FILOMAT (Niš) 9:2 (1995), 187–195

Filomat '94, Niš, October 22–24, 1994
Geometry. Computer Sciences

HOLOMORPHICALLY-PROJECTIVE CONNECTIONS OF A HYPERBOLIC KAEHLERIAN SPACE

Nevena Pušić

ABSTRACT. *We consider the set of connections on a hyperbolic Kaehlerian space which are in holomorphically-projective correspodence to the Levi-Civita connection. We find an invariant tensor of curvature type for all these connections.*

1. About hyperbolic Kaehlerian space

A hyperbolic Kaehlerian space M_n $(n = 2m)$ is a differentiable manifold with indefinite metrics

(1.1) $$ds^2 = g_{ij}dx^i dx^j$$

and so-called strusture($F_j{}^i$)(which is itself a linear transformation of the tangent space, in every point), which satisfies

(1.2) $$F_j{}^s F_s{}^i = \delta^i_j$$

The metrics and the structure are conected in the following way

(1.3) $$F_{ij} = g_{is}F^s{}_j = g_{js}F_i{}^s = -F_{ji}$$

(1.4) $$\check{\nabla}_k F_j{}^i = 0.$$

The tensor (F_{ij}), appearing in (1. 3), which we have formally got from the structure tensor by lowering the upper index, is the covariant structure tensor. The symbol $\check{\nabla}$ denotes the ooperator of covariant differentiation towards the Levi-Civita connection. (1. 4) means that the structure tensor is parallel regarding to the Levi-Civita connection. It is clear that the covariant structure tensor is also parallel. (1. 2) means that the structure is involutive as a linear transformation of the tangent space in every point.

The structure tensor is a real nondegenerate tensor and it has n linearly independent eigenvectors; its matrix has a diagonal expression.

There holds

Lemma 1. *(A) Every tangent vector of a hyperbolic Kaehlerian space is transformed by the structure into an orthogonal vector.*
(B) The scalar square of a vector-original is opposite to the scalar square of the vector-image.

Proof. (A) $a_j F_i{}^j = b_i$

$$a_j b^j = a_j a_s F_t{}^s g^{tj} = a_j a_s F^{js} = -a_j a_s F^{sj} = -a_j b^j = 0$$

$$(B) b_s b^s = b_s b_t g^{ts} = b_s a_j F_t{}^j g^{ts} = b_s a_j F^{sj} =$$
$$= -a_j b_s F^{js} = -a_j a^j . \square$$

The fact that the structure has eigenvectors is enabled by the fact that the metrics is indefinite. We shall give here some features of eigenvalues and eigevectors of the structure.

Lemma 2. *For two different eigenvectors of the structure on a hyperbolic Kaehlerian space either the eigenvalues are mutually iopposite or the eigenvectors are mutually orthogonal.*

Proof. Suppose that u and v are two different eigenvectors for the structure, with eigenvalues λ and κ respectively. Then

$$u_a v^a = u_j v_k g^{jk} = \frac{1}{\kappa} u_j F_k{}^s v_s g^{jk}$$

$$= \frac{1}{\kappa} u_j v_s F^{js} = -\frac{1}{\kappa} v_s u_j F^{sj} = -\frac{\lambda}{\kappa} v_s u^s ,$$

and

$$u_a v^a (1 + \frac{\lambda}{\kappa}) = 0 \tag{1.5}$$

and the Lemma is proved. □

Lemma 3. *If on a hyperbolic Kaehlerian space the vector u is an eigenvector for the structure, then Fu is also an eigenvector for the structure.*

Proof. $v = Fu$, $v_i = F_i{}^a u_a = \lambda u_i$

$$F_j{}^i v_i = F_j{}^i F_i{}^a u_a = u_j = \lambda^2 u_j = \lambda v_j$$

and the Lemma is proved. □

It is obvious from the proof of the Lemma 3. that only eigenvalues of the structure are $\lambda = \pm 1$.

According to the Lemma 1, eigenvectors of the structure tensor are self-orthogonal, i. e. their scalar square vanishes. As the structure has n (dimension of the manifold) linearly independent eigenvectors, there exists a basis of the tangent space which consists of isotropic vectors. We call such a basis an **adapted basis**. Such a basis shows in the simplest way the geometry of a hyperbolic Kaehlerian space. We can construct an adapted basis in the following way: we put on the first $m = \frac{n}{2}$ places those eigenvectors with corresponding eigenvalue 1; on the second m places we put those m eigenvectors with corresponding eigenvalue -1. According to the Lemma 3, there is no other eigenvalues. According to the Lemma 2, in every of these subspaces every basic vector is orthogonal to the every other basic vector; on every of these subspaces induced metrics vanishes identically. Besides, every of these subspaces is invariant under structure isomorphism. This means that a hyperbolic Kaehlerian space is decomposed in very natural way into two totaly geodesic subspaces of same dimension.

We have to mention that, according to the Lemma 1, there exist vectors with positive scalar square (space-like vectors) and those with negative scalar sqare (time-like vectors).

2. Holomorphically planer curves

A two-dimensional submanifold of the manifold M_n with a tangent subspace of the tangent space on M_n, generated by vectors u, Fu we call a holomorphic section of a hyperbolic Kaehlerian space.

A curve $\xi^h(t)$ on M_n satisfying the differential equation

$$\frac{d^2\xi^h}{dt^2} + \Lambda^h_{ji}\frac{d\xi^j}{dt}\frac{d\xi^i}{dt} == \alpha(t)\frac{d\xi^h}{dt} + \beta(t)F_s{}^h\frac{d\xi^s}{dt} \tag{2.1}$$

where $\alpha(t)$ and $\beta(t)$ are functions depending of the parameter t, we call a **holomorphically planer curve**. It can be seen from (2. 1) that a curve is holomorphically planer if and only if holomorphic sections generated by tangent vectors are parallel along the curve.

Two F-connections (satisfying $\nabla_k F = 0$) are said to be mutually holomorphically projective if and only if they have holomorhically planer curves in common.

It is easy to prove that there holds

Proposition 1. *Two symmetric F-connections with coefficients Λ^i_{jk} and $\bar{\Lambda}^i_{jk}$ are holomorphically projective if and only if*

$$\begin{aligned}\bar{\Lambda}^i_{jk} &= \Lambda^i_{jk} + p_j\delta^i_k + p_k\delta^i_j \\ &+p_sF_j{}^sF_k{}^i + p_jF_k{}^sF_j{}^i\end{aligned} \tag{2.2}$$

for some vector field (p_j).

In this article, we shall investigate connections which are holomorphically projective to the Levi-Civita connection.

3. Holomorphically-projective connections

We say that a connection with coefficients Λ^i_{jk} on a hyperbolic Kaehlerian space is holomorphically-projective if its coefficients have the form:

$$\Lambda^i_{jk} = \check{\Gamma}^i_{jk} + p_j\delta^i_k + p_k\delta^i_j + q_jF_k{}^i + q_kF_j{}^i, \tag{3.1}$$

where (p_j) are components of a gradient vector field and (q_j) are components of a vector which is an image of (p_j) under the structure. $\check{\Gamma}$ stands for Christoffel symbols. It is obvious that a holomorphically-projective connection has holomorphically planer curves in common with Levi-Civita connection.

The curvature tensor of the connection (3. 1) has the form

$$R_{ijkl} = K_{ijkl} + g_{ki}p_{lj} - g_{li}p_{kj} + F_{ki}q_{lj} - F_{li}q_{kj} + F_{ji}(q_{kl} - q_{lk}) \tag{3.2}$$

where

$$p_{lj} = \check{\nabla}_l p_j - p_lp_j - q_lq_j \tag{3.3}$$

$$q_{lj} = \check{\nabla}_l q_j - p_lq_j - q_lp_j \tag{3.4}$$

and

$$q_{lj} = F_j{}^a p_{la} \tag{3.5}$$

By K_{ijkl} we denote Riemann-Christoffel tensor of the hyperbolic Kaehlerian space.

In order to eliminate p_{lj} and q_{lj} from the expression (3. 2), we shall suppose that the curvature tensor of the holomorphically-projective connection is invariant under change of places of the first and second pair of indices:

$$R_{ijkl} = R_{klij} \tag{3.6}$$

By (3. 2), we obtain from (3. 6)

$$g_{jk}p_{il} - g_{li}p_{kj} + F_{ki}(q_{lj} + q_{jl}) - F_{li}q_{kj} + F_{jk}q_{il} + F_{ji}(q_{kl} - q_{lk}) - F_{lk}(q_{ij} - q_{ji}) = 0$$

After transvection the upper equality by F^{jk}, we obtain

$$q_{li} + (1-n)q_{il} = F_{il}p^s_s \tag{3.7}$$

where p^s_s stands for $p_{ij}g^{ij}$.

As the covariant structure tensor is skew-symmetric, then the left-hand side of (3. 7) is also skew-symmetric and there holds

$$q_{li} + (1-n)q_{il} = -q_{il} - (1-n)q_{li}$$

what means

$$q_{il} = -q_{li}. \tag{3.8}$$

Now, the curvature tensor of holomorphically-projective connection on a hyperbolic Kaehlerian space has the form

$$R_{ijkl} = K_{ijkl} + g_{ki}p_{lj} - g_{li}p_{kj} + F_{ki}q_{lj} - F_{li}q_{kj} + 2F_{ji}q_{kl}. \tag{3.9}$$

4. HP-curvature tensor

Taking into account equalities (3. 7) and (3. 8), one can easily get

$$q_{li} = -\frac{p_s{}^s}{n} F_{li} \tag{4.1}$$

and by the relation $p_{li} = F_i{}^a q_{la}$,

$$p_{li} = \frac{p_s{}^s}{n} g_{li} \tag{4.2}$$

Using (3. 9), we can find the Ricci tensor of the holomorphically-projective connection

$$p_{li} = \frac{R_{li} - K_{li}}{2-n} \tag{4.3}$$

and

$$p_s{}^s = \frac{R-K}{2-n} \tag{4.4}$$

Then, we have

$$q_{li} = -\frac{R-K}{n(2-n)} F_{li}, \tag{4.5}$$

$$p_{li} = \frac{R-K}{n(2-n)} g_{li}. \tag{4.6}$$

If we substitute (4. 5) and (4. 6) into (3. 9), we obtain

$$R_{ijkl} - \frac{R}{n(2-n)}(g_{ki}g_{lj} - g_{li}g_{kj} - F_{ki}F_{lj} + F_{li}F_{kj} - 2F_{ji}F_{kl}) =$$

$$= K_{ijkl} - \frac{K}{n(2-n)}(g_{ki}g_{lj} - g_{li}g_{kj} - F_{ki}F_{lj} + F_{li}F_{kj} - 2F_{ji}F_{kl}).$$

The tensor on the right-hand side of the upper equality we call the **holomorphically-projective curvature tensor** of a hyperbolic Kaehlerian space. We have proved

Theorem 1. *The tensor*

$$(4.7)\qquad HP_{ijkl} =$$

$$K_{ijkl} - \frac{K}{n(2-n)}(g_{ki}g_{lj} - g_{li}g_{kj} - F_{ki}F_{lj} + F_{li}F_{kj} - 2F_{ji}F_{kl})$$

is an invariant tensor of holomorphically-projective connections on the hyperbolic Kaehlerian space.

We can also prove that there holds

Theorem 2. *The curvature tensor of a holomorphically-projective connection on a hyperbolic Kaehlerian space is skew-symetric in first two indices, but it does not satisfy the first Bianchi identity, except of some special cases.*

Proof. One can easily check, using (3. 9), that R_{ijkl} is skew-symmetric in first two indices.

If we suppose that R_{ijkl} satisfies the first Bianchi identity, then, by (3. 9), we obtain

$$\begin{aligned} 0 = {} & K_{ijkl} + K_{iklj} + K_{iljk} + \\ & + g_{ki}p_{lj} - g_{li}p_{kj} + F_{ki}q_{lj} - F_{li}q_{kj} + 2F_{ji}q_{kl} \\ & + g_{li}p_{jk} - g_{ji}p_{lk} + F_{li}q_{jk} - F_{ji}q_{lk} + 2F_{ki}q_{lj} \\ & + g_{ji}p_{kl} - g_{ki}p_{jl} + F_{ji}q_{kl} - F_{ki}q_{jl} + 2F_{li}q_{jk} = \\ = {} & 4(F_{ki}q_{lj} - F_{li}q_{kj} + F_{ji}q_{kl}), \end{aligned}$$

and, taking into account (4. 1)

$$\frac{p_s{}^s}{n}(F_{ki}F_{lj} - F_{li}F_{kj} + F_{ji}F_{kl} = 0).$$

If we suppose that the expression in parentheses vanishes, then, after contraction by F^{ik},

$$(n-2)F_{lj} = 0,$$

what is senseless. Then $p_s{}^s = 0$ and, regarding to (4. 4), $K = R$, what is a special case. □

Also, we can prove

Theorem 3. *The holomorphically-projective curvature tensor of a hyperbolic Kaehlerian space satisfies the following relations*

$$(a) HP_{ijkl} = -HP_{ijlk};\ HP_{ijkl} = -HP_{jikl};\ HP_{ijkl} = HP_{klij}$$

$$(b) HP_{ijkl} + HP_{iklj} + HP_{iljk} = -4(F_{ki}F_{lj} - F_{li}F_{kj} + F_{ji}F_{kl})$$

$$(c) HP^t{}_{jkt} = K_{jk} - \frac{K}{n}g_{jk}$$

$$(d) HP^i{}_{tkl}F_j{}^t - HP^t{}_{jkl}F_t{}^i = 0.$$

One can easily prove all these properties using the expression (3. 9).

5. Some special cases

There always holds

$$p_s{}^s = \check{\nabla}_s p^s \tag{5.1}$$

according to the Lemma 1. Then also holds

$$R - K = (2-n)\check{\nabla}_s p^s;\ p_{li} = \frac{\check{\nabla}_s p^s}{n} g_{li};\ q_{li} = -\frac{\check{\nabla}_s p^s}{n} F_{li}. \tag{5.2}$$

As the first special case we shall consider that one when the vector field (p^i) generating holomorphically-projective connection is a harmonic vector field, that is

$$\check{\nabla}_s p^s = 0.$$

Then, according to (5. 2), there holds

$$R = K;\ p_{li} = 0;\ q_{li} = 0 \tag{5.3}$$

and then

$$R_{ijkl} = K_{ijkl}$$

and the curvature tensor of the holomorphically-projective connection in this special case will satisfy the first Bianchi identity.

The other special case which we are going to consider here is that one when the generating vector field for the holomorphically-projective connection is an eigenvector for the structure; then the holomorphic section is invariant for the structure. As the only eigenvalues for the structure are ± 1, then holds

$$q_{lj} = \pm p_{li} \tag{5.4}$$

As the tensor (p_{lj}) is symmetric and the tensor (q_{lj}) is skew-symmetric, there will hold

$$q_{lj} = p_{lj} = 0. \tag{5.5}$$

This means that

$$p_s{}^s = \check{\nabla}_s p^s = 0$$

i. e. that the generating vector field is a harmonic one.

If the vector field (p_i) is harmonic or isotropic, then

$$\check{\nabla}_j p_i = p_i p_j + q_i q_j;\ \check{\nabla}_j q_i = p_j q_i + q_j p_i.$$

According to the Ricci identity, there holds

$$\check{\nabla} j \check{\nabla}_k p_i - \check{\nabla}_k \check{\nabla}_j p_i = -K^t{}_{ikj} p_t = 0$$

After contraction by g^{ik}, we obtain

$$-K^{t}{}_{j}p_t = 0$$

or, consequently

$$g^{jk}\check{\nabla}_j\check{\nabla}_k p_i = 0.$$

There holds

Theorem 4. *If the vector which is generating a holomorphically-projective connection of the hyperbolic Kaehlerian space is a harmonic vector field, then the curvature tensor of the hyperbolic Kaehlerian space is equal to the curvature tensor of the holomorphically-projective connection. An example of generating harmonic vector field is a structure eigenvector field. For such a vector field there holds*

$$K^{t}{}_{j}p_t = 0 \text{ and } g^{jk}\check{\nabla}_j\check{\nabla}_k p_i = 0.$$

If the generating vector field has constant scalar square, then the difference between R and K is constant.

Proof. We shall prove just the last statement.

$$\frac{\partial}{\partial x^k}(p_s p^s) = \check{\nabla}_k p_s p^s = p_s\check{\nabla}_k p^s + p^s\check{\nabla}_k p_s.$$

But

$$\text{(5.6)} \qquad \check{\nabla}_k p_s = p_{ks} + p_k p_s + q_k q_s \text{ and } \check{\nabla}_k p^s = p_k^s + p_k p^s + q_k q^s$$

As $p_{ks} = \frac{p_s^s}{n} g_{ks}$ and $p_k^s = \frac{p_s^s}{n}\delta_k^s$, then

$$0 = \frac{\partial}{\partial x^k}(p_s p^s) = 2(\frac{p_s^s}{n} + p_s p^s)p_k$$

and, consequently,

$$p_s p^s = -\frac{p_s^s}{n}.$$

But,

$$R - K = (2-n)p_s^s = n(n-2)p_s p^s$$

and the proof is completed. □

Refrences

[1] M. Prvanović, *Holomorphically-projective transformations in a locally product space*, Mathematica Balcanica, 1(1971) 193–213

[2] N. Pušić, *On invariant tensor of a conformal transformation of a hyperbolic Kaehlerian space*, Zbornik radova Filozofskog fakulteta u Nišu, Serija Matematika, 4(1990) 55–64

[3] K. Yano, *Differential geometry of complex and almost complex spaces*, Pergamon Press, New York, 1965.

Institut za matematiku PMF, 21000 Novi Sad, dr Ilije Djuričića 4, Yugoslavia