FILOMAT (Niš) 9:2 (1995), 131–142
Filomat '94, Niš, October 22-24, 1994
Geometry. Computer Sciences

GEODESIC TUBES AND JACOBI VECTOR FIELDS ON COMPLEX SPACE FORMS

Mirjana Djorić

ABSTRACT. *Studying geodesic variations and associated Jacobi vector fields is very useful for examining the theory of curvature in local and global Riemannian geometry. This is directly connected with the investigation of the geometry of small geodesic spheres and tubes, so it can be used in the analysis of the curvature of the ambient space. In this paper, the explicit expressions for the Jacobi vector fields on complex space forms will be used for calculating the matrix of the shape operator of tubes about geodesics on complex space forms.*

1. Introduction

The study of the curvature of a Riemannian manifold is one of the most interesting topics in Riemannian geometry. As it is well-known, the study of variations of geodesics and the associated Jacobi vector fields is very useful in treating curvature theory in local and global Riemannian geometry. This is directly related to the investigation of the geometry of small geodesic spheres and tubes about curves and submanifolds. The properties of the extrinsic and intrinsic geometry of these geometric objects may be used to study the curvature of the ambient space, as it was done in [1]-[9]. On this occasion we consider only the converse situation, namely, it is quite clear and well-known that when the Riemannian manifold is of a special type (for example, if it has special curvature), then the properties of geometric objects on it are strongly influenced. In [4] the author gave the explicit expressions for the shape operator of tubes about φ-geodesics on Sasakian space forms, while in this paper the special case when the ambient space is a complex

1991 *Mathematics Subject Classification*: 53B20, 53B35, 53C22, 53C55
Key words: Kähler manifold, complex space form, geodesic tube, shape operator, Jacobi vector field.
Supported by Grant 0401A of RFNS through Math. Inst. SANU

space form is considered. Working with Jacobi vector fields, since this falls among the best ways of analyzing the geometry of tubular neighborhoods, the matrix of the shape operator of tubes about geodesics on complex space form is obtained.

We refer to [11] and [14] for a study of tubular neighborhoods and [2] where a more detailed and more complete developement may be found, with an extensive list of references. The article is organized in the following way: Section 2 is devoted to a brief survey of the concepts used throughout the paper and in Section 3 the main results are treated.

2. Preliminaries

Let M be a complex analytic manifold of complex dimension m. By means of charts we may transfer the complex structure of complex m-dimensional Euclidean space C^m to M to obtain an almost complex structure J on M, i.e., a tensor field J on M of type $(1,1)$ such that $J^2 = -I$, where I is the tensor field which is the identity transformation on each tangent space of M. A Riemannian metric g on M is a Hermitian metric if $g(JX, JY) = g(X,Y)$ for any vector fields X and Y on M; M is then called a *Hermitian manifold.* If moreover the almost complex structure J is parallel with respect to the Riemannian connection of g, then J (resp. g) is called a *Kähler structure* (resp. *Kähler metric*); M is then called a *Kähler manifold.* We call a plane which is tangent to M and is invariant by J a *holomorphic plane.* If M is a Kähler manifold, the sectional curvature of a plane p tangent to M will be denoted by $K(p)$ and the sectional curvature of the holomorphic plane generated by a unit tangent vector X will be denoted by $K(X)$. M is said to be of constant holomorphic sectional curvature c if the sectional curvature of every holomorphic tangent plane is equal to c. As a *complex space form* we shall understand a complete Kähler manifold of constant holomorphic sectional curvature and its curvature tensor $R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ is completely determined and given by ([15]):

$$(1) \qquad \begin{aligned} R_{XY}Z = \frac{c}{4}\Big(&g(X,Z)Y - g(Y,Z)X + g(JX,Z)JY - g(JY,Z)JX \\ &+ 2g(JX,Y)JZ\Big). \end{aligned}$$

As is well known, any simply-connected complex space form M is (after multiplying the metric of M by a suitable positive constant) holomorphically isometric to a complex projective space, a complex Euclidean space or a complex hyperbolic space, in dependence of M being of positive, zero or negative holomorphic sectional curvature, respectively ([15]).

We finish these preliminaries by repeating some general facts about tubes. We refer to [2], [11] and [14] for more details and references.

Therefore, let $\sigma : [a, b] \to M$ be a smooth, embedded unit speed curve in a Riemannian manifold M of dimension n and denote by $\sigma^{\perp}$ the normal bundle of σ and by $\exp_\sigma$ the exponential map of this normal bundle, i.e.,

$$\exp_\sigma(\sigma(t), v) = \exp_{\sigma(t)} v$$

for any $t \in [a, b]$ and all $v \in \sigma(t)^{\perp}$. Here $\sigma(t)^{\perp}$ denotes the fiber of $\sigma^{\perp}$ over $\sigma(t)$. Further, let $\mathcal{U}_\sigma(r)$ be the (open) tubular neighborhood or the (open) solid tube of radius r about σ, i.e., the set defined by

$$\mathcal{U}_\sigma(r) = \{\exp_{\sigma(t)} v \mid v \in \sigma(t)^{\perp}, \| v \| < r, t \in [a, b]\}$$

and denote by $N_\sigma(r)$ the (open) solid tube of radius r about the zero section of the normal bundle $\sigma^{\perp}$ of σ. In further text, we shall always assume that the radius r of the tubular neighborhood is smaller than the distance from σ to its nearest focal point. In this case, the exponential map $\exp_\sigma$ is a diffeomorphism between $\mathcal{U}_\sigma(r)$ and $N_\sigma(r)$ and consequently, the set

$$\mathcal{P}_\sigma(s) = \{p \in \mathcal{U}_\sigma(r) \mid d(\sigma, p) = s\},$$

for some $s < r$, is a (smooth) hypersurface in M, called the tube of radius s about σ. If σ is a geodesic on M, the tubes $\mathcal{P}_\sigma$ are called *geodesic tubes* about σ.

For the purpose of describing the geometry of a Riemannian manifold M in the neighborhood of a curve σ we use Fermi coordinates. The *Fermi coordinate system* $(x_1, \dots, x_n)$ with respect to $\sigma(a)$ and relative to a given orthonormal frame field $\{F_1, \dots, F_n\}$ along the curve σ for which $\dot\sigma(t) = (F_1)_{\sigma(t)}$ is defined by

$$(2) \qquad \begin{aligned} x_1\left(\exp_{\sigma(t)}\left(\sum_{j=2}^{n} t_j F_j\right)\right) &= t - a\,, \\ x_i\left(\exp_{\sigma(t)}\left(\sum_{j=2}^{n} t_j F_j\right)\right) &= t_i\,, \quad i = 2, \dots, n, \end{aligned}$$

provided that the numbers $t_2, \dots, t_n$ are small enough in order to have a diffeomorphic $\exp_\sigma$.

Further, if γ is a unit speed geodesic of M normal to σ with $\gamma(0) = m = \sigma(t)$ and $v = \gamma'(0)$, then there is a system of Fermi coordinates $(x_1, \dots, x_n)$ such that for small s we have

$$\left(\frac{\partial}{\partial x_1}\right)_m = \dot\sigma(t), \quad \left(\frac{\partial}{\partial x_i}\right)_m \in \{\dot\sigma(t)\}^{\perp}, \, i = 2, \dots, n-1,$$

$$\left(\frac{\partial}{\partial x_n}\right)_{\gamma(s)} = \gamma'(s)\,.$$

Since $\exp_{\sigma(t)}$ is diffeomorphism on $\mathcal{U}_\sigma(r)$, the equations (2) define a coordinate system near m. It is known ([11]) that the restrictions of the coordinate vector fields $\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right\}$ to σ are orthonormal. In what follows we shall relate the coordinate frame field to a frame field obtained by considering a special set of Jacobi vector fields along γ with a view to obtaining the expression for the shape operator of $\mathcal{P}_\sigma(r)$.

In this aim, let $p = \exp_{\sigma(t)}(rv)$, $v \in \sigma(t)^\perp$, $\|v\| = 1$ be a point of $\mathcal{P}_\sigma(r)$ and let $\gamma : s \mapsto \exp_{\sigma(t)}(sv)$ be the (unique) unit speed geodesic connecting $\sigma(t)$ and p (and cutting σ orthogonally). Denote by $\{E_1,\dots,E_n\}$ the frame field along γ obtained by parallel translation of $\{F_1(t),\dots,F_n(t)\}$ with respect to the Levi Civita connection ∇. Next, if $R = R(s)$ denotes the endomorphism $u \mapsto R_{\gamma'(s)\,u}\gamma'(s)$ of the vector space $\{\gamma'(s)\}^\perp \subset T_{\gamma(s)}M$, then a vector field Y along a geodesic γ is called a *Jacobi vector field* if it satisfies the following second order differential equation- the *Jacobi equation*:

(3) $$Y'' + RY = 0\,,$$

where the prime $'$ denotes covariant differentiation along γ. Next, let $Y_i, i = 1,\dots,n-1$ be the $n-1$ Jacobi vector fields along γ, satisfying the initial conditions

(4) $$\begin{cases} Y_1(0) = F_1(t)\,, & Y_1'(0) = \left(\nabla_{\gamma'}\dfrac{\partial}{\partial x_1}\right)(\sigma(t))\,, \\ Y_i(0) = 0\,, & Y_i'(0) = F_i(t)\,, \qquad i = 2,\dots,n-1 \end{cases}$$

and define

(5) $$Y_i(s) = (B\,E_i)(s)\,, \quad i = 1,\dots,n-1\,.$$

The vector fields $Y_i(s)$ determine a basis for the space $\{\gamma'(s)\}^\perp$ for sufficiently small s and $s \mapsto B(s)$ is an endomorphism-valued function. Then, each $B(s)$ is an endomorphism of the space $\{\gamma'(s)\}^\perp$ and all these spaces may be identified via the parallel translation along γ by using the basis $\{E_i, i = 1,\dots,n\}$. We shall do this at several places without mentioning it explicitly.

Now, from (3), (5) and the initial conditions (4) it follows that B satisfies the Jacobi equation

(6) $$B'' + R \circ B = 0$$

with the initial conditions

$$B(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B'(0) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \tag{7}$$

since we shall focus our attention only to tubes along geodesics.

Finally, we shall write down the matrix of the shape operator S^σ of geodesic tube $\mathcal{P}_\sigma(r)$, using Jacobi vector fields along geodesics orthogonal to σ. Since $\frac{\partial}{\partial s}(p)$ is a unit normal vector of $\mathcal{P}_\sigma(r)$ at $p = \exp_{\sigma(t)}(rv)$, the shape operator S^σ of $\mathcal{P}_\sigma(r)$ at p is defined by

$$(S^\sigma X)(p) = \left(\nabla_X \frac{\partial}{\partial s}\right)(p) \tag{8}$$

for any vector X tangent to $\mathcal{P}_\sigma(r)$ at p. Hence, it is easy to see, by using (5), that the shape operator $S^\sigma(p)$ takes the form

$$S^\sigma(p) = \left(B'B^{-1}\right)(r). \tag{9}$$

3. The main results

In this section we consider complex space forms and we compute the explicit expressions for the shape operator of geodesic tubes in these manifolds. To obtain our results we use here one of the most convenient methods for analyzing the geometry of small geodesic spheres and tubes about curves and submanifolds, by studying the Jacobi vector fields on complex space forms. It is quite natural that the Jacobi vector fields play an important role in this research since it is a well-known fact that the curvature of a Riemannian manifold is geometrically reflected by the behavior of one-parameter families of neighboring geodesics and they are analytically described by Jacobi vector fields. When the manifold is of a special type, the consideration of Jacobi vector fields results in the study of the Jacobi differential equation which has a relatively simple form.

Let m be a point on a complete Kähler manifold M^n of constant holomorphic sectional curvature c and let $\mathcal{P}_\sigma(r)$ be a tube of radius r about a geodesic σ tangent to a unit vector field u. Further, let γ be a geodesic through $m = \gamma(0)$, parametrized by arc length s, with initial velocity vector $\gamma'(0) = v$ and meeting σ orthogonally at $m = \sigma(t)$, with $u = \dot{\sigma}$ at m. Hereafter we shall also write $\gamma'(s) = v$ at any point of γ. For a vector field v the Jacobi equation

$$\nabla_v \nabla_v X + R_{vX} v = 0 \tag{10}$$

for a given complex space form M becomes by virtue of (1)

$$\nabla_v \nabla_v X + \frac{c}{4}\Big(X - 3\, g(JX, v)Jv\Big) = 0\,. \tag{11}$$

Further, we shall distinguish three cases, depending on the position of the point $p = \exp_{\sigma(t)}(rv)$, $v \in \sigma(t)^{\perp}$, $\|v\| = 1$, in the forthcoming three theorems.

First, consider the special points p of the geodesic tubes $\mathcal{P}_\sigma(r)$ on a complex space form M^n, such that $p = \exp_{\sigma(t)}(rv)\,, v \in \sigma(t)^{\perp}$, $v(\sigma(t)) = Ju(\sigma(t))$. As this case has already been investigated in [2] and [6], we give here only the final expression for the matrix of the shape operator. Namely, the following theorem holds:

Theorem 1. ([6]) *Let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature c. Then, at a point $p = \exp_{\sigma(t)}(rv)\,, v \in \sigma(t)^{\perp}$, $v(\sigma(t)) = Ju(\sigma(t))$ of the tube $\mathcal{P}_\sigma(r)$ (along a geodesic $\sigma(t)$ tangent to a vector u), the shape operator $S^\sigma(p)$ can be represented by the following matrix:*

$$S^\sigma(p) = \begin{bmatrix} A(r) & 0 & \dots & 0 \\ 0 & B(r) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B(r) \end{bmatrix} \tag{12}$$

with respect to the basis $\{E_1, \dots, E_{n-1}\}$ defined in Section 2. The explicit expressions for the entries are as follows:

$$A(r) = 0, \qquad B(r) = \frac{1}{r}, \qquad \text{for } c = 0;$$

$$A(r) = -\sqrt{c}\tan\sqrt{c}r, \qquad B(r) = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}r, \qquad \text{for } c > 0;$$

$$A(r) = \sqrt{|c|}\tanh\sqrt{|c|}r, \qquad B(r) = \frac{\sqrt{|c|}}{2}\coth\frac{\sqrt{|c|}}{2}r, \qquad \text{for } c < 0.$$

Now, let us consider sufficiently small tube $\mathcal{P}_\sigma(r)$ about the geodesic σ embedded in a Kähler manifold of constant holomorphic sectional curvature c. Let γ denote the unit-speed geodesic meeting σ orthogonally at $m = \sigma(t)$ and tangent to a vector v such that $g(u(m), Jv(m)) = a$, where $\dot\sigma = u$ at m. To obtain the matrix of the shape operator of $\mathcal{P}_\sigma(r)$ at points $p =$

$\exp_{\sigma(t)}(rv)$, we first choose an orthonormal basis $\{e_1, \dots, e_n\}$ for the tangent space $T_m M$, such that $e_1 = u(m), e_2 = (Jv(m) - au(m))/b, e_n = v(m)$, where $a^2 + b^2 = 1$. Further, let $\{E_1, \dots, E_n\}$ be the basis obtained by parallel translation of the basis $\{e_1, \dots, e_n\}$ along γ. Then it follows that any vector field X orthogonal to the geodesic γ can be written as

$$X = f_1 E_1 + f_2 E_2 + \sum_{i=3}^{n-1} f_i E_i \,. \tag{13}$$

Since, using (1) we obtain

$$\begin{cases} R_{vE_1}v = \dfrac{c}{4}\Big((3a^2+1)E_1 + 3abE_2\Big) \\ R_{vE_2}v = \dfrac{c}{4}\Big(3abE_1 + (3b^2+1)E_2\Big) \\ R_{vE_i}v = \dfrac{c}{4}\,E_i\,, \quad i = 3, \dots, n-1\,, \end{cases} \tag{14}$$

we see that (11) is equivalent to the following system of differential equations:

$$\begin{aligned} &4\,f_1'' + c\,(3a^2+1)f_1 + 3abc\,f_2 = 0\,, \\ &4\,f_2'' + 3abc\,f_1 + c\,(3b^2+1)f_2 = 0\,, \end{aligned} \tag{15}$$

$$4\,f_i'' + c\,f_i = 0\,, \quad i = 3, \dots, n-1\,. \tag{16}$$

Now, consider the substitution

$$\begin{aligned} z_1 &= a\,f_1 + b\,f_2\,, \\ z_2 &= b\,f_1 - a\,f_2\,. \end{aligned} \tag{17}$$

Then the equations (15) take the form

$$\begin{aligned} &a\,z_1'' + b\,z_2'' + c\,a\,z_1 + \frac{c}{4}\,b\,z_2 = 0\,, \\ &b\,z_1'' - a\,z_2'' + c\,b\,z_1 - \frac{c}{4}\,a\,z_2 = 0\,. \end{aligned} \tag{18}$$

In this way, by multiplying the first equation in (18) by a and the second by b and adding the obtained results, we arrive at a differential equation

$$z_1'' + c\,z_1 = 0\,,$$

which is easy to integrate, having in mind that the solutions will depend on the sign of c. Finally, using the standard solutions of the $n-3$ equations (16), we can derive the complete solutions in the three cases we shall need.

Case 1: $c = 0$

Here we find

$$f_1 = (aA + bC)s + aB + bD\,,$$
$$f_2 = (bA - aC)s + bB - aD\,,$$
$$f_i = A_i s + B_i\,, \quad i = 3, \dots, n-1\,,$$

with A, B, C, D, A_i, B_i being constant along γ.

Case 2: $c > 0$

In this case, putting $k = \sqrt{c}$, we obtain

$$f_1 = aF \cos ks + aG \sin ks + bH \cos \tfrac{ks}{2} + bI \sin \tfrac{ks}{2}\,,$$
$$f_2 = bF \cos ks + bG \sin ks - aH \cos \tfrac{ks}{2} - aI \sin \tfrac{ks}{2}\,,$$
$$f_i = F_i \cos \tfrac{ks}{2} + G_i \sin \tfrac{ks}{2}\,, \quad i = 3, \dots, n-1\,,$$

with F, G, H, I, F_i, G_i being constant along γ.

Case 3: $c < 0$

This time we put $k = \sqrt{-c}$. Repeating the same computations, we obtain

$$f_1 = a(Ke^{ks} + Le^{-ks}) + b(Me^{\frac{ks}{2}} + Ne^{-\frac{ks}{2}})\,,$$
$$f_2 = b(Ke^{ks} + Le^{-ks}) - a(Me^{\frac{ks}{2}} + Ne^{-\frac{ks}{2}})\,,$$
$$f_i = K_i e^{\frac{ks}{2}} + L_i e^{-\frac{ks}{2}}\,, \quad i = 3, \dots, n-1\,,$$

with K, L, M, N, K_i, L_i being again constant along γ.

Moreover, we shall need the form of the Jacobi vector fields along a geodesic γ satisfying the following initial conditions:

$$X_1(0) = E_1(0)\,, \quad X_1'(0) = 0\,, \tag{19}$$
$$X_i(0) = 0\,, \qquad X_i'(0) = E_i(0)\,, \quad i = 3, \dots, n-1. \tag{20}$$

We shall therefore compute these special Jacobi fields in the three above-described cases, using the notation $k = \sqrt{c}$ if $c > 0$ and $k = \sqrt{-c}$ when $c < 0$.

Case 1: $c = 0$

$$X_1(s) = E_1(s)\,, \quad X_2(s) = s\,E_2(s)\,, \quad X_i(s) = s\,E_i(s),\, i = 3,\dots,n-1\,.$$

Case 2: c > 0

$$X_1(s)= \left(a^2 \cos ks + b^2 \cos \tfrac{ks}{2}\right) E_1 + a\,b \left(\cos ks - \cos \tfrac{ks}{2}\right) E_2\,,$$

$$X_2(s)= \frac{a\,b}{k} \left(\sin ks - 2 \sin \tfrac{ks}{2}\right) E_1 + \frac{1}{k} \left(b^2 \sin ks + 2a^2 \sin \tfrac{ks}{2}\right) E_2\,,$$

$$X_i(s)= \frac{2}{k} \sin \tfrac{ks}{2}\, E_i(s)\,,\, i = 3,\dots,n-1.$$

Case 3: c < 0

$$X_1(s)= \left(a^2 \cosh ks + b^2 \cosh \tfrac{ks}{2}\right) E_1 + a\,b \left(\cosh ks - \cosh \tfrac{ks}{2}\right) E_2\,,$$

$$X_2(s)= \frac{a\,b}{k} \left(\sinh ks - 2 \sinh \tfrac{ks}{2}\right) E_1 + \frac{1}{k} \left(b^2 \sinh ks + 2a^2 \sinh \tfrac{ks}{2}\right) E_2\,,$$

$$X_i(s)= \frac{2}{k} \sinh \tfrac{ks}{2}\, E_i(s)\,,\, i = 3,\dots,n-1.$$

Finally, using relations (4)-(9) and computed Jacobi vector fields, it follows that the shape operator S^σ can be represented by the following quasi-diagonal matrix:

$$S^\sigma(p) = \begin{bmatrix} A(r) & B(r) & 0 & \dots & 0 \\ B(r) & C(r) & 0 & \dots & 0 \\ 0 & 0 & D(r) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & D(r) \end{bmatrix} \tag{21}$$

with respect to the basis $\{bE_1 - aE_2, Jv, E_3, \dots, E_{n-1}\}$. The explicit expressions for the entries are as follows:

Case 1: c = 0

$$A(r)= B(r) = 0\,,$$
$$C(r)= D(r) = \frac{1}{r}\,.$$

Case 2: c > 0

$$A(r)= \tfrac{1}{2\,\omega} \left(-b^2 \sin \tfrac{kr}{2} \sin kr + 2a^2 \cos \tfrac{kr}{2} \cos kr\right)\,,$$
$$B(r)= -\tfrac{1}{\omega}\, a\, b\,,$$
$$C(r)= \tfrac{1}{\omega} \left(-2a^2 \sin \tfrac{kr}{2} \sin kr + b^2 \cos \tfrac{kr}{2} \cos kr\right)\,,$$
$$D(r)= \tfrac{k}{2} \cot \tfrac{kr}{2}\,,$$

where $\omega = \frac{1}{k}\left(2a^2 \sin\frac{kr}{2}\cos kr + b^2 \sin kr \cos\frac{kr}{2}\right)$.

Case 3: $c < 0$

$$\begin{aligned} A(r) &= \tfrac{1}{2\theta}\left(b^2 \sinh\tfrac{kr}{2}\sinh kr + 2a^2\cosh\tfrac{kr}{2}\cosh kr\right),\\ B(r) &= -\tfrac{1}{\theta}\, a\, b,\\ C(r) &= \tfrac{1}{\theta}\left(2a^2\sinh\tfrac{kr}{2}\sinh kr + b^2\cosh\tfrac{kr}{2}\cosh kr\right),\\ D(r) &= \tfrac{k}{2}\coth\tfrac{kr}{2}, \end{aligned}$$

where $\theta = \frac{1}{k}\left(2a^2\sinh\frac{kr}{2}\cosh kr + b^2\sinh kr\cosh\frac{kr}{2}\right)$.

Therefore, this proves that the following theorem holds:

Theorem 2. *Let (M^n, g, J) be a Kähler manifold of constant holomorphic sectional curvarure c and let $P^\sigma(r)$ be a sufficiently small geodesic tube of radius r around a geodesic σ tangent to a vector u on M^n. Then the shape operator S^σ of tube $P^\sigma(r)$ at points $p = \exp_{\sigma(t)}(rv)$, such that $v(\sigma(t)) \perp u(\sigma(t))$, $g(Jv(\sigma(t)), u(\sigma(t))) = a$, can be represented by the matrix (21).*

Finally, since the case $v(\sigma(t)) \perp Ju(\sigma(t))$ is slightly more difficult than the case $v(\sigma(t)) = Ju(\sigma(t))$, but easier than the general case, where $g(Jv(\sigma(t)), u(\sigma(t))) = a$, we give here only the final result, i.e., the matrix of the shape operator in this case.

Theorem 3. *Let (M^n, g, J) be a Kähler manifold of constant holomorphic sectional curvature c and let $P^\sigma(r)$ be a sufficiently small geodesic tube of radius r around a geodesic σ tangent to a vector u on M^n. Then the shape operator S^σ of tube $P^\sigma(r)$ at points $p = \exp_{\sigma(t)}(rv)$, such that $v(\sigma(t)) = Ju(\sigma(t))$, is given by the following matrix:*

$$(22) \qquad S^\sigma(p) = \begin{bmatrix} A(r) & 0 & 0 & \dots & 0 \\ 0 & B(r) & 0 & \dots & 0 \\ 0 & 0 & C(r) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C(r) \end{bmatrix}$$

with respect to the basis $\{E_1, E_2, \dots, E_{n-1}\}$ defined in Section 2, such that $E_2(\sigma(t)) = Jv(\sigma(t))$. The explicit expressions for the entries are as follows:

Case 1: $\quad c = 0$

$$A(r) = 0\,,\; B(r) = C(r) = \frac{1}{r}\,.$$

Case 2: $\quad c > 0$

$$\begin{aligned} A(r) &= -\tfrac{k}{2}\,\tan\tfrac{kr}{2}\,,\\ B(r) &= k\,\cot kr\,,\\ C(r) &= \tfrac{k}{2}\,\cot\tfrac{kr}{2}\,. \end{aligned}$$

Case 3: $\quad c < 0$

$$\begin{aligned} A(r) &= \tfrac{k}{2}\,\tanh\tfrac{kr}{2}\,,\\ B(r) &= k\,\coth kr\,,\\ C(r) &= \tfrac{k}{2}\,\coth\tfrac{kr}{2}\,. \end{aligned}$$

It is evident that the last result follows either directly from Theorem 2 (by replacing $a = 0\,, b = 1$ in (21)), or following the similar procedure as in Theorem 1 and Theorem 2 (i.e., solving the Jacobi equation (10) and computing the Jacobi vector fields). The author first used the latter method, and then checked the results after having proved Theorem 2.

Remark. After having completed this work, the author was informed by L. Vanhecke, that L. Gheysens derived the complete formulas for S^σ in his dissertation [10] and that the needed material is given in [12].

References

[1] P. Bueken, *Reflections and rotations in contact geometry*, doctoral dissertation, Katholi- eke Universiteit Leuven, 1992.

[2] M. Djorić, *Geometrija geodezijskih sfera i cevi*, doctoral dissertation, Faculty of Mathematics, University of Belgrade, 1994.

[3] M. Djorić, On characterizations of Sasakian space forms and locally φ-symmetric spaces, *Publ. Math. Debrecen* 46(1995), 1-23.

[4] M. Djorić, Geometry of geodesic tubes on Sasakian manifolds, *Proc. Colloquium on Differential Geometry, Debrecen,* 1995, Kluwer Ac. Publ., to appear.

[5] M. Djorić, Geometry of tubes about φ-geodesics on Sasakian manifolds, submitted.

[6] M. Djorić, Characterizations of complex space forms and locally Hermitian symmetric spaces by geodesic tubes, submitted.

[7] M. Djorić and L. Vanhecke, Almost Hermitian geometry, geodesic spheres and symmetries, *Math. Okayama Univ.* 32 (1990), 187-206.

[8] M. Djorić and L. Vanhecke, Geometry of geodesic spheres on Sasakian manifolds, *Rend. Sem. Mat. Univ. Pol. Torino* 49 (1991), 329-357.

[9] M. Djorić and L. Vanhecke, Geometry of tubes about characteristic curves on Sasakian manifolds, *Rend. Circ. Mat. Palermo* XLI (1992), 111-122.

[10] L. Gheysens, *Riemannse differentiaalmeetkunde van buisvormige omgevingen*, doctoral dissertation, Catholic University Leuven, 1981.

[11] A. Gray, *Tubes*, Addison-Wesley Publ. Co., Reading, (1990).

[12] A. Gray and L. Vanhecke, The volumes of tubes in a Riemannian manifold, *Rend. Sem. Mat. Univ. Politec. Torino* 39 (1981), 1-50.

[13] S. Kobayashi and K. Nomizu, *Foundations of Differential geometry*, I,II, Interscience Publ., New York, 1963, 1969.

[14] L. Vanhecke, Geometry in normal and tubular neighborhoods, *Rend. Sem. Fac. Sci. Univ. Cagliari, Supplemento al* Vol.58 (1988), 73-176.

[15] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, 3, World Scientific Publ. Co., Singapore, 1984.

Faculty of Mathematics, University of Belgrade, Studentski trg 16, p.b. 550, 11000 Belgrade, Yugoslavia

E-mail: edjoric@ubbg.etf.bg.ac.yu