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ON A FAMILY OF TENSOR FIELDS IN A GENERALIZED RIEMANNIAN SPACE

Svetislav M. Minčić

ABSTRACT. *In a subspace GR_M of a generalized Riemannian space GR_N we observe a familly of tensor fields (1.1), which contains as particular cases tangent and normal vectors of the subspace as well the curvature vector q^α of a curve in the subspace. Because of non-symmetry of Cristoffel symbols we define four kinds of derivational formulas of the above mentioned familly, as well six integrability conditions of these formulas. As particular cases one obtains Gauss-Codazzi eqations of the subspace and corensponding eqations for q^α. In this manner derivational formulas of Riemannian space are generalized, as well as their integrability conditions, i.e. the Gauss-Codazzi equations.*

0. Introduction

A generalized Riemannian space GR_N in the sense of Eisenhart [1], [2] is a differentiable manifold in which a non-symmetric basic tensor $a_{\alpha\beta} \neq a_{\beta\alpha}$ is introduced. If in the GR_N the coordinates are y^α $(\alpha = 1, ..., N)$, then by the equations

$$y^\alpha = y^\alpha(x^1, ..., x^M) \ \ (M < N) \tag{0.1}$$

a subspace GR_M of the space GR_N is defined. If g_{ij} is the basic tensor of this subspace, then in general $g_{ij} \neq g_{ji}$. In every point of the subspace we can observe $N - M$ unit, mutually orthogonal vectors $N^\alpha_{(\rho)}$ $(\rho = M + 1, ..., N)$, which are also orthogonal to GR_M, i.e. to the vectors (for fixed i)

$$t^\alpha_i = y^\alpha_{,i} = \partial y^\alpha / \partial x^i, \tag{0.2}$$

where the comma (,) signifies a partial derivative. We remark that in this work the Greek indices take values $1, ..., N$ and the Latin values $1, ..., M$ $(M < N)$, except indices in brackets, which take values $M + 1, ..., N$.

Let $g_{\underline{ij}}$ signify the symmetrisation, and $g_{\underset{\vee}{ij}}$ antisymmetrsation with respect to i, j and analogically in other cases. Then ([7], [8]):

$$a_{\alpha\beta} t^\alpha_i t^\beta_j = g_{ij}, \tag{0.3}$$

$$a_{\underline{\alpha\beta}} t^\alpha_i t^\beta_j = g_{\underline{ij}}, \qquad a_{\underset{\vee}{\alpha\beta}} t^\alpha_i t^\beta_j = g_{\underset{\vee}{ij}}, \tag{0.4 a,b}$$

$$a_{\underline{\alpha\pi}} a^{\underline{\pi\beta}} = \delta^\beta_\alpha, \qquad g_{\underline{ip}} g^{\underline{pj}} = \delta^j_i, \tag{0.5 a,b}$$

$$a_{\underline{\alpha\beta}} N^\alpha_{(\rho)} N^\beta_{(\sigma)} = e_{(\rho)} \delta_{(\rho\sigma)} \qquad (e_{(\rho)} = \pm 1), \quad a_{\underline{\alpha\beta}} N^\alpha_{(\rho)} t^\beta_j = 0. \tag{0.6 a,b}$$

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The Cristoffel symbols of the GR_N are given by

$$\Gamma_{\alpha.\beta\gamma} = \frac{1}{2}(a_{\beta\alpha,\gamma} - a_{\beta\gamma,\alpha} + a_{\alpha\gamma,\beta}), \quad \Gamma^{\alpha}_{\beta\gamma} = a^{\underline{\alpha\pi}}\Gamma_{\pi.\beta\gamma} \tag{0.7a, b}$$

and analogically for GR_M by g_{ij}. Then we have, for example,

$$\Gamma_{\alpha.\beta\gamma} \neq \Gamma_{\alpha.\gamma\beta}, \quad \Gamma^{\alpha}_{\beta\gamma} \neq \Gamma^{\alpha}_{\gamma\beta}.$$

Because of non-summetry of Cristoffel symbols, we can define 4 kinds of covariant derivative [3], [4]. For example,

$$\begin{aligned} t^{\alpha}_{i\underset{1}{|}m} &= t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\pi\mu} t^{\pi}_{i} t^{\mu}_{m} - \Gamma^{p}_{im} t^{\alpha}_{p}, \\ t^{\alpha}_{i\underset{2}{|}m} &= t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\mu\pi} t^{\pi}_{i} t^{\mu}_{m} - \Gamma^{p}_{mi} t^{\alpha}_{p}, \\ t^{\alpha}_{i\underset{3}{|}m} &= t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\pi\mu} t^{\pi}_{i} t^{\mu}_{m} - \Gamma^{p}_{mi} t^{\alpha}_{p}, \\ t^{\alpha}_{i\underset{4}{|}m} &= t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\mu\pi} t^{\pi}_{i} t^{\mu}_{m} - \Gamma^{p}_{im} t^{\alpha}_{p}, \end{aligned} \tag{0.8a}$$

$$\begin{aligned} N^{\alpha}_{(\rho)\underset{1}{|}m} &= N^{\alpha}_{(\rho)\underset{3}{|}m} = N^{\alpha}_{(\rho),m} + \Gamma^{\alpha}_{\pi\mu} N^{\pi}_{(\rho)} t^{\mu}_{m}, \\ N^{\alpha}_{(\rho)\underset{2}{|}m} &= N^{\alpha}_{(\rho)\underset{4}{|}m} = N^{\alpha}_{(\rho),m} + \Gamma^{\alpha}_{\mu\pi} N^{\pi}_{(\rho)} t^{\mu}_{m} \end{aligned} \tag{0.8b}$$

We also obtain 4 kinds of derivational formulas (see (16) and (37′) in [5]):

$$t^{\alpha}_{i\underset{\theta}{|}m} = \underset{\theta}{\Phi}{}^{p}_{im} t^{\alpha}_{p} + \sum_{\rho} \underset{\theta}{\Omega}_{(\rho)im} N^{\alpha}_{(\rho)}, \tag{0.9a}$$

$$N^{\alpha}_{(\sigma)\underset{\theta}{|}m} = -e_{(\sigma)} g^{\underline{ps}} \underset{\theta}{\Omega}_{(\sigma)sm} t^{\alpha}_{p} + \sum_{\rho} \underset{\theta}{\psi}_{(\rho\sigma)m} N^{\alpha}_{(\rho)}, \quad \underset{\theta}{\psi}_{(\sigma\sigma)m} = 0, \tag{0.9b}$$

where $\theta = 1, 2, 3, 4$ signifies the kind of covariant derivative.

On the base of (0.8b) it is

$$\underset{1}{\Omega}_{(\rho)ij} = \underset{3}{\Omega}_{(\rho)ij}, \quad \underset{2}{\Omega}_{(\rho)ij} = \underset{4}{\Omega}_{(\rho)ij}, \tag{0.10a}$$

$$\underset{1}{\psi}_{(\rho\sigma)m} = \underset{3}{\psi}_{(\rho\sigma)m}, \quad \underset{2}{\psi}_{(\rho\sigma)m} = \underset{4}{\psi}_{(\rho\sigma)m}, \tag{0.10b}$$

and with respect to (48′, 24′) in [5] we have

$$\underset{2}{\Phi}{}^{h}_{im} = -\underset{1}{\Phi}{}^{h}_{im}, \quad \underset{3}{\Phi}{}^{h}_{im} = \underset{1}{\Phi}{}^{h}_{im} + 2\Gamma^{h}_{\underset{\vee}{im}}, \quad \underset{4}{\Phi}{}^{h}_{im} = -\underset{1}{\Phi}{}^{h}_{im} - 2\Gamma^{h}_{\underset{\vee}{im}} \tag{0.11}$$

1. Derivational formula of the field and integrability conditions

Suppouse that in the points of the GR_M a family of tensor fields is defined:

$$\lambda^{\alpha}_{(\tau)i} = b^{s}_{(\tau)i} t^{\alpha}_{s} + \sum_{\rho=M+1}^{N} c_{(\rho\tau)i} N^{\alpha}_{(\rho)}. \tag{1.1}$$

Applying to (1.1) the covariant derivative of kind $\mu \in \{1,2,3,4\}$ with respect to x^m and using (0.9), we obtain

$$\lambda^{\alpha}_{(\tau)i\underset{\mu}{|}m} = \underset{\mu}{q}^{p}_{(\tau)im} t^{\alpha}_{p} + \sum_{\rho} \underset{\mu}{r}_{(\rho\tau)im} N^{\alpha}_{(\rho)}, \tag{1.2}$$

where

$$\underset{\mu}{q}^{p}_{(\tau)im} = b^{p}_{(\tau)i\underset{\mu}{|}m} + b^{s}_{(\tau)i} \underset{\mu}{\Phi}^{p}_{sm} - \sum_{\rho} e_{(\rho)} c_{(\rho\tau)i} g^{\underline{ps}} \underset{\mu}{\Omega}_{(\rho)sm}, \tag{1.3a}$$

$$\underset{\mu}{r}_{(\rho\tau)im} = b^{s}_{(\tau)i} \underset{\mu}{\Omega}_{(\rho)sm} + c_{(\rho\tau)i\underset{\mu}{|}m} + \sum_{\sigma} c_{(\sigma\tau)i} \underset{\mu}{\psi}_{(\rho\sigma)m}. \tag{1.3b}$$

The formula (1.2) is *derivational formula of the field* $\lambda^{\alpha}_{(\tau)i}$.

Applying to (1.2) covariant derivative of kind $\nu \in \{1,2,3,4\}$ with respect to x^n and using (0.9) repeatedly, we get

$$\begin{aligned} \lambda^{\alpha}_{(\tau)i\underset{\mu}{|}m\underset{\nu}{|}n} = {} & (\underset{\mu}{q}^{p}_{(\tau)im\underset{\nu}{|}n} + \underset{\mu}{q}^{s}_{(\tau)im} \underset{\nu}{\Phi}^{p}_{sn} - \sum_{\sigma} e_{(\sigma)} g^{\underline{ps}} \underset{\mu}{r}_{(\sigma\tau)im} \underset{\nu}{\Omega}_{(\sigma)sn}) t^{\alpha}_{p} + \\ & + \sum_{\rho} (\underset{\mu}{q}^{p}_{(\tau)im} \underset{\nu}{\Omega}_{(\rho)pn} + \underset{\mu}{r}_{(\rho\tau)im\underset{\nu}{|}n} + \sum_{\sigma} \underset{\mu}{r}_{(\sigma\tau)im} \underset{\nu}{\psi}_{(\rho\sigma)n}) N^{\alpha}_{(\rho)}, \end{aligned} \tag{1.4}$$

where the tensors $\underset{\mu}{q}^{p}_{(\tau)im}$, $\underset{\mu}{r}_{(\rho\tau)im}$ are given by (1.3). From here we obtain

$$\begin{aligned} \lambda^{\alpha}_{(\tau)i\underset{\mu}{|}m\underset{\nu}{|}n} - \lambda^{\alpha}_{(\tau)i\underset{\nu}{|}n\underset{\mu}{|}m} = {} & [\underset{\mu}{q}^{p}_{(\tau)im\underset{\nu}{|}n} - \underset{\nu}{q}^{p}_{(\tau)in\underset{\mu}{|}m} + \underset{\mu}{q}^{s}_{(\tau)im} \underset{\nu}{\Phi}^{p}_{sn} - \underset{\nu}{q}^{s}_{(\tau)in} \underset{\mu}{\Phi}^{p}_{sm} - \\ & - \sum_{\sigma} e_{(\sigma)} g^{\underline{ps}} (\underset{\mu}{r}_{(\sigma\tau)im} \underset{\nu}{\Omega}_{(\sigma)sn} - \underset{\nu}{r}_{(\sigma\tau)in} \underset{\mu}{\Omega}_{(\sigma)sm})] t^{\alpha}_{p} + \\ & + \sum_{\rho} [\underset{\mu}{q}^{p}_{(\tau)im} \underset{\nu}{\Omega}_{(\rho)pn} - \underset{\nu}{q}^{p}_{(\tau)in} \underset{\mu}{\Omega}_{(\rho)pm} + \underset{\mu}{r}_{(\rho\tau)im\underset{\nu}{|}n} - \underset{\nu}{r}_{(\rho\tau)in\underset{\mu}{|}m} + \\ & + \sum_{\sigma} (\underset{\mu}{r}_{(\sigma\tau)im} \underset{\nu}{\psi}_{(\rho\sigma)n} - \underset{\nu}{r}_{(\sigma\tau)in} \underset{\mu}{\psi}_{(\rho\sigma)m})] N^{\alpha}_{(\rho)}. \end{aligned} \tag{1.5}$$

Applying the identities of Ricci-type (7), (11), (56) from [3] and (12), (13), (46) from [4] to the family of tensor fields $\lambda^{\alpha}_{(\tau)i}$, we get

$$(1.6)\qquad \lambda^{\alpha}_{(\tau)i\underset{1}{|}m\underset{1}{|}n}-\lambda^{\alpha}_{(\tau)i\underset{1}{|}n\underset{1}{|}m}=\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{1}{R}{}^{p}_{imn}\lambda^{\alpha}_{(\tau)p}-2\Gamma^{p}_{\underset{\vee}{mn}}\lambda^{\alpha}_{(\tau)i\underset{1}{|}p},$$

$$(1.7)\qquad \lambda^{\alpha}_{(\tau)i\underset{2}{|}m\underset{2}{|}n}-\lambda^{\alpha}_{(\tau)i\underset{2}{|}n\underset{2}{|}m}=\underset{2}{R}{}^{\alpha}_{\pi\mu\nu}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{2}{R}{}^{p}_{imn}\lambda^{\alpha}_{(\tau)p}+2\Gamma^{p}_{\underset{\vee}{mn}}\lambda^{\alpha}_{(\tau)i\underset{2}{|}p},$$

$$(1.8)\qquad \lambda^{\alpha}_{(\tau)i\underset{1}{|}m\underset{2}{|}n}-\lambda^{\alpha}_{(\tau)i\underset{2}{|}n\underset{1}{|}m}=\underset{3}{R}{}^{\alpha}_{\pi mn}\lambda^{\pi}_{(\tau)i}-\underset{3}{R}{}^{p}_{imn}\lambda^{\alpha}_{(\tau)p},$$

$$(1.9)\qquad \lambda^{\alpha}_{(\tau)i\underset{3}{|}m\underset{3}{|}n}-\lambda^{\alpha}_{(\tau)i\underset{3}{|}n\underset{3}{|}m}=\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{2}{R}{}^{p}_{imn}\lambda^{\alpha}_{(\tau)p}+2\Gamma^{p}_{\underset{\vee}{mn}}\lambda^{\alpha}_{(\tau)i\underset{3}{|}p},$$

$$(1.10)\qquad \lambda^{\alpha}_{(\tau)i\underset{4}{|}m\underset{4}{|}n}-\lambda^{\alpha}_{(\tau)i\underset{4}{|}n\underset{4}{|}m}=\underset{2}{R}{}^{\alpha}_{\pi\mu\nu}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{1}{R}{}^{p}_{imn}\lambda^{\alpha}_{(\tau)p}-2\Gamma^{p}_{\underset{\vee}{mn}}\lambda^{\alpha}_{(\tau)i\underset{4}{|}p},$$

$$(1.11)\qquad \lambda^{\alpha}_{(\tau)i\underset{3}{|}m\underset{4}{|}n}-\lambda^{\alpha}_{(\tau)i\underset{4}{|}n\underset{3}{|}m}=\underset{4}{R}{}^{\alpha}_{\pi mn}\lambda^{\pi}_{(\tau)i}+\underset{3}{R}{}^{p}_{inm}\lambda^{\alpha}_{(\tau)p},$$

where

$$(1.12)\qquad \underset{1}{R}{}^{i}_{jmn}=\Gamma^{i}_{jm,n}-\Gamma^{i}_{jn,m}+\Gamma^{p}_{jm}\Gamma^{i}_{pn}-\Gamma^{p}_{jn}\Gamma^{i}_{pm},$$

$$(1.13)\qquad \underset{2}{R}{}^{i}_{jmn}=\Gamma^{i}_{mj,n}-\Gamma^{i}_{nj,m}+\Gamma^{p}_{mj}\Gamma^{i}_{np}-\Gamma^{p}_{nj}\Gamma^{i}_{mp},$$

$$(1.14)\qquad \underset{3}{R}{}^{i}_{jmn}=\Gamma^{i}_{jm,n}-\Gamma^{i}_{nj,m}+\Gamma^{p}_{jm}\Gamma^{i}_{np}-\Gamma^{p}_{nj}\Gamma^{i}_{pm}+\Gamma^{p}_{nm}(\Gamma^{i}_{pj}-\Gamma^{i}_{jp}),$$

$$(1.15)\qquad \begin{aligned}\underset{3}{R}{}^{\alpha}_{\beta mn}=&(\Gamma^{\alpha}_{\beta\mu,\nu}-\Gamma^{\alpha}_{\nu\beta,\mu}+\Gamma^{\pi}_{\beta\mu}\Gamma^{\alpha}_{\nu\pi}-\Gamma^{\pi}_{\nu\beta}\Gamma^{\alpha}_{\pi\mu})t^{\mu}_{m}t^{\nu}_{n}+\\&+2\Gamma^{\alpha}_{\underset{\vee}{\beta\mu}}(y^{\mu}_{,mn}-\Gamma^{p}_{nm}t^{\mu}_{p}),\end{aligned}$$

$$(1.16)\qquad \begin{aligned}\underset{4}{R}{}^{\alpha}_{\beta mn}=&(\Gamma^{\alpha}_{\beta\mu,\nu}-\Gamma^{\alpha}_{\nu\beta,\mu}+\Gamma^{\pi}_{\beta\mu}\Gamma^{\alpha}_{\nu\pi}-\Gamma^{\pi}_{\nu\beta}\Gamma^{\alpha}_{\pi\mu})t^{\mu}_{m}t^{\nu}_{n}+\\&+2\Gamma^{\alpha}_{\underset{\vee}{\beta\mu}}(y^{\mu}_{,mn}-\Gamma^{p}_{mn}t^{\mu}_{p}),\end{aligned}$$

The magnitudes $\underset{t}{R}{}^{\alpha}_{\beta\mu\nu}$, $\underset{t}{R}{}^{i}_{jmn}$, $(t=1,2,3)$ are tensors and we call them curvature tensors of the space, and of the subspace, respectively the 1st, the 2nd and the 3rd kind. The magnitudes $\underset{3}{R}{}^{\alpha}_{\beta mn}$, $\underset{4}{R}{}^{\alpha}_{\beta mn}$ are tensors too and we call them curvature tensors of the 3rd, of the 4th kind of the space GR_N in relation to the subspace GR_M.

1.1. Taking in (1.5) $\mu=\nu=1$, in (1.6) replacing $\lambda^{\alpha}_{(\tau)i\underset{1}{|}p}$ by virtue of (1.2) and equating the obtained results, we obtain the *first integrability condition of the* *derivational formula*(1.2):

$$(1.17)\quad \begin{aligned}&\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n} - \underset{1}{R}{}^{p}_{imn}\lambda^{\alpha}_{(\tau)p} - 2\Gamma^{p}_{\underset{\vee}{mn}}\Big(\underset{1}{q}{}^{s}_{(\tau)ip}t^{\alpha}_{s} + \sum_{\rho}\underset{1}{r}_{(\rho\tau)ip}N^{\alpha}_{(\rho)}\Big) =\\ &= \Big[\underset{1}{q}{}^{p}_{(\tau)im\underset{1}{|}n} - \underset{1}{q}{}^{p}_{(\tau)in\underset{1}{|}m} + \underset{1}{q}{}^{s}_{(\tau)im}\underset{1}{\Phi}{}^{p}_{sn} - \underset{1}{q}{}^{s}_{(\tau)in}\underset{1}{\Phi}{}^{p}_{sm} -\\ &-\sum_{\sigma}e_{(\sigma)}g^{\underline{ps}}\big(\underset{1}{r}_{(\sigma\tau)im}\underset{1}{\Omega}_{(\sigma)sn} - \underset{1}{r}_{(\sigma\tau)in}\underset{1}{\Omega}_{(\sigma)sm}\big)\Big]t^{\alpha}_{p} +\\ &+\sum_{\rho}\Big[\underset{1}{q}{}^{p}_{(\tau)im}\underset{1}{\Omega}_{(\rho)pn} - \underset{1}{q}{}^{p}_{(\tau)in}\underset{1}{\Omega}_{(\rho)pm} + \underset{1}{r}_{(\rho\tau)im\underset{1}{|}n} - \underset{1}{r}_{(\rho\tau)in\underset{1}{|}m} +\\ &+\sum_{\sigma}\big(\underset{1}{r}_{(\sigma\tau)im}\underset{1}{\psi}_{(\rho\sigma)n} - \underset{1}{r}_{(\sigma\tau)in}\underset{1}{\psi}_{(\rho\sigma)m}\big)\Big]N^{\alpha}_{(\rho)}.\end{aligned}$$

Multiplying (1.17) by $a_{\underline{\alpha\beta}}t^{\beta}_{h}$ and using (0.4a), (0.6b), (0.5b), we obtain

$$(1.17')\quad \begin{aligned}&\underset{1}{R}_{\beta\pi\mu\nu}t^{\beta}_{h}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n} - \underset{1}{R}{}^{p}_{imn}a_{\underline{\alpha\beta}}t^{\beta}_{h}\lambda^{\alpha}_{(\tau)p} - 2\Gamma^{p}_{\underset{\vee}{mn}}\underset{1}{q}{}^{s}_{(\tau)ip}g_{\underline{hs}} =\\ &= \big(\underset{1}{q}{}^{p}_{(\tau)im\underset{1}{|}n} - \underset{1}{q}{}^{p}_{(\tau)in\underset{1}{|}m} + \underset{1}{q}{}^{s}_{(\tau)im}\underset{1}{\Phi}{}^{p}_{sn} - \underset{1}{q}{}^{s}_{(\tau)in}\underset{1}{\Phi}{}^{p}_{sm}\big)g_{\underline{hp}} -\\ &-\sum_{\sigma}e_{(\sigma)}\big(\underset{1}{r}_{(\sigma\tau)im}\underset{1}{\Omega}_{(\sigma)hn} - \underset{1}{r}_{(\sigma\tau)in}\underset{1}{\Omega}_{(\sigma)hm}\big).\end{aligned}$$

If we multiply (1.17) by $a_{\underline{\alpha\beta}}N^{\beta}_{(\varphi)}$ and take into consideration (0.6), we get

$$(1.17'')\quad \begin{aligned}&\underset{1}{R}_{\beta\pi\mu\nu}N^{\beta}_{(\varphi)}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n} - \underset{1}{R}{}^{p}_{imn}a_{\underline{\alpha\beta}}\lambda^{\alpha}_{(\tau)p}N^{\beta}_{(\varphi)} - 2\Gamma^{p}_{\underset{\vee}{mn}}\underset{1}{r}_{(\varphi\tau)ip}e_{(\varphi)} =\\ &= e_{(\varphi)}\Big[\underset{1}{q}{}^{p}_{(\tau)im}\underset{1}{\Omega}_{(\varphi)pn} - \underset{1}{q}{}^{p}_{(\tau)in}\underset{1}{\Omega}_{(\varphi)pm} + \underset{1}{r}_{(\varphi\tau)im\underset{1}{|}n} - \underset{1}{r}_{(\varphi\tau)in\underset{1}{|}m} +\\ &+\sum_{\sigma}\big(\underset{1}{r}_{(\sigma\tau)im}\underset{1}{\psi}_{(\varphi\sigma)n} - \underset{1}{r}_{(\sigma\tau)in}\underset{1}{\psi}_{(\varphi\sigma)m}\big)\Big].\end{aligned}$$

1.2. Taking $\mu = \nu = 2$ in (1.5) and equating with (1.7), using (1.2), we obtain the *second integrability condition of the derivational formula*(1.2):

$$(1.18)\quad \begin{aligned}&\underset{2}{R}{}^{\alpha}_{\pi\mu\nu}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n} - \underset{2}{R}{}^{p}_{imn}\lambda^{\alpha}_{(\tau)p} + 2\Gamma^{p}_{\underset{\vee}{mn}}\Big(\underset{1}{q}{}^{s}_{(\tau)ip}t^{\alpha}_{s} + \sum_{\rho}\underset{2}{r}_{(\rho\tau)ip}N^{\alpha}_{(\rho)}\Big) =\\ &= \Big[\underset{2}{q}{}^{p}_{(\tau)im\underset{2}{|}n} - \underset{2}{q}{}^{p}_{(\tau)in\underset{2}{|}m} + \underset{2}{q}{}^{s}_{(\tau)im}\underset{2}{\Phi}{}^{p}_{sn} - \underset{2}{q}{}^{s}_{(\tau)in}\underset{2}{\Phi}{}^{p}_{sm} -\\ &-\sum_{\sigma}e_{(\sigma)}g^{\underline{ps}}\big(\underset{2}{r}_{(\sigma\tau)im}\underset{2}{\Omega}_{(\sigma)sn} - \underset{2}{r}_{(\sigma\tau)in}\underset{2}{\Omega}_{(\sigma)sm}\big)\Big]t^{\alpha}_{p} +\\ &+\sum_{\rho}\Big[\underset{2}{q}{}^{p}_{(\tau)im}\underset{2}{\Omega}_{(\rho)pn} - \underset{2}{q}{}^{p}_{(\tau)in}\underset{2}{\Omega}_{(\rho)pm} + \underset{2}{r}_{(\rho\tau)im\underset{2}{|}n} - \underset{2}{r}_{(\rho\tau)in\underset{2}{|}m} +\\ &+\sum_{\sigma}\big(\underset{2}{r}_{(\sigma\tau)im}\underset{2}{\psi}_{(\rho\sigma)n} - \underset{2}{r}_{(\sigma\tau)in}\underset{2}{\psi}_{(\rho\sigma)m}\big)\Big]N^{\alpha}_{(\rho)}.\end{aligned}$$

Multiplying (1.18) by $a_{\underline{\alpha\beta}}t^{\beta}_{h}$, we obtain

$$\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu}t^{\beta}_{h}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{2}{R}^{p}_{imn}a_{\underline{\alpha\beta}}t^{\beta}_{h}\lambda^{\alpha}_{(\tau)p}+2\Gamma^{p}_{\underset{\vee}{mn}}\underset{2}{q}^{s}_{(\tau)ip}g_{\underline{hs}}=\\
&=(\underset{2}{q}^{p}_{(\tau)im\underset{2}{|}n}-\underset{2}{q}^{p}_{(\tau)in\underset{2}{|}m}+\underset{2}{q}^{s}_{(\tau)im}\underset{2}{\Phi}^{p}_{sn}-\underset{2}{q}^{s}_{(\tau)in}\underset{2}{\Phi}^{p}_{sm})g_{\underline{hp}}-\\
&-\sum_{\sigma}e_{(\sigma)}(\underset{2}{r}_{(\sigma\tau)im}\underset{2}{\Omega}_{(\sigma)hn}-\underset{2}{r}_{(\sigma\tau)in}\underset{2}{\Omega}_{(\sigma)hm}),
\end{aligned}\tag{1.18'}$$

and multiplying the same equation by $a_{\underline{\alpha\beta}}N^{\beta}_{(\varphi)}$:

$$\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu}N^{\beta}_{(\varphi)}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{2}{R}^{p}_{imn}a_{\underline{\alpha\beta}}\lambda^{\alpha}_{(\tau)p}N^{\beta}_{(\varphi)}+2\Gamma^{p}_{\underset{\vee}{mn}}\underset{2}{r}_{(\varphi\tau)ip}e_{(\varphi)}=\\
&=e_{(\varphi)}[\underset{2}{q}^{p}_{(\tau)im}\underset{2}{\Omega}_{(\varphi)pn}-\underset{2}{q}^{p}_{(\tau)in}\underset{2}{\Omega}_{(\varphi)pm}+\underset{2}{r}_{(\varphi\tau)im\underset{2}{|}n}-\underset{2}{r}_{(\varphi\tau)in\underset{2}{|}m}+\\
&+\sum_{\sigma}(\underset{2}{r}_{(\sigma\tau)im}\underset{2}{\psi}_{(\varphi\sigma)n}-\underset{2}{r}_{(\sigma\tau)in}\underset{2}{\psi}_{(\varphi\sigma)m})].
\end{aligned}\tag{1.18''}$$

1.3. If in (1.5) we replace $\mu=1$, $\nu=2$ and use (1.8), we get *the third integrability condition of the derivational formula* (1.2):

$$\begin{aligned}
&\underset{3}{R}^{\alpha}_{\pi mn}\lambda^{\pi}_{(\tau)i}-\underset{3}{R}^{p}_{imn}\lambda^{\alpha}_{(\tau)p}=\\
&=[\underset{1}{q}^{p}_{(\tau)im\underset{2}{|}n}-\underset{2}{q}^{p}_{(\tau)in\underset{1}{|}m}+\underset{1}{q}^{s}_{(\tau)im}\underset{2}{\Phi}^{p}_{sn}-\underset{2}{q}^{s}_{(\tau)in}\underset{1}{\Phi}^{p}_{sm}-\\
&-\sum_{\sigma}e_{(\sigma)}g^{\underline{ps}}(\underset{1}{r}_{(\sigma\tau)im}\underset{2}{\Omega}_{(\sigma)sn}-\underset{2}{r}_{(\sigma\tau)in}\underset{1}{\Omega}_{(\sigma)sm})]t^{\alpha}_{p}+\\
&+\sum_{\rho}[\underset{1}{q}^{p}_{(\tau)im}\underset{2}{\Omega}_{(\rho)pn}-\underset{2}{q}^{p}_{(\tau)in}\underset{1}{\Omega}_{(\rho)pm}+\underset{1}{r}_{(\rho\tau)im\underset{2}{|}n}-\underset{2}{r}_{(\rho\tau)in\underset{1}{|}m}+\\
&+\sum_{\sigma}(\underset{1}{r}_{(\sigma\tau)im}\underset{2}{\psi}_{(\rho\sigma)n}-\underset{2}{r}_{(\sigma\tau)in}\underset{1}{\psi}_{(\rho\sigma)m})]N^{\alpha}_{(\rho)}.
\end{aligned}\tag{1.19}$$

Multiplying the pervious equation by $a_{\underline{\alpha\beta}}t^{\beta}_{h}$, we obtain

$$\begin{aligned}
&\underset{3}{R}_{\beta\pi mn}t^{\beta}_{h}\lambda^{\pi}_{(\tau)i}-\underset{3}{R}^{p}_{imn}a_{\underline{\alpha\beta}}\lambda^{\alpha}_{(\tau)p}t^{\beta}_{h}=\\
&=(\underset{1}{q}^{p}_{(\tau)im\underset{2}{|}n}-\underset{2}{q}^{p}_{(\tau)in\underset{1}{|}m}+\underset{1}{q}^{s}_{(\tau)im}\underset{2}{\Phi}^{p}_{sn}-\underset{2}{q}^{s}_{(\tau)in}\underset{1}{\Phi}^{p}_{sm})g_{\underline{hp}}-\\
&-\sum_{\sigma}e_{(\sigma)}(\underset{1}{r}_{(\sigma\tau)im}\underset{2}{\Omega}_{(\sigma)hn}-\underset{2}{r}_{(\sigma\tau)in}\underset{1}{\Omega}_{(\sigma)hm}).
\end{aligned}\tag{1.19'}$$

where $\underset{3}{R}_{\beta\pi mn}=a_{\underline{\alpha\beta}}\underset{3}{R}^{\alpha}_{\pi mn}$, and $\underset{3}{R}^{\alpha}_{\pi mn}$ is given by (1.15).

Multipling (1.19) by $a_{\underline{\alpha\beta}}N^{\beta}_{(\varphi)}$, we get

$$\begin{aligned}
&\underset{3}{R}_{\beta\pi mn}N^{\beta}_{(\varphi)}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{3}{R}^{p}_{imn}a_{\underline{\alpha\beta}}\lambda^{\alpha}_{(\tau)p}N^{\beta}_{(\varphi)}=\\
(1.19")\qquad &=e_{(\varphi)}[\underset{1}{q}^{p}_{(\tau)im}\underset{2}{\Omega}_{(\varphi)pn}-\underset{2}{q}^{p}_{(\tau)in}\underset{1}{\Omega}_{(\varphi)pm}+\underset{1}{r}_{(\varphi\tau)im\underset{2}{|}n}-\underset{2}{r}_{(\varphi\tau)in\underset{1}{|}m}+\\
&+\sum_{\sigma}(\underset{1}{r}_{(\sigma\tau)im}\underset{2}{\psi}_{(\varphi\sigma)n}-\underset{2}{r}_{(\sigma\tau)in}\underset{1}{\psi}_{(\varphi\sigma)m})].
\end{aligned}$$

1.4. By replacing $\mu=\nu=3$ in (1.5) and applying (1.9), we obtain *the fourth integrability condition of the derivational formula* (1.2):

$$\begin{aligned}
&\underset{1}{R}^{\alpha}_{\pi\mu\nu}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{2}{R}^{p}_{imn}\lambda^{\alpha}_{(\tau)p}+2\Gamma^{p}_{\underset{\vee}{mn}}(\underset{3}{q}^{s}_{(\tau)ip}t^{\alpha}_{s}+\sum_{\rho}\underset{3}{r}_{(\rho\tau)ip}N^{\alpha}_{(\rho)})=\\
&[\underset{3}{q}^{p}_{(\tau)im\underset{3}{|}n}-\underset{3}{q}^{p}_{(\tau)in\underset{3}{|}m}+\underset{3}{q}^{s}_{(\tau)im}\underset{3}{\Phi}^{p}_{sn}-\underset{3}{q}^{s}_{(\tau)in}\underset{3}{\Phi}^{p}_{sm}-\\
(1.20)\qquad &-\sum_{\sigma}e_{(\sigma)}g^{\underline{ps}}(\underset{3}{r}_{(\sigma\tau)im}\underset{3}{\Omega}_{(\sigma)sn}-\underset{3}{r}_{(\sigma\tau)in}\underset{3}{\Omega}_{(\sigma)sm})]t^{\alpha}_{p}+\\
&+\sum_{\rho}[\underset{3}{q}^{p}_{(\tau)im}\underset{3}{\Omega}_{(\rho)pn}-\underset{3}{q}^{p}_{(\tau)in}\underset{3}{\Omega}_{(\rho)pm}+\underset{3}{r}_{(\rho\tau)im\underset{3}{|}n}-\underset{3}{r}_{(\rho\tau)in\underset{3}{|}m}+\\
&+\sum_{\sigma}(\underset{3}{r}_{(\sigma\tau)im}\underset{3}{\psi}_{(\rho\sigma)n}-\underset{3}{r}_{(\sigma\tau)in}\underset{3}{\psi}_{(\rho\sigma)m})]N^{\alpha}_{(\rho)}.
\end{aligned}$$

If we multiply the pervious equation by $a_{\underline{\alpha\beta}}t^{\beta}_{h}$, we have

$$\begin{aligned}
&\underset{1}{R}_{\beta\pi mn}t^{\beta}_{h}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{2}{R}^{p}_{imn}a_{\underline{\alpha\beta}}\lambda^{\alpha}_{(\tau)p}t^{\beta}_{h}+2\Gamma^{p}_{\underset{\vee}{mn}}\underset{3}{q}^{s}_{(\tau)ip}g_{\underline{hs}}=\\
(1.20')\qquad &=(\underset{3}{q}^{p}_{(\tau)im\underset{3}{|}n}-\underset{3}{q}^{p}_{(\tau)in\underset{3}{|}m}+\underset{3}{q}^{s}_{(\tau)im}\underset{3}{\Phi}^{p}_{sn}-\underset{3}{q}^{s}_{(\tau)in}\underset{3}{\Phi}^{p}_{sm})g_{\underline{hp}}-\\
&-\sum_{\sigma}e_{(\sigma)}(\underset{3}{r}_{(\sigma\tau)im}\underset{3}{\Omega}_{(\sigma)hn}-\underset{3}{r}_{(\sigma\tau)in}\underset{3}{\Omega}_{(\sigma)hm}).
\end{aligned}$$

Multiplying (1.20) by $a_{\underline{\alpha\beta}}N^{\beta}_{(\varphi)}$, we get

$$\begin{aligned}
&\underset{1}{R}_{\beta\pi\mu\nu}N^{\beta}_{(\varphi)}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n}-\underset{2}{R}^{p}_{imn}a_{\underline{\alpha\beta}}\lambda^{\alpha}_{(\tau)p}N^{\beta}_{(\varphi)}+2\Gamma^{p}_{\underset{\vee}{mn}}\underset{3}{r}_{(\varphi\tau)ip}e_{(\varphi)}=\\
(1.20")\qquad &=e_{(\varphi)}[\underset{3}{q}^{p}_{(\tau)im}\underset{3}{\Omega}_{(\varphi)pn}-\underset{3}{q}^{p}_{(\tau)in}\underset{3}{\Omega}_{(\varphi)pm}+\underset{3}{r}_{(\varphi\tau)im\underset{3}{|}n}-\underset{3}{r}_{(\varphi\tau)in\underset{3}{|}m}+\\
&+\sum_{\sigma}(\underset{3}{r}_{(\sigma\tau)im}\underset{3}{\psi}_{(\varphi\sigma)n}-\underset{3}{r}_{(\sigma\tau)in}\underset{3}{\psi}_{(\varphi\sigma)m})].
\end{aligned}$$

1.5 By equalizing the right sides in (1.10) and (1.5) for $\mu=\nu=4$, we get *the* *fifth integrability condition of the derivational formula* (1.2):

$$(1.21)\quad \begin{aligned} &\underset{2}{R}^{\alpha}_{\pi\mu\nu}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n} - \underset{1}{R}^{p}_{imn}\lambda^{\alpha}_{(\tau)p} - 2\Gamma^{p}_{\underset{\vee}{mn}}(\underset{4}{q}^{s}_{(\tau)ip}t^{\alpha}_{s} + \sum_{\rho}\underset{4}{r}_{(\rho\tau)ip}N^{\alpha}_{(\rho)}) = \\ &[\underset{4}{q}^{p}_{(\tau)im\underset{4}{|}n} - \underset{4}{q}^{p}_{(\tau)in\underset{4}{|}m} + \underset{4}{q}^{s}_{(\tau)im}\underset{4}{\Phi}^{p}_{sn} - \underset{4}{q}^{s}_{(\tau)in}\underset{4}{\Phi}^{p}_{sm} - \\ &-\sum_{\sigma}e_{(\sigma)}g^{\underline{ps}}(\underset{4}{r}_{(\sigma\tau)im}\underset{4}{\Omega}_{(\sigma)sn} - \underset{4}{r}_{(\sigma\tau)in}\underset{4}{\Omega}_{(\sigma)sm})]t^{\alpha}_{p} + \\ &+\sum_{\rho}[\underset{4}{q}^{p}_{(\tau)im}\underset{4}{\Omega}_{(\rho)pn} - \underset{4}{q}^{p}_{(\tau)in}\underset{4}{\Omega}_{(\rho)pm} + \underset{4}{r}_{(\rho\tau)im\underset{4}{|}n} - \underset{4}{r}_{(\rho\tau)in\underset{4}{|}m} + \\ &+\sum_{\sigma}(\underset{4}{r}_{(\sigma\tau)im}\underset{4}{\psi}_{(\rho\sigma)n} - \underset{4}{r}_{(\sigma\tau)in}\underset{4}{\psi}_{(\rho\sigma)m})]N^{\alpha}_{(\rho)}. \end{aligned}$$

Multiplying the pervious equation by $a_{\underline{\alpha\beta}}t^{\beta}_{h}$, we obtain

$$(1.21')\quad \begin{aligned} &\underset{2}{R}_{\beta\pi\mu\nu}t^{\beta}_{h}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n} - \underset{1}{R}^{p}_{imn}a_{\underline{\alpha\beta}}\lambda^{\alpha}_{(\tau)p}t^{\beta}_{h} - 2\Gamma^{p}_{\underset{\vee}{mn}}\underset{4}{q}^{s}_{(\tau)ip}g_{\underline{hs}} = \\ &= (\underset{4}{q}^{p}_{(\tau)im\underset{4}{|}n} - \underset{4}{q}^{p}_{(\tau)in\underset{4}{|}m} + \underset{4}{q}^{s}_{(\tau)im}\underset{4}{\Phi}^{p}_{sn} - \underset{4}{q}^{s}_{(\tau)in}\underset{4}{\Phi}^{p}_{sm})g_{\underline{hp}} - \\ &-\sum_{\sigma}e_{(\sigma)}(\underset{4}{r}_{(\sigma\tau)im}\underset{4}{\Omega}_{(\sigma)hn} - \underset{4}{r}_{(\sigma\tau)in}\underset{4}{\Omega}_{(\sigma)hm}). \end{aligned}$$

and multiplying the same equation by $a_{\underline{\alpha\beta}}N^{\beta}_{(\varphi)}$:

$$(1.21'')\quad \begin{aligned} &\underset{2}{R}_{\beta\pi\mu\nu}N^{\beta}_{(\varphi)}\lambda^{\pi}_{(\tau)i}t^{\mu}_{m}t^{\nu}_{n} - \underset{1}{R}^{p}_{imn}a_{\underline{\alpha\beta}}\lambda^{\alpha}_{(\tau)p}N^{\beta}_{(\varphi)} - 2\Gamma^{p}_{\underset{\vee}{mn}}\underset{4}{r}_{(\varphi\tau)ip}e_{(\varphi)} = \\ &= e_{(\varphi)}[\underset{4}{q}^{p}_{(\tau)im}\underset{4}{\Omega}_{(\varphi)pn} - \underset{4}{q}^{p}_{(\tau)in}\underset{4}{\Omega}_{(\varphi)pm} + \underset{4}{r}_{(\varphi\tau)im\underset{4}{|}n} - \underset{4}{r}_{(\varphi\tau)in\underset{4}{|}m} + \\ &+\sum_{\sigma}(\underset{4}{r}_{(\sigma\tau)im}\underset{4}{\psi}_{(\varphi\sigma)n} - \underset{4}{r}_{(\sigma\tau)in}\underset{4}{\psi}_{(\varphi\sigma)m})]. \end{aligned}$$

1.6 Finally, if we equilaize the right sides in (1.11) and (1.5) for $\mu = 3$ $\nu = 4$, we obtain *the sixth integrability condition of the derivational formula* (1.2):

$$(1.22)\quad \begin{aligned} &\underset{4}{R}^{\alpha}_{\pi mn}\lambda^{\pi}_{(\tau)i} + \underset{3}{R}^{p}_{inm}\lambda^{\alpha}_{(\tau)p} = \\ &[\underset{3}{q}^{p}_{(\tau)im\underset{4}{|}n} - \underset{4}{q}^{p}_{(\tau)in\underset{3}{|}m} + \underset{3}{q}^{s}_{(\tau)im}\underset{4}{\Phi}^{p}_{sn} - \underset{4}{q}^{s}_{(\tau)in}\underset{3}{\Phi}^{p}_{sm} - \\ &-\sum_{\sigma}e_{\sigma}g^{\underline{ps}}(\underset{3}{r}_{(\sigma\tau)im}\underset{4}{\Omega}_{(\sigma)sn} - \underset{4}{r}_{(\sigma\tau)in}\underset{3}{\Omega}_{(\sigma)sm})]t^{\alpha}_{p} + \\ &+\sum_{\rho}[\underset{3}{q}^{p}_{(\tau)im}\underset{4}{\Omega}_{(\rho)pn} - \underset{4}{q}^{p}_{(\tau)in}\underset{3}{\Omega}_{(\rho)pm} + \underset{3}{r}_{(\rho\tau)im\underset{4}{|}n} - \underset{4}{r}_{(\rho\tau)in\underset{3}{|}m} + \\ &+\sum_{\sigma}(\underset{3}{r}_{(\sigma\tau)im}\underset{4}{\psi}_{(\rho\sigma)n} - \underset{4}{r}_{(\sigma\tau)in}\underset{3}{\psi}_{(\rho\sigma)m})]N^{\alpha}_{(\rho)}. \end{aligned}$$

If we multiply this equaton by $a_{\underline{\alpha\beta}} t^\beta_h$, we obtain

$$
\begin{aligned}
&\underset{4}{R}_{\beta\pi mn} t^\beta_h \lambda^\pi_{(\tau)i} + \underset{3}{R}^p_{inm} a_{\underline{\alpha\beta}} \lambda^\alpha_{(\tau)p} t^\beta_h = \\
(1.22') \qquad &= (\underset{3}{q}^p_{(\tau)im\underset{4}{|}n} - \underset{4}{q}^p_{(\tau)in\underset{3}{|}m} + \underset{3}{q}^s_{(\tau)im} \underset{4}{\Phi}^p_{sn} - \underset{4}{q}^s_{(\tau)in} \underset{3}{\Phi}^p_{sm}) g_{\underline{hp}} - \\
&-\sum_\sigma e_{(\sigma)} (\underset{3}{r}_{(\sigma\tau)im} \underset{4}{\Omega}_{(\sigma)hn} - \underset{4}{r}_{(\sigma\tau)in} \underset{3}{\Omega}_{(\sigma)hm}).
\end{aligned}
$$

where $\underset{4}{R}_{\beta\pi mn} = a_{\underline{\alpha\beta}} \underset{4}{R}^\alpha_{\pi mn}$ and $\underset{4}{R}^\alpha_{\pi mn}$ is given in relation to (1.16).

Multiplying (1.22) by $a_{\underline{\alpha\beta}} N^\beta_{(\varphi)}$, we have

$$
\begin{aligned}
&\underset{4}{R}_{\beta\pi mn} N^\beta_{(\varphi)} \lambda^\pi_{(\tau)i} + \underset{3}{R}^p_{inm} a_{\underline{\alpha\beta}} \lambda^\alpha_{(\tau)p} N^\beta_{(\varphi)} = \\
(1.22'') \qquad &= e_{(\varphi)} [\underset{3}{q}^p_{(\tau)im} \underset{4}{\Omega}_{(\varphi)pn} - \underset{4}{q}^p_{(\tau)in} \underset{3}{\Omega}_{(\varphi)pm} + \underset{3}{r}_{(\varphi\tau)im\underset{4}{|}n} - \underset{4}{r}_{(\varphi\tau)in\underset{3}{|}m} + \\
&+\sum_\sigma (\underset{3}{r}_{(\sigma\tau)im} \underset{4}{\psi}_{(\varphi\sigma)n} - \underset{4}{r}_{(\sigma\tau)in} \underset{3}{\psi}_{(\varphi\sigma)m})].
\end{aligned}
$$

2. Some special cases

For some fixed values of coeffitients b, c we obtain from (1.1) special cases, some of which are very importtant.

2.1. If

$$
(2.1) \qquad b^s_{(\tau)i} = \delta^s_i \quad \& \; c_{(\rho\tau)i} = 0, \quad \text{then} \quad \lambda^\alpha_{(\tau)i} = t^\alpha_i = y^\alpha_{,i}.
$$

In this case from (1.3) we obtain

$$
\underset{\mu}{q}^p_{(\tau)im} = \delta^s_{i\underset{\mu}{|}m} + \delta^s_i \underset{\mu}{\Phi}^p_{sm} = \underset{\mu}{\Phi}^p_{im}, \quad \underset{\mu}{r}_{(\rho\tau)im} = \delta^s_i \underset{\mu}{\Omega}_{(\rho)sm} = \underset{\mu}{\Omega}_{(\rho)im},
$$

and (1.2) gives (0.9a) i.e. the first derivational formula of a subspace of a generalized Riemannian space. In this case integrability conditions of the first derivational formula of the field (1.1) reduce to integrability conditions of the first derivational formula, from which we obtain several equations of the Gauss-Codazzi type. For example, in this case (1.19') becomes

$$
\begin{aligned}
&\underset{3}{R}_{\beta\pi mn} t^\beta_h t^\pi_i - \underset{3}{R}_{himn} = \\
(2.2) \qquad &= (\underset{1}{\Phi}^p_{im\underset{2}{|}n} - \underset{2}{\Phi}^p_{in\underset{1}{|}m} + \underset{1}{\Phi}^s_{im} \underset{2}{\Phi}^p_{sn} - \underset{2}{\Phi}^s_{in} \underset{1}{\Phi}^p_{sn}) g_{\underline{hp}} - \\
&-\sum_\sigma e_{(\sigma)} (\underset{1}{\Omega}_{(\sigma)im} \underset{2}{\Omega}_{(\sigma)hn} - \underset{2}{\Omega}_{(\sigma)in} \underset{1}{\Omega}_{(\sigma)hm}),
\end{aligned}
$$

and this is the third kind of the Gauss equation of the subspace GR_M. Now from (1.19") we obtain

$$(2.3)\qquad \begin{aligned} &\underset{3}{R}_{\beta\pi mn} N^\beta_{(\varphi)} t^\pi_i = \\ &= e_{(\varphi)}[\underset{1}{\Phi}^p_{im} \underset{2}{\Omega}_{(\varphi)pn} - \underset{2}{\Phi}^p_{in} \underset{1}{\Omega}_{(\varphi)pm} + \underset{1}{\Omega}_{\varphi)im\underset{2}{|}n} - \underset{2}{\Omega}_{\varphi in\underset{1}{|}m} + \\ &+ \sum_\sigma (\underset{1}{\Omega}_{(\sigma)im} \underset{2}{\psi}_{(\varphi\sigma)n} - \underset{2}{\Omega}_{(\sigma)in} \underset{1}{\psi}_{(\varphi\sigma)m})]. \end{aligned}$$

and this is the first Codazzi eqation of the third kind.

2.2. The next special case is

$$(2.4)\qquad b^s_{(\tau)i} = 0 \quad \& \quad c_{(\rho\tau)i} = \delta_{\rho\tau} \quad \Rightarrow \quad \lambda^\alpha_{(\tau)i} = N^\alpha_{(\tau)}.$$

Now, it is

$$\underset{\mu}{q}^p_{(\tau)im} = -\sum_\rho e_{(\rho)} \delta_{\rho\tau} g^{ps} \underset{\mu}{\Omega}_{(\rho)sm} = -g^{ps} \underset{\mu}{\Omega}_{(\tau)sm} e_{(\tau)}$$

$$\underset{\mu}{r}_{(\rho\tau)im} = \sum_\sigma \delta_{\sigma\tau} \underset{\mu}{\psi}_{(\rho\sigma)m} = \underset{\mu}{\psi}_{(\rho\tau)m},$$

and (1.2) results in (0.9b), i.e. in the second derivational formula of the subspace . In this case from (1.19') one obtains the equation which is equivalent to (2.3). The equation (1.19") in this case becomes

$$\begin{aligned} \underset{3}{R}_{\beta\pi mn} N^\beta_{(\varphi)} N^\pi_{(\tau)} &= e_{(\varphi)}[-g^{ps} \underset{1}{\Omega}_{(\tau)sm} e_{(\tau)} \underset{2}{\Omega}_{(\varphi)pn} + g^{ps} \underset{2}{\Omega}_{(\tau)sn} e_{(\tau)} \underset{1}{\Omega}_{(\varphi)pm} + \\ &+ \underset{1}{\psi}_{(\varphi\tau)m\underset{2}{|}n} - \underset{2}{\psi}_{(\varphi\tau)n\underset{1}{|}m} + \sum_\sigma (\underset{1}{\psi}_{(\sigma\tau)m} \underset{2}{\psi}_{(\varphi\sigma)n} - \underset{2}{\psi}_{(\sigma\tau)n} \underset{1}{\psi}_{(\varphi\sigma)m})] \end{aligned}$$

and this is the second Codazzi equation of the third kind.

2.3. Curvature vector of a curve C in a subspace GR_M of a space GR_N is determined in the same way [6], as in the usual Riemannian space, i.e.

$$(2.5)\qquad q^\alpha = t^\alpha_i p^i + \sum_\rho K_{(\rho)} N^\alpha_{(\rho)}$$

where $K_{(\rho)}$ is normal curvature of C, which corresponds to the normal $N^\alpha_{(\rho)}$. Now from (1.1) one obtains

$$(2.6)\qquad b^s_{(\tau)i} = p^s \quad \& \quad c_{(\rho\tau)i} = K_{(\rho)} \quad \Rightarrow \quad \lambda^\alpha_{(\tau)i} = q^\alpha.$$

So the field q^α is a special case of the field $\lambda^\alpha_{(\tau)i}$ (1.1). Therefore, from integrability conditions of derivational formula of the field $\lambda^\alpha_{(\tau)i}$ one obtains corensponding equations for q^α.

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DEPARTMENT OF MATHEMATICS, FACULTY OF PHILOSOPHY, ĆIRILA I METODIJA 2, 18 000 NIŠ, YUGOSLAVIA.