

MODELING OF RATIONAL CURVES BY INTERPOLATION

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ABSTRACT. *The algorithm for modeling shapes with (n,n) -rational curves is proposed. It is based on interpolation by rational functions using continued fraction numerical technique. The converse algorithm for transformation of rational curve into a parametric continued fraction form is also given. The direct algorithm is illustrated through several examples.*

1. Introduction

The Bézier curve of degree n is defined by the control points B_0, \dots, B_n through

$$P_n(t) = \sum_{i=0}^n B_i b_i^n(t), \quad t \in [0, 1],$$

where $b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are Bernstein basis polynomials. An example of a third order Bézier curve is shown in Figure 1(a).

A natural generalization of this model is the rational Bézier curve (of degree n) that, besides the control points B_0, \dots, B_n involves the weights $\omega_0, \dots, \omega_n$ as shape parameters

$$(1) \quad R_n(t) = \frac{\sum_{i=0}^n B_i \omega_i b_i^n(t)}{\sum_{i=0}^n \omega_i b_i^n(t)}, \quad t \in [0, 1].$$

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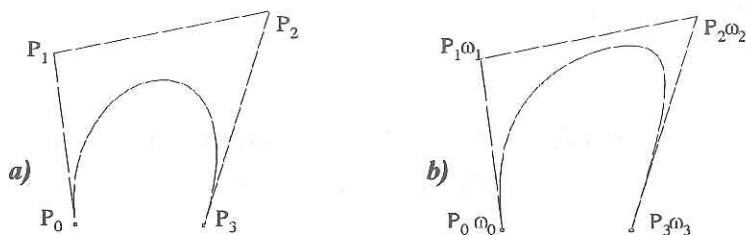


Figure 1. Bézier curve (a) and rational Bézier curve (b)

If some weight is relatively large comparing to others, the corresponding control point "pulls" the curve toward it. Figure 1(b) shows the rational curve with $\omega_0 = \omega_3 = 1$, $\omega_1 = 3$ and $\omega_2 = 6$.

The rational scheme reveals many useful properties. The most important of them are:

- the possibility of exact modeling of conic sections;
- continuous changing of weights results in continuous adjustment of curve form.

On the other hand, all good properties of the polynomial Bézier curves maintains, except subdivision which can not be carried over without weights being changed.

It is customary in free form curve modeling to use some interpolation model as an initiator. The Bézier curve modeling is preceded by the Lagrange or spline interpolation model. For the rational Bézier curve, it is recommendable to start with rational interpolant. The most natural approach is to represent such (n,n) -rational interpolation curve via the Bernstein basis, for each coordinate axis separately. For example for x-axis:

$$(2) \quad R_n(x) = \frac{\sum_{i=0}^n B_i \omega_i b_i^n(x)}{\sum_{i=0}^n \omega_i b_i^n(x)}, \quad x \in [0, 1],$$

where the ordinates B_i and weights ω_i are to be determined so that $R_n(x)$ interpolates the data $\{(x_i, y_i)\}_{i=0}^\nu$, i.e.

$$(3) \quad R_n(x_i) = y_i, \quad i = 0, \dots, \nu,$$

with ν chosen so that there are enough equations to determine B_i, ω_i in (2), with one arbitrary weight.

A variant of this problem is considered by Piegl [8], but for piecewise cubic rational curve.

By introducing (3) in (2), the following linear system is obtained

$$\begin{bmatrix} b_0^n(x_0) & \cdots & b_n^n(x_0) & -y_0 b_1^n(x_0) & \cdots & -y_0 b_n^n(x_0) \\ b_0^n(x_1) & \cdots & b_n^n(x_1) & -y_1 b_1^n(x_1) & \cdots & -y_1 b_n^n(x_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_0^n(x_\nu) & \cdots & b_n^n(x_\nu) & -y_\nu b_1^n(x_\nu) & \cdots & -y_\nu b_n^n(x_\nu) \end{bmatrix} \begin{bmatrix} B_0 \omega_0 \\ B_1 \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} y_0 b_0^n(x_0) \\ y_1 b_0^n(x_1) \\ \vdots \\ y_\nu b_0^n(x_\nu) \end{bmatrix}.$$

Unfortunately, this system has no such nice behaviour as in the case of Lagrange interpolation (Vandermonde determinant $\neq 0$). Here, the singularity can occur. Next, the interpolant may not exist in spite of regularity of the system (see Mayers [6]).

In this paper only the case when interpolant exists is considered.

For interpolant construction, the inverted differences are used while the interpolant has continued fraction form

$$(4) \quad R_n(x) = c_0 + \frac{x - x_0}{c_1} + \frac{x - x_1}{c_2} + \cdots + \frac{x - x_{\nu-1}}{c_\nu},$$

or more conveniently

$$(5) \quad R_n(x) = [c_0; \frac{x - x_0}{c_1}, \frac{x - x_1}{c_2}, \dots, \frac{x - x_{\nu-1}}{c_\nu}],$$

where

$$(6) \quad c_i = \phi(x_0, \dots, x_i), \quad i = 0, \dots, \nu,$$

are inverted differences given by

$$(7) \quad \begin{aligned} \phi(x_0) &= y_0, \quad \phi(x_0, x_1) = \frac{x_1 - x_0}{y_1 - y_0}, \\ \phi(x_0, \dots, x_i) &= \frac{x_i - x_{i-1}}{\phi(x_0, \dots, x_{i-2}, x_i) - \phi(x_0, \dots, x_{i-2}, x_{i-1})}, \quad i = 2, 3, \dots \end{aligned}$$

Now, the continued fraction $R_n(x)$ can be expressed in the form $\frac{P_n(x)}{Q_n(x)}$. By the using of the known transformation from monomial to Bernstein basis, the

rational form (2) can be obtained. But, this transformation is numerically unstable (Farouki, Rajan [3]). So it is better to find the algorithm for direct expression of the continued fraction in Bernstein form.

2. Algorithms

Here, two algorithms are proposed. One, for transformation the continued fraction in rational Bézier form and, the second, for inverse transformation back to the continued fraction form.

For continued fraction $R_n(x)$, given by (4), the rational function (8)

$$r_k(x) = \frac{P_k(x)}{Q_k(x)} = \left[c_0; \frac{x-x_0}{c_1}, \frac{x-x_1}{c_2}, \dots, \frac{x-x_{k-1}}{c_k} \right], \quad k = 0, \dots, \nu - 1,$$

refers to as k -th convergent of $R_n(x)$. Obviously, $r_n(x) = R_n(x)$.

It is known that polynomials P_k and Q_k satisfy three term recurrence relation

$$z_k = c_k z_{k-1} + (x - x_{k-1}) z_{k-2}, \quad k = 1, \dots, n,$$

where the sequence $\{P_k\}$ is initialized by $P_{-1}(x) = 1$, $P_0(x) = c_0$ and $\{Q_k\}$ by $Q_{-1}(x) = 0$, $Q_0(x) = 1$, see [2], [4], [5], [7].

The Algorithm 1 is given by the following theorem:

Theorem 1. *Let the set of points in the plane $\{(x_i, y_i)\}_{i=0}^n$ be given such that $0 = x_0 < x_1 < \dots < x_n = 1$. Let the k -th convergent of $R_n(x)$ be given in Bézier form*

$$(9) \quad r_k(x) = \frac{\sum_{i=0}^k p_i^k b_i^k(x)}{\sum_{i=0}^k q_i^k b_i^k(x)},$$

by the coefficients p_i^k and q_i^k ($i = 0, \dots, k$) satisfy the recurrence relation

$$(10) \quad \begin{aligned} s_i^k &= A_0 s_i^{k-1} + A_1 s_{i-1}^{k-1} + A_2 s_i^{k-2} + A_3 s_{i-1}^{k-2} + A_4 s_{i-2}^{k-2}, \\ i &= 0, \dots, k, \quad k = 2, 3, \dots, n, \end{aligned}$$

with initial conditions for $\{P_k\}$, given by $p_0^0 = c_0$, $p_0^1 = c_0 c_1 - x_0$, $p_1^1 = c_0 c_1 - x_0 + 1$, and $q_0^0 = 1$, $q_0^1 = q_1^1 = c_1$, for $\{Q_k\}$. The constants A_i in (10) are given by

$$A_0 = \frac{(k-i)c_k}{k}, \quad A_1 = \frac{ic_k}{k}, \quad A_2 = -\frac{(k-i)(k-i-1)x_{k-1}}{k(k-1)},$$

$$A_3 = -\frac{i(k-1)(1-2x_{k-1})}{k(k-1)}, \quad A_4 = -\frac{i(i-1)(1-x_{k-1})}{k(k-1)}.$$

In practical calculations the coefficients A_i are replaced with

$$A_0 = (k-1)(k-i)c_k, \quad A_1 = (k-1)ic_k, \quad A_2 = -(k-i)(k-i-1)x_{k-1},$$

$$A_3 = -i(k-1)(1-2x_{k-1}), \quad A_4 = -i(i-1)(1-x_{k-1}).$$

In this way, one can avoid operation of division, which results in improving the numerical stability of the algorithm.

Finally, the control points B_i and weights ω_i can be determined from the system

$$(11) \quad B_i \omega_i = p_i^n, \quad \omega_i = q_i^n, \quad i = 0, \dots, n.$$

In the case $\omega_i = 0$, the control point B_i is an arbitrary constant.

The truncation error estimates as (see [4])

$$(12) \quad R_n(x) - r_k(x) = K[r_k(x) - r_{k-1}(x)], \quad x \in [0, 1],$$

with $K = -d_k(x)/(1 + d_k(x))$, where

$$(13) \quad d_k(x) = \frac{(x - x_k)Q_{k-1}(x)}{\phi_{k+1}(x_0, \dots, x_k, x)Q_k(x)},$$

and ϕ_{k+1} are inverted differences given by (7) with y_i being replaced by $R_n(x_i)$. Note that the constant K in (12) can be easily approximated using extended de Casteljau algorithm [1], which allows to compute both $Q_k(x)$ and ϕ_{k+1} .

Converse algorithm

Conversely, the Bézier curve (2) can be transformed in the form

$$(14) \quad R_n(x) = \left[a_0; \frac{1}{\beta_1(x)}, \frac{1}{\beta_2(x)}, \dots, \frac{1}{\beta_n(x)} \right],$$

where

$$(15) \quad \beta_k(x) = C_{n-k+1} b_0^1(x) + D_{n-k+1} b_1^1(x), \quad k = 1, \dots, n,$$

are the Bernstein polynomials of first order. This procedure of the Algorithm 2 is given by the following theorem:

Theorem 2. In (14), the constants a_0 , C_k and D_k , $k = 1, \dots, n$ are given by

$$(16) \quad \begin{cases} p_0^n = a_0 q_0^n + p_0^{n-1}, \\ p_i^n = a_0 q_i^n + \frac{i}{k} p_{i-1}^{n-1} + \left(1 - \frac{i}{k}\right) p_i^{n-1}, \quad i = 1, 2, \dots, n-1, \\ p_n^n = a_0 q_n^n + p_{n-1}^{n-1}, \end{cases}$$

$$(17) \quad \begin{cases} q_i^k = \frac{k-1}{k} C_{n-k+1} p_i^{k-1} + \frac{i}{k} D_{n-k+1} p_{i-1}^{k-1} + \frac{(k-1)(k-i-1)}{k(k-1)} r_i^{k-2} \\ \quad + \frac{2i(k-i)}{k(k-1)} r_{i-2}^{k-2} + \frac{i(i-1)}{k(k-1)} r_{i-2}^{k-2}, \quad i = 0, \dots, k \quad k = n, n-1, \dots, 1. \end{cases}$$

Proof. After division in $R_n(x)$ one obtains

$$(18) \quad R_n(t) = \frac{\sum_{i=0}^n p_i^n b_i^n(x)}{\sum_{i=0}^n q_i^n b_i^n(x)} = a_0 + \frac{\sum_{i=0}^{n-1} p_i^{n-1} b_i^{n-1}(x)}{\sum_{i=0}^n q_i^n b_i^n(x)},$$

where, according to [3 eq. 48], constants a_0 , p_i^n , q_i^n and p_i^{n-1} are connected as in (16). Note that a_0 can be expressed explicitly as

$$a_0 = \frac{\sum_{i=0}^n (-1)^{n-1} \binom{n}{i} p_i^n}{\sum_{i=0}^n (-1)^{n-1} \binom{n}{i} q_i^n}.$$

After k -th division in (18), by the rule of continued fraction, one gets

$$\begin{aligned} R_n(x) &= a_0 + \frac{1}{\beta_1(x) + \frac{\sum_{i=0}^k p_i^{k-1} b_i^{k-1}(x)}{\sum_{i=0}^k q_i^k b_i^k(x)}} \\ &= a_0 + \frac{1}{\beta_1(x) + \dots + \beta_n(x) + \frac{1}{\frac{\sum r_i^{k-2} b_i^{k-2}(x)}{\sum p_i^{k-1} b_i^{k-1}(x)}}}, \end{aligned}$$

i.e.

$$(19) \quad \sum_{i=0}^k q_i^k b_i^k(x) = \beta_k(x) \sum_{i=0}^{k-1} p_i^{k-1} b_i^{k-1}(x) + \sum_{i=0}^{k-2} r_i^{k-2} b_i^{k-2}(x).$$

Relations (17) follow from (19) by comparing coefficients after replacing $\beta_k(x)$ by (15), then by using identities $(1-x)b_i^{n-1}(x) = \frac{n-i}{n}b_i^n(x)$, $xb_i^{n-1}(x) = \frac{i+1}{n}b_{i+1}^n(x)$, and, after that, elevating degree of $\sum_{i=0}^{k-2} r_i^{k-2} b_i^{k-2}(x)$ for two. \square

3. Applications and examples

The main application of Algorithm 1 is in modeling. Using interpolating points, the initial interpolation model is found by calculating the control points B_i and weights ω_i and the one can continue the modeling process by the standard interactive technique.

Second application is in recognition the parameters for same free form curve. For example, if one knows that some curve is rational but does not know its control points or weights they can be retrieved by the Algorithm 1.

The Algorithm 2 carries over the Bézier rational curve into continued fraction form. It may be important for further processing of such curve, like for approximation or data reduction.

The following examples illustrate our Algorithm 1

Example 1. The data $(x_i, y_i): (2.5, 6.875)(5.0, 2.23)(20.0, 0.283)(40.0, 0.143)$ results the curve in Figure 2(a), while the more complete data $(x_i, y_i): (2.5, 6.875)(5.0, 2.23)(10.0, 0.751)(15.0, 0.416)(20.0, 0.283)(25.0, 0.219)(30.0, 0.182)(40.0, 0.143)$ gives curve in Figure 2(b).

Example 2. Here, the data (7.99, 0.0) (8.09, 0.000027643) (8.19, 0.0437488) (8.7, 0.169183) (9.2, 0.46428) (10.0, 0.943740) (12, 0.998636) (15.0, 0.999919) (20.0, 0.9999 94) are used. The corresponding rational curve is shown in Figure 2(c).

Example 3. Figure 2(d) shows the result of applying Algorithm 1 on the data (-4.0, -1.0) (-3.0, -1.0) (-2.0, -1.0) (-1.0, -1.0) (0.0, 0.0) (1.0, 1.0) (2.0, 1.0) (3.0, 1.0) (4.0, 1.0).



Figure 2. Rational Bézier interpolants for various data

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