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# ON WARPED PRODUCT MANIFOLDS

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ABSTRACT. *This is a survey article on warped product manifolds and contains: applications in some relativistic theories (Schwarzschild spacetime and Robertson-Walker spacetime), subprojective spaces, the invariant way characterizing warped products and the geometry of warped product in terms of warping function and the geometies of the base and the fiber.*

## 1. Definition and the first example

Let $(\overline{M}, \overline{g})$ and $(\overset{*}{M}, \overset{*}{g})$ be two Riemannian manifolds such that $dim\,\overline{M} = q$, $dim\,\overset{*}{M} = n - q$, $1 < q < n$. Let $F$ be a positive $C^\infty$ function on $\overline{M}$.

**Definition.** *([23],p.204). The warped product $M = \overline{M} \times_F \overset{*}{M}$ of $(\overline{M}, \overline{g})$ and $(\overset{*}{M}, (\overset{*}{g})$ is the manifold $M = \overline{M} \times \overset{*}{M}$ whith the metric $g = \overline{g} \times_F \overset{*}{g}$. More precisely*

$$g = \overline{g} \times F\overset{*}{g} = \pi_1^*\overline{g} + (F \circ \pi_1)\pi_2^*\overset{*}{g},$$

*where $\pi_1 : \overline{M} \times \overset{*}{M} \to \overline{M}$, $\pi_2 : \overline{M} \times \overset{*}{M} \to \overset{*}{M}$ are natural projections. The manifold $\overline{M}$ is caled the base manifold, while $\overset{*}{M}$ is the fiber.*

For each $(\overline{m}, \overset{*}{m}) \in M$ the subset $\overline{M} \times \overset{*}{m}$ is a totally geodesic submanifold of warped product and all such submanifolds are isometrically related; the submanifolds $\overline{m} \times \overset{*}{M}$ are totally umbilic and the map $\pi_2|_{\overline{m} \times \overset{*}{M}}$ is a positive homotety onto $\overset{*}{M}$ scale factor $\dfrac{1}{F(\overline{m})}$. For each $(\overline{m}, \overset{*}{m}) \in M$, the submanifolds $\overline{M} \times \overset{*}{m}$ and $\overline{m} \times \overset{*}{M}$ are orthogonal at $(\overline{m}, \overset{*}{m})$. The converse is also true, that is we have the following theorem:

**Theorem.** *([23], [44]). A Riemannian space is a warped product manifold if and only if it can be decomposed into two families of mutualy orthogonal submanifolds, one family consisting of totally geodesic and the other of totally umbilic submanifolds.*

If $F = const.$ then $F$ can be incorporate in the metric $\overset{*}{g}$ and $M = \overline{M} \times \overset{*}{M}$ reduces to a productmanifold, both $\overline{M} \times \overset{*}{m}$ and $\overline{m} \times \overset{*}{M}$ being totally geodesic.

Thus, the class of warped products contains the class of product manifolds and is its generalization.

The class of warped product contains all Riemannian manifolds of constant curvature. In fact, for each point of such a manifold there exsist a neighbourhood in which, with respect to the polar coordinats $r$, $\phi^1, \dots, \phi^{n-1}$ the metric is

$$ds^2 = dr^2 + sin^2\sqrt{k}r\, d\overset{*}{s}^2(\phi^1, \dots, \phi^{n-1}) \text{ for } k > 0$$

and

$$ds^2 = dr^2 + sin^2\sqrt{-k}r\, d\overset{*}{s}^2(\phi^1, \dots, \phi^{n-1}) \text{ for } k < 0,$$

where $k \neq 0$ is the constant curvature of $M$ and $d\overset{*}{s}^2$ is the metric of the unit $(n-1)$–dimensional sphere $S^{n-1}$. We note that the manifold of constant curvature $k \neq 0$ can not be a product manifold. If $k = 0$, then the manifold can be represented as a product manifold on many ways. But for example, for $R^3\backslash 0$, with respect to the spherical coordinates, we have

$$ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta\, d\varphi^2).$$

It means that $R^3\backslash 0$ can be identified with warped $R^+ \times_r S^2$ with a ray from the origin as a basis and the spheres $S^2(r)$, $r > 0$ as the fibers.

The surface of revolution is an other example of Warped products. Let $C$ be the curve in $R^3$ whose parametric representation is

$$x = g(u), \qquad y = 0, \qquad z = F(u).$$

If $z$ is the axis of revolution and $v$ is the angle of rotation, then we have

$$ds^2 = [(g'(u))^2 + (F')^2]du^2 + F^2 dv^2.$$

Thus, the surface of revolution is warped product $C \times_F S^1$, where the curve $C$ is the basis manifold and the circles of revolution the fibers.

The four dimensional warped products are very important in the construction of simple models of some relativistic theories.Thus Schwartzschild spacetime is the simplest relativistic model of a universe containing a single star. The star is assumed to be static and spherically symmetric and to be the only source of gravitation for the spacetime. It follows from these assumptions that Schwartzschild spacetime is warped product $P \times_r S^2$, where the fiber $S^2$ is the unit sphere and the base space $P = R \times R^+$ is a half-plane $r > 0$ in the $rt$-plane endowed with the metric

$$-(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2, \tag{1.1}$$

where $m$ is a constant identified with the mass of the star. The function $1 - \frac{2m}{r}$ increases from limit $-\infty$ at $r = 0$ toward limit 1 at $r = \infty$. But $1 - \frac{2m}{r} = 0$

at $r = 2m$, that is the metric (1.1), and therefore the metric of Warped product $M = P \times_r S^2$, degenerates at $r = 2m$. So, we have to consider two Schwartzschild spacetimes:

1) Schwartzschild exterior spacetime $M = P_I \times_r S^2$, where $P_I$ is the region $r > 2m$;

2) Schwartzschild black hole $M = P_{II} \times_r S^2$, where $P_{II}$ is the region $0 < r < 2m$.

The star is characterized by its mass $m$ and its radius $R$. For the spacetime around the star we have $r > R$. For an ordinary star we have $R > 2m$, that is the surface of the star is in Schwarzschild exterior spacetime. But if $R < 2m$, then $R$ can be only 0; the star dissapears and the warped product $P_{II} \times_r S^2$ becomes black hole. ([23],Chapter 13).

According to the astronomical evidences, the universe can be modeled as a spacetime containing a perfect fluid whose "molecules" are the galaxies. Also, the galaxies, taking into account the large scale appropiate to cosmology, appear to be distributed the same in all directions. Starting with this isotropy condition and using the physical assumptions about the galactic flow, it is possible to construct a simple cosmological model, so called Robertson-Waker spacetime ([23]). This model is the warpedproduct

$$M = M(k,\, F) = I \times_F S, \tag{1.2}$$

where $I$ is an open interval in $R^1$ and $S$ is a three-dimensional manifold of constant curvature $k = -1$, 0 or 1. The metric of the manifold (1.2) is

$$ds^2 = -(dt)^2 + F\, d\overset{*}{s}{}^2,$$

where $d\overset{*}{s}{}^2$ is the metric of the mabifold $S$. It can be proved that the Ricci curvature for Robertson-Walker spacetime $M(k,\, F)$ with flow vector field $\mathcal{U} = \partial_t$ is given by

$$Ric\,(\mathcal{U},\mathcal{U}) = -\frac{3F''}{F}, \qquad Ric\,(\mathcal{U},\, \mathcal{X}) = 0,$$
$$Ric\,(\mathcal{X},\, \mathcal{Y}) = \left[2\left(\frac{F'}{F}\right) + \frac{2k}{F^2} + \frac{F''}{F}\right] < \mathcal{X},\, \mathcal{Y} > \ \text{if}\ \mathcal{X},\, \mathcal{Y} \perp \mathcal{U}$$

([23], p.345).

Also, if $\mathcal{U}$ is the flow vector field a Robertson-Walker spacetime $M(k,\, F)$, then $(\mathcal{U},\, p,\, \rho)$ is a perfect fluid with energy density $\rho$ and the pressure $p$ given by

$$\frac{8\pi}{3}\rho = \left(\frac{F'}{F}\right) + \frac{k}{F^2}, \quad -8\pi p = 2\frac{F''}{F} + \left(\frac{F'}{F}\right) + \frac{k}{F^2}$$

(see [23],p.345).

Acording the astronomical estimates, the spaces $S(t)$ are expanding, i.e. currently $F$ has positive derivative. The following theorem considers the past and the future.

**Theorem.** *([23],p.348). Let* $m(k, F) = I \times_F S and H(t) = \dfrac{F'(t)}{F(t)}$. *If* $H_0 = H_0(t_0) > 0$ *for some* $T_0$, *and* $\rho + 3p > 0$, *then* $I$ *has an initial endpoint* $t_*$ *with*

$$t_0 - H_0^{-1} < t_* n < t_0$$

*and eighter (1)* $F' > 0$ *or (2)* $F$ *has a maximum point after* $t_0$ *and* $I$ *is a finite interval* $(t_*, t_{**})$.

It means thet the universe had the definite beginning and eighter continues expanding, or after conctracting for a while, comes to an end. Using some additional dates, it can be concluded that our universe began in a colossal explosoin.

## 2. Subprojective spaces

The warped product appears also in the investigations of the subprojective and generalized subprojective spaces.

The subprojective spaces were first defined and investigated by V.F.Kagan ([20],-[21],[39]). With respect to the projective properties, these spaces are a natural generalization of the Riemannian spaces of constant curvature. Namely, according the well known Beltrami's theorem, the spaces of constant curvature and only such spaces, admit a mapping on an euclidean space such that the geodesics corenspond to the streit lines. But if the space allows mapping on the flat space such that each of its geodesic corresponds to a plane curve and all such planes contain the same point or are parallel, then we say that the space is subprojective one. A geodesic can also be considered as an autoparallel line, i.e. it is an object defined by the connection only. Thus a subprojective space need not to be Riemannian. It is sufficient that it is a diferentiable manifold endowed with an affine connection. As for Riemannian subprojective spaces, all of them are known in the sense that their metrics are known ([32],[39],[43]). In fact, with respect to the special local coordinates, the metric of the subprojective space has the form

$$ds^2 = (dx^1)^2 + F(x^1)d\overset{*}{s}{}^2\,(x^2, ....x^n), \tag{2.1}$$

where $d\overset{*}{s}{}^2$ is the metric of (n-1)-dimensional space of constant curvature, or the form

$$ds^2 = 2dx^1dx^2 + F(x^1)d\overset{*}{s}{}^2\,(x^3, ....x^n), \tag{2.2}$$

where $d\overset{*}{s}{}^2$ is the (n-2)-dimensional euclidean metric. The metric (2.1) is positive definite, and (2.2) is not.

We see from (2.1) and (2.2) that every subprojective space is a Warped product manifold.

It is intresting to compare subprojective Riemannian spaces to the spaces of constant curvature with respect to the group of motions. It is wel known ( see for example [15] or [38]) that the group of motions of an $n$-dimensional Riemannian space has at most $\frac{n(n+1)}{2}$ parametres. Such a group is transitive and a space admits a group of motions of maximum order $\frac{n(n+1)}{2}$ if and only if it is a space of constant curvature.

The intransitive group has a most $\frac{n(n-1)}{2}$ parametres and all Riemannian spaces admiting such group of motions are subprojective ( see [41],[45]). Conversely, every subprojective space admits intransitive group of motions of order $\frac{n(n-1)}{2}$.In both cases (2.1) and (2.2), this group acts as the transitive group on the hypersurfaces $x^1 = const.$ In the case (2.2), they are isotropic. In some casees, this intransitive group of motions becomes the transitive group of order $\frac{1}{2}n(n-1)+1$. In the case (2.1) this happens only if $F = const.$, that is if the subprojective space is decomposable. Namely, $\frac{1}{2}n(n-1)+1$ is the order of the transitive group of motions of $(n-1)$-dimensional space of constant curvature. Then we add one parameter group of motions along the curve $x^1$

The subprojective space of type (2.2) also admits, in some special cases, the transitive group of motions of order $\frac{1}{2}n(n-1)+1$. First, we note that, with respect to the conformally euclidean coordinates, (2.2) can be rewritten as follows

$$ds^2 = e^{-2\mu(x^1)}\left[2dx^1dx^2 + \epsilon_i(dx^1)^2\right], \; i = 3, ..., n; \; \epsilon_i = \pm 1. \tag{2.3}$$

It was proved in [41] that that the manifold endowed with metric (2.3) admits the transitive group of motions of order $\frac{1}{2}n(n-1)+1$ if and only if the function $\mu = \mu(x^1)$ satisfies

$$\frac{d\mu}{dx^1} = \frac{Ax^1 + B}{A(x^1)^2 + Cx^1 + D},$$

where $A$,$B$,$C$ and $D$ are constants.

Thus, while the Riemannian space of constat curvature are characterized by the property that they admit the transitive group of motions of maximum order, the subprojective spaces are characterized by the property that they admit the intransitive group of motions of maximum order.

The group of motions of general warped products are investigated in [40]. Here, we cite the following theorem.

**Theorem.** *([40]). If the intransitive group of motions of a Riemannian space $M$ is of order $\frac{1}{2}q(q+1)$, $q \geq 2$ and has $q$-dimensional nonisotropic surfaces of the transitivity, then $M$ is the warped product $M = \overline{M} \times_F \overset{*}{M}$ such that $dim\overset{*}{M} = q$.*

This is one of the theorems used for proving the above properties of the subprojective spaces.

We say that a manifold endowed with an affine connection is a generalized subprojective manifold if it admits a mapping on an euclidean space such that every autoparallel corresponds to the curve belonging to a $(q+1)$-dimensional plane $(1 \leq q \leq n-2)$, all this planes containing finite or infinite $(q-1)$-dimensional plane. For $q = 1$, this definition of the subprojective spaces.

G.Vranceanu ([34],[35]) proved that a generalized Riemannian subprojective space with positive definite metric is the warped product $M = \overline{M} \times_F \overset{*}{M}$, where $\overset{*}{M}$ is the space of constant curvature. Conversely, each such a warped product is a generalized subprojective space. But, in the case of indefinite metric, a generalized subprojective space need not be a warped product ([36]).

## 3. The invariant way characterizing the warped product manifolds

Let $\overline{U} : x^1, ..., x^q$ be a local chart for the manifold $\overline{M}$ and $\overset{*}{U} : x^{q+1}, ..., x^n$ that for $\overset{*}{M}$. Then $\overline{U} \times \overset{*}{U} : x^1, ..., x^n$ is a local chart for the warped product $M = \overline{M} \times_F \overset{*}{M}$. With respect to this local chart, we have

$$\overline{g}_{ab} = \overline{g}_{ab}(x^c), \quad \overset{*}{g}_{\alpha\beta} = \overset{*}{g}_{\alpha\beta}(x^\gamma), \quad F = F(x^a), \tag{3.1}$$

while for the metric tensor $g$ of warped product $M = \overline{M} \times_F \overset{*}{M}$, we have

$$g_{ij} = \begin{cases} \overline{g}_{ab} & \text{for} \quad i = a,\ j = b; \\ F\overset{*}{g}_{\alpha\beta} & \text{for} \quad i = \alpha,\ j = \beta; \\ 0 & \text{for all other cases.} \end{cases} \tag{3.2}$$

Here and the seqel the letters $a$, $b$, $c$ range over the indices $1, ..., q$, greak letters $\alpha$, $\beta$, $\gamma$ over the indices $q+1, ..., n$ and letters $i$, $j$, $k$ - over the indices $1, ..., n$.

The definition given in §1 shows that a Riemannian manifold is a warped product if the coordinates can be chosen such that the metric tensor takes the form (3.2) where (3.1) is satisfied. We shall see in §4 that many interesting properties of the warped product manifolds can be obtained using so adapted local coordinates.

There exsist also tensor equations, that is an invariant way, characterizing the warped products. They are contained in the following theorem.

**Theorem.** *([44]).- A Riemannian manifold is warped product if and only if there exsists a symmetric tensor $A_{ij}$, not proportional to the metric tensor, and gradient vector field $u_i$such that*

$$\nabla_k A_{ij} = \frac{1}{2}(u_i A_{kj} + u_j A_{ik}), \quad A_{ij} A^j{}_k = A_{ik}. \tag{3.3}$$

*then* $u_i = \dfrac{\partial}{\partial x^i} log\, F$.

Here and in the sequel, $\nabla$ is the operator of covariant differentatiion with respect to Levi-Civita connection.

If $F = const.$, then (3.3) reduces to

$$\nabla_k A_{ij} = 0, \qquad A_{ij} A^j{}_k = A_{ik},$$

and this conditions, given by P.A.Schirokov ([46]), for a Riemannian space to be decomposable.

For a warped product manifold with $l$-dimensional base or with $l$-dimensional fiber, we have theorems:

**Theorem.** *([37],[44]). A Riemannian manifold is a warped product with l-dimensional base if and only if the equations*

$$\nabla_j f_i = \varphi g_{ij}, \qquad f_i = \frac{\partial f}{\partial x^i}, \qquad \varphi = \varphi(f)$$

*admit solution $f \neq const.$*

**Theorem.** *([44]).-A Riemannian manifold is a warped product with l-dimensional fiber if and only if there exists a nonisotropic vector field $A_i$ which, together with the gradient $u_i = \dfrac{\partial}{\partial x^i} log\, F$ satisfies*

$$\nabla_j = \frac{1}{2}(A_i u_j - A_j u_i).$$

## 4. The geometry of the warped product in terms of warping function $F$ and the geometries of $\overline{M}$ and $\overset{*}{M}$

There are many papers dedicated to the investigation of the geometry of the warped product $M = \overline{M} \times_F \overset{*}{M}$ in terms of warping function $F$ and the geometries of $\overline{M}$ and $\overset{*}{M}$. In this section we quote some examples.

**4.1** From now on, we suppose $F = const.$ and we use the local coordinates with respect to which the relation (3.1) and (3.2) are satisfied. We assume that each object denoted by a dash is formed $\overline{g}_{ab}$ and each object denoted by a star using $\overset{*}{g}_{\alpha\beta}$. Then the local components $\Gamma^h_{ij}$ of the Levi-Civita connection on $\overline{M} \times_F \overset{*}{M}$ are the following (see for example [11], [17], [44]):

$$(4.1) \qquad \begin{cases} \Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, & \Gamma^a_{\beta\gamma} = -\dfrac{1}{2}\overline{g}^{ab} F_b \overset{*}{G}_{\alpha\beta}, & \Gamma^\alpha_{\beta\gamma} = \overset{*}{\Gamma}{}^\alpha_{\beta\gamma}, \\ \Gamma^\alpha_{a\beta} = \dfrac{1}{2F} F_a \delta^\alpha_\beta, & \Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0, & F_a = \dfrac{\partial F}{\partial x^a}. \end{cases}$$

The local components $R_{hijk}$ of the curvature tensor of $M = \overline{M} \times_F \overset{*}{M}$ which in general do not vanish identicaly, are the following

$$(4.2) \qquad \begin{cases} R_{abcd} = \overline{R}_{abcd}, & R_{\alpha ab\beta} = -\dfrac{1}{2} T_{ab} \overset{*}{G}_{\alpha\beta}, \\ R_{\alpha\beta\gamma\delta} = F \overset{*}{R}_{\alpha\beta\gamma\delta} - \dfrac{1}{4}\Delta_1 F \overset{*}{G}_{\alpha\beta\gamma\delta}, & \end{cases}$$

where $T$ is the (0.2) tensor with the local components $T_{ij}$ defined by

$$T_{\alpha\beta} = T_{\alpha a} = 0, \qquad T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F} F_a F_b, \tag{4.3}$$

and

$$\Delta_1 F = \overline{g}^{ab} F_a F_b,$$
$$\overset{*}{G}_{\alpha\beta\gamma\delta} = \overset{*}{g}_{\alpha\delta}\overset{*}{g}_{\beta\gamma} - \overset{*}{g}_{\alpha\gamma}\overset{*}{g}_{\beta\delta}.$$

In view of (4.1) and (4.2), we get

$$\tag{4.4} \begin{cases} \nabla_e R_{abcd} = \overline{\nabla}_e \overline{R}_{abcd}, \\ \nabla_\alpha R_{abcd} = \nabla_d R_{abc\alpha} = \nabla_\gamma R_{\alpha ab\beta} = 0, \\ \nabla_c R_{ab\alpha\beta} = \nabla_\gamma R_{ab\alpha\beta} = \nabla_b R_{\alpha\beta\gamma a} = 0, \\ \nabla_\beta R_{\alpha abc} = \frac{1}{2}\left[F_e \overline{R}^e{}_{abc} - \frac{1}{2F}(F_b T_{ac} - F_c T_{ab})\right] \overset{*}{g}_{\alpha\beta}, \\ \nabla_b R_{\alpha ac\beta} = -\frac{1}{2}(\nabla_b T_{ac})\overset{*}{g}_{\alpha\beta}, \\ \nabla_\delta R_{\alpha\beta\gamma a} = -\frac{1}{2}\left[F_a \overset{*}{R}_{\alpha\beta\gamma\delta} + \frac{1}{2}(F_e T^e{}_a - \frac{1}{2F} F_a \Delta_1 F)\overset{*}{G}_{\alpha\beta\gamma\delta}\right], \\ \nabla_a R_{\alpha\beta\gamma\delta} = -F_a \overset{*}{R}_{\alpha\beta\gamma\delta} + \frac{1}{2}\left[\frac{F_a}{F}\Delta_1 F - \frac{1}{2}\partial_a(\Delta_1 F)\right] \overset{*}{G}_{\alpha\beta\gamma\delta}, \\ \nabla_\rho R_{\alpha\beta\gamma\delta} = F \overset{*}{\nabla}_\rho \overset{*}{R}_{\alpha\beta\gamma\delta}. \end{cases}$$

The local components $S_{ij} = R^r{}_{ijr}$ of the Ricci tensor of $\overline{M} \times_F \overset{*}{M}$,which in general do not vanish identically, are the following

$$\tag{4.5} \begin{cases} S_{ab} = \overline{S}_{ab} - \frac{n-q}{2F} T_{ab}, \\ \overset{*}{S}_{\alpha\beta} = \overset{*}{S}_{\alpha\beta} - \frac{1}{2}\left[tr(T) + \frac{n-q-1}{2F}\Delta_1 F\right] \overset{*}{g}_{\alpha\beta}, \quad tr9T) = \overline{g}^{ab} T_{ab}. \end{cases}$$

The scalar curvature $R$ of the metric $\overline{g} \times_F \overset{*}{g}$ satisfies the eqation

$$R = \overline{R} + \frac{1}{R}\overset{*}{R} - \frac{n-q}{F}\left(tr(T) + \frac{n-q-1}{4F}\Delta_1 F\right). \tag{4.6}$$

Therefore, Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}S_{hk} - g_{ik}S_{hj} + g_{hk}S_{ij} - g_{hj}S_{ik}) +$$
$$+ \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj})$$

has the following components

$$
(4.7)\quad \begin{cases}
C_{abcd} = \overline{R}_{abcd} - \dfrac{1}{n-2}(\overline{g}_{ad}\overline{S}_{bc} - \overline{g}_{ac}\overline{S}_{bd} + \overline{g}_{bc}\overline{S}_{ad} - \overline{g}_{bd}\overline{S}_{ac}) + \\
\quad + \dfrac{n-q}{2(n-2)F}(\overline{g}_{ad}T_{bc} - \overline{g}_{ac}T_{bd} + \overline{g}_{bc}T_{ad} - \overline{g}_{bd}T_{ac}) + \\
\quad + \dfrac{R}{(n-1)(n-2)}\overline{G}_{abcd}, \\
C_{\alpha ab\beta} = -\dfrac{1}{n-2}\left(\dfrac{q-2}{2}T_{ab} + F\overline{S}_{ab}\right)\overset{*}{g}_{\alpha\beta} - \dfrac{1}{n-2}\overline{g}_{ab}\overset{*}{S}\alpha\beta + \\
\quad + \dfrac{1}{(n-1)(n-2)}\left[F\overline{R} + \overset{*}{R} - \dfrac{n-2q+1}{2}tr(T) + \right. \\
\quad \left. + \dfrac{(q-1)(n-q-1)}{4F}\Delta_1 F\right]\overline{g}_{ab}\overset{*}{g}_{\alpha\beta}, \\
C_{\alpha\beta\gamma\delta} = F\overset{*}{R}_{\alpha\beta\gamma\delta} - \dfrac{F}{n-2}(\overset{*}{g}_{\alpha\delta}\overset{*}{S}_{\beta\gamma} - \overset{*}{g}_{\alpha\gamma}\overset{*}{S}_{\beta\delta} + \overset{*}{g}_{\beta\gamma}\overset{*}{S}_{\alpha\delta} - \overset{*}{g}_{\beta\delta}\overset{*}{S}_{\alpha\gamma}) + \\
\quad + \dfrac{F}{n-2}\left[\dfrac{FR}{n-1} + tr(T) + \dfrac{n-2q}{4F}\Delta_1 F\right]\overset{*}{G}_{\alpha\beta\gamma\delta}, \\
C_{abc\beta} = C_{ab\alpha\beta} = C_{a\beta\gamma\delta} = 0.
\end{cases}
$$

Moreover, from (4.1) and (4.5), we find

$$
(4.8)\quad \begin{cases}
\nabla_c S_{ab} = \overline{\nabla}_c\overline{S}_{ab} - (n-q)\nabla_c\left(\dfrac{1}{2F}T_{ab}\right), \\
\nabla_\alpha S_{ab} = \nabla_b S_{a\alpha} = 0, \\
\nabla_\beta S_{a\alpha} = -\dfrac{F_a}{2F}\left[\overset{*}{S}_{\alpha\beta} - \dfrac{1}{2}\left(tr(T) + \dfrac{n-q-1}{2F}\nabla_1 F\right)\overset{*}{g}_{\alpha\beta}\right] + \\
\quad + \dfrac{1}{2}F_e\left[\overline{S}^{e}{}_{a} - \dfrac{n-q}{2F}T^{e}{}_{a}\right]\overset{*}{g}_{\alpha\beta}, \qquad T^{E}{}_{a} = \overline{g}^{eb}T_{ab}, \\
\nabla_a S_{\alpha\beta} = -\dfrac{F_a}{F}\overset{*}{S}_{\alpha\beta} + \dfrac{1}{2}\dfrac{F_a}{F}\left[tr(T) + \dfrac{n-q-1}{2F}\nabla_1 F\right]\overset{*}{g}_{\alpha\beta} - \\
\quad - \dfrac{1}{2}\partial_a\left[tr(T) + \dfrac{n-q-1}{2F}\Delta_1 F\right]\overset{*}{g}_{\alpha\beta}, \\
\nabla_\delta S_{\alpha\beta} = \overset{*}{\nabla}_\delta\overset{*}{S}_{\alpha\beta}.
\end{cases}
$$

Using (4.4), (4.6), (4.7) and (4.8) we can get the local components of $\nabla_r C_{ijkh}$ and $\nabla_r\nabla_s C_{ijkh}$.

**4.2** In view of (4.4) we can state

**Theorem.** *( [22], Th.1)-If a warped product $M = \overline{M} \times_F \overset{*}{M}$ with $n \neq q+1$ is Cartan-symmetric (i.e. if $\nabla_r R_{ijkh} = 0$ ). then $\overline{M}$ is Cartan-symmetric and $\overset{*}{M}$ is of a constant curvature.*

Simmilary, it follows from (4.7) that if $M$ is conformally flat (i.e. if $C_{ijkh} = 0$), then $\overset{*}{M}$ is a space of constant curvature. Conversely is not true, but we have

**Theorem.** *-Let $\overline{M}$ be an open interval of $R$ with metric $g_{11} = \epsilon$, $\epsilon \in \{-1,\ 1\}$. Let $F$ be a positive $C^{\infty}$ function on $\overline{M}$ and let $dim\,\overset{*}{M} \geq 2$. Then the warped product $M = \overline{M} \times_F \overset{*}{M}$ is conformally flat if and only if $\overset{*}{M}$ is a manifold of constant curvature.*

If a recurrent space ($\nabla_r R_{ijkh} = A_r R_{ijkh}$ ) is a locally decomposable, then one of the decomposition space is flat and other is a recurrent space ( [31,p164). The non-decomposable recurrent Riemannian spaces are all known. Some of them are warped products. For 2-recurrent ($\nabla_r \nabla_s R_{ijkh} = A_{rs} R_{ijkh}$ ), conformally symmetric ($\nabla_r C_{ijkh}$ ), conformally birecurrent ($\nabla_r \nabla_s C_{ijkh} = A_{rs} C_{ijkh}$ ) Riemannian spaces, we have

**Theorem.** *([13], [19], [22]).- If a 2-recurent (conformally symmetric, conformally recurrent, conformally birecurrent ) Riemannian space is a warped product and $dim\,\overline{M} > 3$, then $\overline{M}$ is 2-recurent (conformally symmetric, conformally recurrent, conformally birecurrent ) and $\overset{*}{M}$ is a space of constant curvature.*

M.C.Chaki and G.Kumer ([6]) generalized this theorem for the space satisfying

$$\nabla_r \nabla_s C_{ijkh} = A_r \nabla_s C_{ijkh} + B_{rs} C_{ijkh}.$$

One generalization of a recurrent space is the Riemannian manifold satisfying ([2],-[4],[5],[16],[25],[26],[33] ):

$$\nabla_r R_{ijkh} = A_r R_{ijkh} + B_i R_{rjkh} + B_j R_{irkh} + B_k R_{ijrh} + B_h R_{ijkr}. \tag{4.9}$$

Here, we shall give an example of warped product manifolds satisfying (4.9).

Let $\overline{M}$, $dim\,\overline{M} \geq 2$, be equiped with the metric

$$\overline{g}_{ab} dx^a dx^b = \sum_{a=1}^{q} e_a (dx^a)^2, \qquad e_a = \pm 1$$

and let

$$F = (C_0 = C_1 x^1 +, \cdots, C_q x^q)^2,$$

where $C_0$, $C_1$,...,$C_q$ are constants such that

$$\sum_{a=1}^{q} e_a (C_a)^2 = 0.$$

Then

$$\left\{ \begin{array}{l} F_a = 2C_a (C_0 = C_1 x^1 +, \cdots, C_q x^q), \quad \overline{\nabla}_b F_a = 2C_a C_b, \\ T_{ab} = 0, \qquad \nabla_1 F = 0. \end{array} \right. \tag{4.10}$$

In view of (4.2), (4.4) and (4.10), it follows that the only components of $R_{ijkh}$ and $\nabla_r R_{ijkh}$ not identically equal to zero are those related to

$$\begin{aligned} &R_{\alpha\beta\gamma\delta} = F\overset{*}{R}_{\alpha\beta\gamma\delta},\\ &\nabla_\rho R_{\alpha\beta\gamma\delta} = F\overset{*}{\nabla}_\rho \overset{*}{R}_{\alpha\beta\gamma\delta},\\ &\nabla_a R_{\alpha\beta\gamma\delta} = -F_a\overset{*}{R}_{\alpha\beta\gamma\delta}, \quad \nabla_\rho R_{\alpha\beta\gamma a} = -\frac{1}{2}F_a\overset{*}{R}_{\alpha\beta\gamma\rho}. \end{aligned}$$

Now, if $\overset{*}{M}$ is a recurrent space, i.e. if $\overset{*}{\nabla}_\rho \overset{*}{R}_{\alpha\beta\gamma\delta} = A_\rho \overset{*}{R}_{\alpha\beta\gamma\delta}$, then

$$\begin{aligned} &\nabla_\rho R_{\alpha\beta\gamma\delta} = A_\rho R_{\alpha\beta\gamma\delta},\\ &\nabla_a R_{\alpha\beta\gamma\delta} = -\frac{F_a}{F}R_{\alpha\beta\gamma\delta}, \quad \nabla_\rho R_{\alpha\beta\gamma a} = -\frac{1}{2}\frac{F_a}{F}R_{\alpha\beta\gamma\rho}, \end{aligned}$$

and warped product$M = \overline{M} \times_F \overset{*}{M}$ satisfies (4.9), the vector fields $A$ and $B$having the localcomponents

$$\text{(4.11)} \qquad \begin{cases} A : A_\alpha, \quad A_a = -\dfrac{F_a}{F};\\ B : B_\alpha = 0, \quad B_a = -\dfrac{1}{2}\dfrac{F_a}{F}. \end{cases}$$

If $\overset{*}{M}$ is Cartan-symmetric, $A_\alpha = 0$, and $A = 2B$. The Ricci tensor of considered warped product satisfies

$$\text{(4.12)} \qquad \nabla_k S_{ij} = A_k S_{ij} + B_i S_{kj} + B_j S_{ik},$$

or ( in the case $\overset{*}{M}$ is Cartan-symmetric )

$$\text{(1.13)} \qquad \nabla_k S_{ij} = 2B_k S_{ij} + B_i S_{kj} + B_j S_{ik}.$$

Indeed, in view of (4.10), we can reduce the relations (4.5) and (4.8) as follows

$$\begin{aligned} &S_{ab} = S_{a\beta} = 0, \qquad S_{\alpha\beta} = \overset{*}{S}_{\alpha\beta};\\ &\nabla_c S_{ab} = \nabla_\beta S_{ab} = \nabla_b S_{a\alpha} = 0,\\ &\nabla_\beta S_{a\alpha} = -\frac{1}{2}\frac{F_a}{F}\overset{*}{S}_{\alpha\beta}, \quad \nabla_a S_{\alpha\beta} = -12\frac{F_a}{F}\overset{*}{S}_{\alpha\beta},\\ &\nabla_\delta S_{\alpha\beta} = \overset{*}{\nabla}_\delta \overset{*}{S}_{\alpha\beta}. \end{aligned}$$

If $\overset{*}{M}$ is a reccurent manifold, then $\overset{*}{\nabla}_\rho \overset{*}{S}_{\alpha\beta} = A_\rho \overset{*}{S}_{\alpha\beta}$, i.e. it is also Ricci-recurrent and taking into account (4.11), we have (4.12).

$\overset{*}{M}$ can be Ricci-reccurent and not recurrent. For example, W.Roter determined in [29] and [30] the metrics of conformally symmetric and conformally recurrent

Ricci recurrent manifolds which are not recurrent. In this way we obtain new examples of Riemannian manifolds satisfying (4.12). The Riemannian manifolds satisfying (4.13) was introduced by M.C.Chaki ([3]) and further investigated in [4] and [7].

S.Ewert-Krzemieniewski ([16]) determined the subprojective spaces satisfying (4.9) with $A = 2B$. More precisely, he determined the function $F$ in (2.1) such that the condition (4.9) is fulfilled for $A = 2B$.

N.Pušić ([27],[28]) investigated Ricci-recurrent warped product manifolds. Among others, she proved that if $M = \overline{M} \times_F \overset{*}{M}$ is Ricci-recurrent, then $\overset{*}{M}$ is an Einstein space.

**4.3** An $n$-dimensional ($n \geq 4$) Riemannian manifold is said to have harmonic Weyl conformal curvature tensor ([1],p440) or to be nearly conformally symmetric ([17]) if its Ricci tensor satisfies the condition

$$\nabla_k S_{ij} - \nabla_j S_{ik} = \frac{1}{2(n-1)}(g_{ij}\nabla_k R - g_{ik}\nabla_j R). \tag{4.14}$$

Namely, it is easy to check that for every conformally symmetric manifold the condition (4.14) holds.

If the Ricci tensor satisfies

$$\nabla_k S_{ij} = \frac{n}{(n-1)(n+2)} g_{ij}\nabla_k R + \frac{n-2}{2(n-1)(n+2)}(g_{kj}\nabla_i R + g_{ik}\nabla_j R), \tag{4.15}$$

then it satisfies the condition (4.14), too.

Finaly, for Einstein manifold ($S_{ij} = \frac{R}{n} g_{ij}$), if $n > 2$ then $R = const.$ and both conditions (4.14) and (4.15) are identically satisfied.

If the warped product manifold $M = \overline{M} \times_F \overset{*}{M}$ satisfies (4.14), then $\overset{*}{M}$ is an Einstein space with a constant scalar curvature. The converse is not true. But if $dim\,\overline{M} = 1$, we have

**Theorem.** *([17]).-Let $dim\,\overline{M} = 1$ and $g_{11} = 1$. then the warped product $M = \overline{M} \times_F \overset{*}{M}$ satisfies (4.14) if and only if $\overset{*}{M}$ is an Einstein space and its scalar curvature is constant.*

Furthermore, if the function $f^2 = \frac{1}{F}$ is a solution of ordinary differential equation

$$\frac{d^2 f}{(dx^1)^2} - \frac{2\overset{*}{R} f^3}{(n-1)(n-2)} = const.,$$

the Ricci tensor satisfies the condition (4.15). (This is the example of manifolds satisfying (4.15), given in [1],p.433)

But, if $F$ is given by one of the following formulas

$$\text{if } \overset{*}{R} > 0, \quad F^2 = \begin{cases} \dfrac{4}{a}\dfrac{\overset{*}{R}}{(n-1)(n-2)}\, sh^2\dfrac{\sqrt{a}(x^1+b)}{2}, & a > 0, \\ \dfrac{\overset{*}{R}}{(n-1)(n-2)}\,(x^1+b)^2, & \\ -\dfrac{4}{a}\dfrac{\overset{*}{R}}{(n-1)(n-2)}\, sin^2\dfrac{\sqrt{-a}(x^1+b)}{2}, & a < 0; \end{cases}$$

$$\text{if } \overset{*}{R} = 0, \quad F^2 = be^{ax^1}, \quad a \neq 0;$$

$$\text{if } \overset{*}{R} < 0, \quad F^2 = -\frac{4}{a}\frac{\overset{*}{R}}{(n-1)(n-2)}\, cos\, h^2\frac{\sqrt{a}(x^1+b)}{2}, \quad a > 0;$$

where $a$ and $b$ are constans, then warped product $M = \overline{M} \times_F \overset{*}{M}$ is an Einstein space ([18]).

It is interesting to note that warped product manifold provided with the metric

$$ds^2 = -(dx^1)^2 + F\overset{*}{g}_{\alpha\beta}dx^\alpha dx^\beta$$

satisfies (4.14) if and only if $\overset{*}{g}_{\alpha\beta}dx^\alpha dx^\beta$ is the metric of an Einstein space with constant scalar curvature and the function $F$ has the form $F = e^{bx+a}$, where $a$ and $b$ are constants.

**4.4** The Riemannian space is said to be semi-symmetric if its curvature tensor satisfies

$$R \cdot R = 0, \tag{4.16}$$

where the first tensor acts on the second as a derivation.

There are many various possibilities to obtain curvature conditions weaker that (4.16). To expres them, let $\widetilde{R}(X, Y)$ and $X \wedge_A Y$ be defined by

$$\widetilde{R}(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$
$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$

respectively, where $X$, $Y$, $Z$ are vector fields and $A$ is an $(0, 2)$ tensor field on $(M, g)$. For (0, k)-tensor field $P$ on $M$, $k \geq 1$, we define tensors $R \cdot P$ and $Q(A, P)$ by the formulas

$$(R \cdot P)(X_1, ..., X_k; X, Y) = -P(\widetilde{R}(X, Y)X_1, \cdots, X_k) - \cdots - P(X_1, \cdots, X_{k-1}, \widetilde{R}(X, Y)X_k),$$

$$Q(A, P)(X_1, ..., X_k; X, Y) = P((X \wedge_A Y)X_1, \cdots, X_k) + \cdots + P(X_1, \cdots, X_{k-1}, (X \wedge Y)X_k).$$

Then, the desired conditions weaker then (4.16) are

(4.18) $$R \cdot R = \mathcal{L}Q(g,\, R),$$

(4.19) $$R \cdot R = Q(S,\, R),$$

(4.19) $$C \cdot C = \mathcal{L}Q(g,\, C),$$

(4.20) $$R \cdot R = Q(S,\, R) + \mathcal{L}Q(g,\, C),$$

where $C$ is the conformal curvature tensor and $\mathcal{L}$ is a function on $M$.

There exsists many examples of warped product manifolds satisfying one of these conditions. We cite some of them.

**Theorem.** *([11]).- Let $M$ be an open interval of* $\mathbf{R}$ *with the metric $g_{11} = \epsilon$, $\epsilon \in \{-1,\, 1\}$, $F$ a positive $C^\infty$ function on $\overline{M}$ and na $* M$ a manifold of constant curvature. Then the warped prodoct $M = \overline{M} \times_F \overset{*}{M}$ satisfied (4.17).*

**Theorem.** *([11]).- Let $\overline{M}$ be an open subset of* $\mathbf{R}^q \setminus \{0, ..., 0\}$, $q \geq 2$, *eqiped with the metric $\overline{g}_{ab} = \delta_{ab}$, $F(x^1, ..., x^q) = \dfrac{1}{4}[(x^1)^2 + ... + (x^q)^2]^2$ and na $* M$ ($dim\, \overset{*}{M} \geq 2$) a localy flat manifold. Then the warped prodoct $M = \overline{M} \times_F \overset{*}{M}$ satisfied (4.17).*

**Theorem.** *([8]).- Let $\overline{M} = \{(x^1,\, x^2) \in \mathbf{R}^2$ and $x^1 > 0,\, x^2 > 0\}$ be a 2-dimensional manifold with the metric $\overline{g}$ defined by $\overline{g}_{ab} = \epsilon_a$, $\epsilon_a = \pm 1$. Let $\overset{*}{M}$, $dim\, \overset{*}{M} \geq 2$ be a manifold of constsnt curvature and let*

$$F(x^1,\, x^2) = (x^1)^{\overline{\frac{c+1}{c}}} \cdot (x^2)^{\overline{\frac{c-1}{c}}},$$

*where $c$ is nonzero constant. Then the warped product $M = \overline{M} \times_F \overset{*}{M}$ satisfied (4.18).*

**Theorem.** *([14]).- Let $\overline{M}$ be a l-dimensional manifold and let $\overset{*}{M}$ be a 3-dimensional manifold or (if $dim\, M \geq 4$) conformally flat. Then the warped product $M = \overline{M} \times_F \overset{*}{M}$ satisfied (4.19) if and only if*

$$\overset{*}{S}_{\alpha\beta} = \mu \overset{*}{g}_{\alpha\beta} + \nu u_\alpha u_\beta,$$

*where $\mu$ and $\nu$ are function and $u_\alpha$ is a vector field on $\overset{*}{M}$.*

**Theorem.** *([10]).- Let $\overline{M}$, $dim\, \overline{M} = q \geq 2$ and $\overset{*}{M}$, $dim\, \overset{*}{M}$ be two Riemannian manifolds of constant curvature and $F$ a positive smoth function on $\overline{M}$. Then the* *warped product* $M = \overline{M} \times_F \overset{*}{M}$ *satisfied (4.20) with* $\mathcal{L} = -\dfrac{n-2}{q(q-1)}\overline{R}$ *if and only if, at every point of* $\overline{M}$, *the condition*

$$rank\left(\frac{1}{2}T_{ab} - \frac{F\mathcal{L}}{n-2}\overline{g}_{ab}\right) \leq 1$$

*is satisfied. (The (0, 2) tensor* $T$ *is defined by (4. 3) and* $\overline{R}$ *is the scalar curvature of* $\overline{M}$.*)*

**Theorem.** *([10]).- Let* $F$ *be a positive smooth function on 2-dimensional Riemannian manifold* $\overline{M}$ *such that the tensor* $T$ *is proportional to* $\overline{g}$. *Moreover, let* $\overset{*}{M}$, $dim\,\overset{*}{M} \geq 2$, *be a manifold of constant curvature. Let the function* $\mathcal{L}$ *defined by*

$$\mathcal{L} = \frac{n-3}{4}\frac{tr\,(T)}{F} - \frac{\overline{R}}{2}$$

*satisfied* $\mathcal{L} = -\dfrac{n-2}{2}\overline{R}$. *Then* $M = \overline{M} \times_F \overset{*}{M}$ *is a manifold fulfilling (4.20).*

**Theorem.** *([9]).- An Einstein manifold* $(M, g)$, $dim\,M \geq 4$, *satisfying (4.17), satisfies the condition (4.19), too.*

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