of A respectively.

Now, we describe algorithm for computation of $DET_{(R,k)}(A)$ of a rectangular matrix $A \in \mathbb{C}_k^{n\times k}$, such that $r_c(A) = k$. In the algorithm a combination $1 \le q_1 < \ldots < q_k \le n$ of rows or columns of A is fixed.

procedure $D_1(n, k, x, y, lg)$

◇ $n, k \le n$: The number of rows and number of columns.

◇ x, y: The internal forms of A and R respectively.

◇ lg: The indikator.

begin

Step 1: $p(1:k) \longleftarrow (1:k)$;

Step 2: A **while** cycle which terminates when all the combinations $1 \le p_1 < \ldots < p_k \le n$ are generaed.

Step 2.1: Compute $det(M)$ and $det(M_1)$, using x and y, where

$$M = \begin{cases} A\left[\begin{smallmatrix} p_1 & \cdots & p_k \\ 1 & \cdots & k \end{smallmatrix}\right], & lg = 1 \\ A\begin{bmatrix} 1 & \cdots & k \\ p_1 & \cdots & p_k \end{bmatrix}, & lg = 2 \\ A\begin{bmatrix} p_1 & \cdots & p_k \\ q_1 & \cdots & q_k \end{bmatrix}, & lg = 3 \end{cases}, \quad M_1 = \begin{cases} R\left[\begin{smallmatrix} p_1 & \cdots & p_k \\ 1 & \cdots & k \end{smallmatrix}\right], & lg = 1 \\ R\begin{bmatrix} 1 & \cdots & k \\ p_1 & \cdots & p_k \end{bmatrix}, & lg = 2 \\ R\begin{bmatrix} p_1 & \cdots & p_k \\ q_1 & \cdots & q_k \end{bmatrix}, & lg = 3 . \end{cases}$$

Step 2.2: $S \longleftarrow S \oplus det\left(\overline{M_1}\right) \otimes det(M)$.

Step 2.3: Generate a new combination $1 \le p_1 < \ldots < p_k \le n$.

end D_1

In the following procedure D_2 is described the algorithm for computation of $DET_{(R,l)}(A)$, where $A \in \mathbb{C}^{m\times n}$ is a matrix, such that $l = r_c(A) < \min\{m,n\}$. The main part of this algorithm is a cycle generating all combinations $1 \le q_1 < \ldots < q_l \le n$ and calling the procedure $D_1(m,l,x,y,3)$.

procedure $D_2(m,n,l,x,y)$

◇ m,n : The number of rows and the number of columns respectively.

◇ $l = r_c(A) < \min\{m,n\}$.

◇ x, y: The internal forms of A and R, respectively.

begin

Step 1: $q(1:l) \longleftarrow (1:l)$;

Step 2: An while cycle, which terminates when all of the combinations $1 \le q_1 < \ldots < q_l \le n$ are formed. In the cycle perform:

Step 2.1: $D_1(m,l,x,y,3)$;

Step 2.2: Generate a new combination $1 \le q_1 < \ldots < q_l \le n$.

end D_2

Finally, the algorithms D_1 and D_2 are used in the following procedure D, which computes $DET_{(R,t)}(A)$, for $t = r_c(A)$.

procedure $D(l,m,n,x,y)$

◇ $l = r_c(A)$: Dimensions of square submatrices of A and R.

◇ m,n : Dimensions of the given matrix A.

begin

$S \longleftarrow 0$

if $l = n < m$ **then** $D_1(m,n,x,y,1)$

else if $l = m < n$ **then** $D_1(n,m,x,y,2)$

else $D_2(m,n,l,x,y)$

end D

In the following procedure I, we describe the algorithm for exact computation of generalized inverses.

procedure $I(m,n,x,y,G)$ { Computing the generalized inverse G of A.}

◇ m,n : The number of rows and number of columns of A, respectively.

◇ x, y: The internal forms of A and R, respectively.

◇ $G = (g_{ij})$: The internal form of computed generalized inverse of A.

begin

Step 1: $t \longleftarrow rank(A) + 1$

repeat

$t \longleftarrow t - 1$; $D(t,m,n,x,y)$

until $S \neq 0$

Step 2: $p(1:t) \longleftarrow (1:t)$; $q(1:t) \longleftarrow (1:t)$;

Step 3:
for $w = 1 : n$ **do**
for $v = 1 : m$ **do**
Step 3.1: $suma \longleftarrow 0$
Step 3.2:
A **while** cycle over the combinations $1 \le p_1 < \ldots < p_t \le m$
A **while** cycle over the combinations $1 \le q_1 < \ldots < q_t \le n$
In the while cycles perform step a, step b and step c.
Step a: **if** $(q[k] = w)$ **and** $(p[l] = v)$ **then**
$\{1 \le k \le t\,,\ 1 \le l \le t\}$
Step a.1: Form $\overline{R}\begin{bmatrix} p_1 & \ldots & p_t \\ q_1 & \ldots & q_t \end{bmatrix}$, $A_{vw}\begin{bmatrix} p_1 & \ldots & p_t \\ q_1 & \ldots & q_t \end{bmatrix}$, using y and x.
Step a.2: $suma \longleftarrow suma \oplus \overline{R}\begin{pmatrix} p_1 & \ldots & p_t \\ q_1 & \ldots & q_t \end{pmatrix} \otimes A_{vw}\begin{pmatrix} p_1 & \ldots & p_t \\ q_1 & \ldots & q_t \end{pmatrix}$
Step b: Form a new combination $1 \le q_1 < \ldots q_t \le n$
Step c: Form a new combination $1 \le p_1 < \ldots p_t \le m$
Step 3.3: $g_{wv} \longleftarrow suma \oslash S$

end I

4. Numerical examples

If a matrix R runs over the set of m by n matrices, in (2.1) we get various definitions of generalized inverses.

1. If $r = r_c(A)$ and a matrix R satisfies condition (1), then $A^{-1}_{(R,r)}$ is equal to the *Stojaković's inverse*, i.e. the equivalent *Joshi's inverse*, in the case $\epsilon = 1$ and the *Radić's inverse*, in the case $\epsilon = -1$.

For example, consider the matrix $A = \begin{pmatrix} \frac{11}{2} & \frac{23}{15} & 1 \\ \frac{3}{20} & -\frac{2}{7} & \frac{234}{233} \end{pmatrix}$. Using $R = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 1 & 0 \end{pmatrix}$ we get the following *Stojaković's inverse* of A:

$$A^{-1}_{(R,2)} = \begin{pmatrix} \frac{58600}{440191} & \frac{619780}{1320573} \\ \frac{139335}{880382} & \frac{366975}{440191} \\ \frac{22135}{880382} & \frac{1720705}{1320573} \end{pmatrix}.$$

Using fixed point representation for the elements in A, i.e.

$$A = \begin{pmatrix} 5.50000000000000000 & 1.53333333333333344 & 1.00000000000000000 \\ 0.14999999999999999 & -0.28571428571428569 & 1.00429184549356232 \end{pmatrix},$$

and the same matrix R we get the following *Stojaković's inverse* of A:

$$A^{-1}_{(R,2)} = \begin{pmatrix} 0.1331240302505049270 & -0.4693265726317288330 \\ 0.1582665252129189510 & 0.8336722013853078430 \\ 0.0251424949624140422 & 1.3029987740170365700 \end{pmatrix}.$$

2. Furthermore, if $R = A$ satisfies (1), then $A^{-1}_{(R,r)} = A^{\dagger}$, and both generalized inverses are identical to the *Stojaković's* or the *Radić's generalized inverse*.

Concertly, for $R = A = \begin{pmatrix} \frac{5729}{327} & \frac{5729}{327} & 0 \\ 0 & \frac{5729}{327} & \frac{5729}{327} \\ -\frac{5729}{327} & 0 & -\frac{5729}{327} \end{pmatrix}$ we get the following *Moore-Penrose inverse* of A, which is identical to the *Stojaković's inverse* of A:

$$A^{-1}_{(R,2)} = A^{\dagger} = \begin{pmatrix} \frac{2008044837}{256295929} & 0 & -\frac{2008044837}{256295929} \\ \frac{2008044837}{256295929} & \frac{2008044837}{256295929} & 0 \\ 0 & \frac{2008044837}{256295929} & \frac{2008044837}{256295929} \end{pmatrix},$$

3. If $A \in \mathbb{C}_r^{m\times n}$ and $R = A$ we get $A^{-1}_{(R,r)} = A^{\dagger}$.

For example, if we use $R = A = \begin{pmatrix} \frac{175}{23} & 0 & \frac{175}{23} \\ 0 & \frac{1}{13} & \frac{175}{23} \\ \frac{175}{46} & \frac{1}{13} & \frac{525}{46} \\ 0 & \frac{1}{13} & \frac{175}{23} \end{pmatrix}$, then is obtained

$$A^{-1}_{(R,2)} = A^{\dagger} = \begin{pmatrix} \frac{192878339}{497627891} & -\frac{201395239}{995255782} & -\frac{4258450}{497627891} & -\frac{201395239}{995255782} \\ \frac{1684865000}{497627891} & -\frac{1263648750}{497627891} & -\frac{421216250}{497627891} & -\frac{2263648750}{497627891} \\ -\frac{655721205}{497627891} & -\frac{1075979571}{995255782} & \frac{1281633260}{995255782} & -\frac{1075979571}{995255782} \end{pmatrix}.$$

4. For a square matrix A, such taht $ind(A) = 1$ and $R = A^*$ we get $A^{-1}_{(R,m)} = A^{\#}$. For example, let

$$A = \begin{pmatrix} 21.93 - 3i & 4. & \frac{275}{35917} & 9.13570 + 2950.84725i \\ 11.35 & 35.75 - 2i & 0 & 1257420 \\ 257384 & \frac{91584}{23} & 12 + 15i & \frac{213574}{5762403} \\ 159384 - 135i & 109825.23 & \frac{183294}{7359} & 0.000579 \end{pmatrix}, \quad i = \sqrt{-1}.$$

Using $R = A^*$, we get the following group inverse, approximately:

$$A^{\#} = \begin{pmatrix} -0.006 - 0.011i & -0.00003 + 0.00001i & 0.000004 + 0.000001i & 0 \\ -0.029 + 0.011i & 0.00002 + 0.00007i & -0.000004 + 0.000001i & 0.00001 \\ 167.245 - 23.231i & 0.05330 - 0.39265i & -0.0109 + 0.0004i & -0.0057 - 0.00073i \\ 0.000001 & 0.000001 & 0 & 0 \end{pmatrix}.$$

5. For $A \in \mathbb{C}^{m\times n}$ using $R = \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I}_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \in \mathbb{C}^{m\times n}$, we obtain

$$DET_{(R,r)}(A) = A\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix},$$

and the following *algebraic complement* of the element a_{ij}:

$$A_{ij}^{(R,r)} = \begin{cases} 0, & \text{for } j > r \text{ or } i > r \\ A_{ji}\begin{pmatrix} 1 & \dots & i & \dots & r \\ 1 & \dots & j & \dots & r \end{pmatrix}, & \text{for } j, i \le r . \end{cases}$$

Generalized inverse of A is equal to

$$A_{(R,r)}^{-1} = \frac{1}{A\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix}} \cdot \begin{pmatrix} A_{11}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & \dots & A_{r1}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{1r}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & \dots & A_{rr}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Concretly, for $A = \begin{pmatrix} \frac{13}{56} & 115 & \frac{476}{13} \\ \frac{1}{3} & -372 & \frac{23}{26} \\ -3 & \frac{14}{3} & \frac{21}{17} \\ \frac{12}{13} & 1 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ the following *right generalized inverse* of A is obtained :

$$A_{(R,2)}^{-1} = \begin{pmatrix} \frac{10652600}{6188751} & -\frac{8558144}{80453763} & -\frac{1364947612}{26817921} & 0 \\ \frac{35980}{2062917} & -\frac{3615752}{8939307} & -\frac{7448669}{160907526} & 0 \\ \frac{76388480}{18566253} & \frac{7857808}{6188751} & -\frac{1097248}{6188751} & 0 \end{pmatrix}.$$

5. Conclusion

The memory requirements of the above presented procedures for $A \in \mathbb{C}_r^{m\times n}$ are two $r_c(A) \times r_c(A)$ matrices. The advantage of the presented algorithms is in their generality, induced by theoretical weight of Theorem 2.1. The efficiency of these algorithms is identical to the efficiency of the algorithms presented in [12].

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