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$F(2k+1,1)$-STRUCTURE ON THE LAGRANGIAN SPACE

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ABSTRACT. *If almost product P or almost complex structure J on the tangent space $T(E)=T_V(E)+T_H(E)$ of Lagrangian $2n$ dimensional manifold E are defined, and if $f_v(2k+1,1)$-structure on $T_V(E)$ is defined, then $f_p(2k+1,1)$ and $f_j(2k+1,1)$-structures on $T_H(E)$ are defined in the natural way. We can define $F_p(2k+1,1)$, $F_j(2k+1,1)$-structures on $T(E)$. The condition is given for the reduction of the structural group of such manifolds.*

1. Introduction

Let M be an n dimensional and E $2n$ dimensional differentiable manifold and let $\eta=(E,\pi,M)$ be vector bundles and $\pi E=M$. The differential structures (U,ϕ,R^{2n}) are vector charts of the vector bundles η. Hence the canonical coordinates on $\pi^{-1}(U)$ are $(x^1,\dots,x^n,y^1,\dots,y^n)=(x^i,y^a)$, $i=1,2,\dots,n$ $a=1,\dots,n$. Transformation maps on E are

$$\begin{aligned} x^{i'} &= x^{i'}(x^1,x^2,\dots,x^n)\\ y^{a'} &= M_a^{a'}(x^1,\dots,x^n)y^a = M_a^{a'}(x^i)y^a\\ \operatorname{rank}\left[\frac{\partial x^{i'}}{\partial x^i}\right] &= n,\ \operatorname{rank}\left[\frac{\partial y^{a'}}{\partial y^a}\right] = \operatorname{rank} M_a^{a'} = n. \end{aligned}$$

The inverse transformations are

$$\begin{aligned} x^i &= x^i(x^{1'},x^{2'},\dots,x^{n'})\\ y^a &= M_{a'}^a(x^{i'},\dots,x^{n'})y^{a'} \end{aligned}$$

where $M_{a'}^a M_a^{a'} = \delta_b^a$.

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The local natural bases of the tangent space $T(E)$ are $\{\partial_i, \partial_a\}$

$$\partial_a = \frac{\partial}{\partial y^a} = M_a^{a'}(x^i)\partial_{a'}$$

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\partial_i + (\partial_i M_b^{a'}(x))y^b\partial_{a'}.$$

The nonlinear connection on E is distribution

$$N : u \in E \to N_u \subset T_u(E)$$

which is supplementary to the distribution V,

$$T_u(E) = N_u \oplus V_u, \quad \forall_u \in E. \tag{1.1}$$

They are localy determined by $\delta_i = \partial_i - N_i^a\partial_a$. The local bases adapted to decompositions in (1.1) is $\{\delta_i, \partial_a\}$.

It is easy to prove that on $\{\delta_i, \partial_a\}$

$$\delta_{i'} = \delta_i \frac{\partial x^i}{\partial x^{i'}}, \quad \partial_{a'} = \frac{\partial y^a}{\partial y^{a'}}\partial_a.$$

The subspace of $T(E)$ spaned by $\{\delta_i\}$ will be denoted by $T_H(E)$ and the subspace spaned by $\{\partial_a\}$ will be denoted by $T_V(E)$, $T(E) = T_H(E) \oplus T_V(E)$, $\dim T_H(E) = n = \dim T_V(E)$.

Definition 1.1. *If the Riemannian metrical structure on $T(E)$ is given by $G = g_{ij}(x^i, y^a)dx^i \otimes dx^j + g_{ab}(x^i, y^a)\delta y^a \otimes \delta y^b$ where $g_{ij}(x^i, y^a) = g_{ij}(x^i)$, $g_{ab} = \frac{1}{2}\partial_a\partial_b L(x^i, y^a)$ and $L(x^i, y^a)$ is a Lagrange function, then such a space we call Lagrangian space.*

Let $X \in T(E)$, then $X = X^i\delta_i + \bar{X}^a\partial_a$ and the automorphism $P : \mathcal{X}(T(E)) \to \mathcal{X}(T(E))$ defined by

$$PX = \bar{X}^i\delta_i + X^a\partial_a$$

is the natural almost product structure on $T(E)$. i.e, $P^2 = I$. If we denote by v and h the projection morphism of $T(E)$ to $T_V(E)$ and $T_H(E)$ respectively, we have

$$P \circ h = v \circ P.$$

The automorphism

$$JX = -\bar{X}^i\delta_i + X^a\partial_a$$

is the natural almost complex structure on $T(E)$.

2. $f(2k+1,1)$-structures

Definition 2.1. *We call Lagrange vertical $f_v(2k+1,1)$-structure of rank r on $T_V(E)$ a non-null tensor field f_v of type (1,1) and of class C^∞ such that $f_v^{2k+1}+f_v=0$, $k\in N$, and rank $f_v=r$, where r is constant everywhere.*

Definition 2.2. *We call Lagrange horizontal $f_h(2k+1,1)$-structure on $T_H(E)$ a non-null tensor field f_h on $T_H(E)$ of type (1,1) of class C^∞ satisfying $f_h^{2k+1}+f_h=0$, $k\in N$, rank $f_h=r$, where r is constant everywhere.*

An $F(2k+1,1)$-structure on $T(E)$ is a non-null tensor field F of type $\binom{11}{11}$ such that $F^{2k+1}+F=0$, $k\in N$, rank $F=2r$ =const.

For our study it is very convenient to consider f_v or f_h as morphism of vectors bundles.

$$f_v:\mathcal{X}T_V(E)\to\mathcal{X}T_V(E)$$
$$f_h:\mathcal{X}T_H(E)\to\mathcal{X}T_H(E).$$

Let f_v be a Lagrange vertical $f_v(2k+1,1)$-structure of rank r. We define the morphisms

$$l=-f_v^{2k}\ \text{ and }\ m=f_v^{2k}+I_{T_V(E)}$$

where $I_{T_V(E)}$ denotes the identity morphism on $T_V(E)$.

It is clear thet $l+m=I$. Also we have

$$l\,m=m\,l=-f_v^{4k}-f_v^{2k}=-f_v^{2k-1}(f_v^{2k+1}+f_v)=0,$$
$$m^2=m,\ \ l^2,=l.$$

Hence the morphisms l, m applied to the $\mathcal{X}(T_V(E))$ are complementaly projection morphisms, then there exist complementary distributions VL and VM corresponding to the projection morphisms l and m respectively such that dim $VL=r$ and dim $VM=n-r$.

It is easily to see that

$$lf_v=f_vl=f_v,\ \ mf_v=f_vm=0,\ \ f_v^{2k}m=0, \tag{2.1}$$

$$f_v^{2k}l=-l.$$

Proposition 2.1. *If a Lagrange $f_v(2k+1,1)$-structure of rank r defined on $T_V(E)$, then the horizontal $f_h(2k+1,1)$-structure of rank r is defined on $T_H(E)$ by the natural almost product structure of $T(E)$, as f_p, or by the almost complex natural complex structure of $T(E)$, as f_j.*

Proof. If we put

$$f_pX = Pf_vPX, \ \ \forall X \in T_H(E) \tag{2.2}$$

$$f_jX = -Jf_vJX, \ \ \forall X \in T_H(E) \tag{2.3}$$

it is easy to see that

$$f_p^{2k+1}X = Pf_v^{2k+1}PX, \ \ f_j^{2k+1}X = -Jf_v^{2k+1}JX$$

and

$$f_p^{2k+1} + f_p = 0, \ \ f_j^{2k+1} + f_j = 0$$

and rank f_p = rank $f_j = r$. It is easy to see that $f_p = f_j = f_h$.

Proposition 2.2. *If a Lagrange $f_v(2k+1,1)$-structure of rank r is defined on $T_V(E)$, then an $F_p(2k+1,1)$-structure or $F_j(2k+1,1)$-structure are defined on $T(E)$ by the natural almost product or natural almost complex structure of $T(E)$.*

Proof. We put

$$F_p = f_ph + f_vv,$$
$$F_j = f_jh + f_vv,$$

where f_p, f_j are defined by (2.2), (2.3) and h, v are the projection morphisms of $T(E)$ to $T_H(E)$ and $T_V(E)$. Then it is easy to check that

$$F_p^2 = f_p^2h + f_v^2v, \ \ F_p^{2k+1} = f_p^{2k+1}h + f_v^{2k+1}v.$$

Thus $F_p^{2k+1} + F_p = 0$. Similary $F_j^{2k+1} + F_j = 0$. It is clear that rank $F_p = \text{rank} F_j = 2r$.

If l_p, m_p are complementary projection morphisms of the horizontal $f_p(2k+1,1)$-structure, which is defined by the natural almost product structure of $T(E)$, we have

$$l_pX = -f_p^{2k}X = -Pf_v^{2k}PX = PlPX, \forall X \in T_H(E)$$

$$m_pX = f_p^{2k} + I_{T_V(E)}X = Pf_v^{2k}PX + PI_{T_V(E)}PX = PmPX, \forall X \in T_H(E).$$

If L_p, M_p are complementary projection morphism of the $F_p(2k+1,1)$ structure on $T(E)$, then we have

$$\begin{aligned} L_p = -F_p^{2k} &= -f_p^{2k}h - f_v^{2k}v = l_ph + lv \\ M_p = F_p^{2k} + I_{T(E)} &= f_p^{2k}h + f_v^{2k} + I_{T_H(E)}h + I_{T_V(E)}v = \\ &= m_ph + mv. \end{aligned} \tag{2.4}$$

Thus, if there is given a Lagrange $f_v(2k+1,1)$-structure on $T_V(E)$ of rank r, then there exist complementary distributions HL_p, HM_p of $T_H(E)$, corresponding to the morphisms l_p, m_p such that

$$HL_p = PVL, HM_p = PVM. \tag{2.5}$$

Thus we have the decompositions

$$T(E) = T_H(E) \oplus T_V(E) = PVL \oplus PVM \oplus VL \oplus VM.$$

If TL_p, TM_p denote complementary distributions corresponding to the morphisms L_p, M_p respectively, then from (2.4) and (2.5) we have

$$TL_p = PVL \oplus VL \quad , \quad TM_p = PVM \oplus VM.$$

Let $\bar{g}$ is a pseudo-Riemannian metric tensor, which is symmetric, bilinear and non-degenerate on $T_V(E)$.

$$\bar{g} : \mathcal{X}(T_V(E)) \times \mathcal{X}(T_V(E)) \to \mathcal{F}(T(E)).$$

(for examples $\bar{g}$ can be the vertical part of Lagrange metric structure).

The mapping

$$a : \mathcal{X}(T_V(E)) \times \mathcal{X}(T_V(E) \to \mathcal{F}(T(E))$$

which is defined by

$$a(X,Y) = \frac{1}{2}[\bar{g}(lX,lY) + \bar{g}(mX,mY)] \ \forall X,Y \in \mathcal{X}T_V(E)$$

is a pseudo-Riemannian structure on $T(E)$ such that $a(X,Y) = 0$, $\forall X \in \mathcal{X}(T(VL))$, $Y \in \mathcal{X}(T(VM))$.

Theorem 2.1. *If a Lagrange $f_v(2k+1,1)$-structure $k \geq 1$ of rank r is defined on $T_V(E)$ then there exist a pseudo-Riemannian structure of $T_V(E)$ with respect to the complementary distributions VL and VM are orthogonal and f_v is an isometry on $T_V(E)$.*

Proof. If we put

$$g(X,Y) = \frac{1}{2k}[a(X,Y) + a(f_vX, f_vY) + \cdots + a(f_v^{2k-1}X, f_v^{2k-1}Y)]$$

it is easy to see that

$$g(X,Y) = 0 \quad \forall X \in \mathcal{X}(VL), \quad Y \in \mathcal{X}(VM).$$

Using (2.1) we get

$$g(f_vX, f_vY) = \frac{1}{2k}[a(f_vX, f_vY) + a(f_v^2X, f_v^2Y) + \cdots + a(X,Y)].$$

Thus f_v is an isometry with respect to g.

Let $X \in \mathcal{X}(T(VL))$ then $f_v X$, $f_v^2 X, \ldots, f^{2k} X \in \mathcal{X}(T(VL))$ and

$$g(X, f_v^k X) = g(f_v X, f_v^{k+1} X) = \ldots = g(f_v^k X, f_v^{2k} X) = -g(f_v^k X, X).$$

Consequently

$$g(X, f_v^k X) = g(f_v X, f_v^{k+1} X) = \ldots = g(f_v^k X, f_v^{2k} X) = 0$$

and $r = 2km$.

Thus we can chose in $\mathcal{X}(T(VL))$ $r = 2km$ mutualy orthogonal unit vector fields such that

$$\begin{aligned} f(X_a) &= X_{\alpha+m} & \alpha &= 1, 2, \ldots, (2k-1)m, \\ f(X_\alpha) &= -X_{-(2k-1)m+\alpha}, & \alpha &= (2k-1)m+1, \ldots, 2km. \end{aligned}$$

An adapted frame of the Lagrange $f_v(2k+1,1)$-structure on $T_V(E)$ is the orthogonal frame $R = \{X_\alpha, X_\beta\}$, where X_β is an orthogonal frame of $\mathcal{X}(T(VM))$.

Let $\bar{R} = \{\bar{X}_\alpha, \bar{X}_\beta\}$ be another adapted frame of the Lagrange $f_v(2k+1,1)$-structure, and $\bar{R} = AR$, then orthogonal matrix A is an element of the group $U_{(km)} \times O_{(n-2km)}$.

Theorem 2.2. *A necessary and suficient condition for $T_V(E)$ to admit Lagrange $f_v(2k+1,1)$-structure, $k \geq 1$ of rank r is that $r = 2km$ and the structure group of the tangent bundle of the manifold be reduced to the group $U_{(km)} \times O_{(n-2km)}$.*

We can define a maping g_p:

$$g_p(X, Y) = g(PX, PY), \quad \forall X, Y \in \mathcal{X}(T_H(E))$$

g_p is a metric structure on $T_H(E)$. Using (2.5) the distributions HL_p, HM_p are orthogonal with respect to g_p and the horizontal $f_p(2k+1,1)$-structure which is define by $f_p X = P f_v P X$, $\forall X \in \mathcal{X}(T_H(E))$ is an isometry on $T_H(E)$ with respect to g_p.

Proposition 2.3. *If $\{X_\alpha, X_\beta\}$ is an adapted frame of a given Lagrange $f_v(2k+1,1)$-structure f_v on $T_V(E)$ with respect to g, then the frame $\{PX_\alpha, PX_\beta\}$ is an adapted frame of the horizontal $f_p(2k+1,1)$-structure with respect to g_p.*

It is clear that the frames $\{PX_\alpha, PX_\beta, X_\alpha, X_\beta\}$ are adapted frames to the decomposition

$$T(E) = HL_p \oplus HM_p \oplus VL \oplus VM.$$

Theorem 2.3. *If a Lagrange $f_v(2k+1,1)$-structure is defined on $T_V(E)$ with pseudo-Riemannian structure g, then the structure group of the tangent bundle on $T(E)$ be reduced to $U_{(km)} \times O_{(n-2km)} \times U_{(km)} \times O_{(n-2km)}$.*

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