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AN INEQUALITY FOR THE TRIANGLE

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ABSTRACT. *Inequalities for the triangle in the most of cases become equalities for the equilateral triangle [2], [5]. In this article is given an inequality with unique property that it becomes equality for isoscales and rectangular triangles. Also, an inequality connected with Karamata's inequality is given.*

Theorem 1. *Let a, b, c, α, β, γ are the sides and angles of a triangle respectively and R the radius of its circumcircle. Then*

$$2R \geq \frac{b^2+c^2}{\sqrt{2b^2+2c^2-a^2}}, \tag{1}$$

equality holds if and only if $b = c$ or $\alpha = \frac{\pi}{2}$.

$$2R \geq \max\left\{\frac{b^2+c^2}{\sqrt{2b^2+2c^2-a^2}}, \frac{c^2+a^2}{\sqrt{2c^2+2a^2-b^2}}, \frac{a^2+b^2}{\sqrt{2a^2+2b^2-c^2}}\right\}, \tag{2}$$

equality holds if and only if the triangle is isosceles or rectangular.

Lemma.

$$\begin{aligned} &|\cos\alpha| \geq \frac{|b^2+c^2-a^2|}{b^2+c^2}, \quad \sin\alpha \leq a\frac{\sqrt{2b^2+2c^2-a^2}}{b^2+c^2}, \\ &|\tan\alpha| \leq a\frac{\sqrt{2b^2+2c^2-a^2}}{|b^2+c^2-a^2|}. \end{aligned} \tag{3}$$

If $\alpha \leq \frac{\pi}{2}$, then

$$\begin{aligned} &\cos\frac{\alpha}{2} \geq \frac{\sqrt{2b^2+2c^2-a^2}}{\sqrt{2b^2+2c^2}}, \quad \sin\frac{\alpha}{2} \leq \frac{a}{\sqrt{2b^2+2c^2}}, \\ &\tan\frac{\alpha}{2} \leq \frac{a}{\sqrt{2b^2+2c^2-a^2}}, \end{aligned} \tag{4}$$

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and conversely for $\alpha \geq \frac{\pi}{2}$. *Equalities hold if and only if* $b = c$ *or* $\alpha = \frac{\pi}{2}$.

Proof. Inequality (3.1) is equivalent to $|\cos\alpha|(b-c)^2 \geq 0$, which becomes equality if and only if $b = c$ or $\alpha = \frac{\pi}{2}$. (3.2) is equivalent to (3.1), and (3.3) is their consequence. Using $2R\sin\alpha = a$ one obtains that (1) is equivalent to (3.2). Since $\cos\alpha \geq (b^2+c^2-a^2)/(b^2+c^2)$ if $\alpha \leq \frac{\pi}{2}$, and conversely if $\alpha \geq \frac{\pi}{2}$, inequalities (4) follow. □

The inequality of I. J. Schoenberg [4] for the two-dimensional euclidean space reads as follows: If λ_1, λ_2, λ_3 are real numbers, then

$$(5)\qquad (\lambda_1+\lambda_2+\lambda_3)^2R^2 \geq \lambda_2\lambda_3a^2+\lambda_3\lambda_1b^2+\lambda_1\lambda_2c^2.$$

Introduce the functional

$$f(\lambda_1,\lambda_2,\lambda_3) = \lambda_2\lambda_3a^2+\lambda_3\lambda_1b^2+\lambda_1\lambda_2c^2$$

and consider now the inequality (5) with two equal parameters. The functional $f(\lambda_1,\lambda_2,\lambda_2)$, $\lambda_1+2\lambda_2 =$ const. has a maximum if

$$\lambda_1(b^2+c^2)+2\lambda_2(a^2-b^2-c^2) = 0,$$

$$\lambda_1 = 2\lambda_2\frac{b^2+c^2-a^2}{b^2+c^2}.$$

For this value (5) becomes

$$\left(2\frac{b^2+c^2-a^2}{b^2+c^2}+2\right)^2R^2 \geq a^2+2(b^2+c^2-a^2),$$

as $2b^2+2c^2-a^2 > (b-c)^2 > 0$, follows (1).

We now give the necessary and sufficient conditions for parameters in Schoenberg's inequality for holding equality, what led to the given thorem. Let

$$(6)\qquad \lambda_1+\lambda_2+\lambda_3 = \lambda.$$

The functional f has a maximum, with the condition $\lambda_2+\lambda_3 =$ const., similarly $\lambda_3+\lambda_1 =$ const. and $\lambda_1+\lambda_2 =$ const., if

$$(7)\qquad (\lambda_2-\lambda_3)a^2+(b^2-c^2)\lambda_1 = 0, \qquad \text{cycl.}$$

The system of linear equations (6-7) has solution

$$\mu_1 = ka^2(b^2+c^2-a^2), \qquad \text{cycl.},$$

where

$$k = \lambda\left(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4\right)^{-1}.$$

Using formulas for area F of a triangle, Heron's and $4FR = abc$, we get equality

$$f(\mu_1, \mu_2, \mu_3) = k^2a^2b^2c^2\left(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4\right) = \lambda^2R^2.$$

By special values of the λ's several inequalities for the triangle, including the well-known formulas of Weitzenböck, Finsler and Hadwiger, can be deduced [4], [2].

Remark. *Equality in (5) holds if and only if* $\sin 2\alpha = r\lambda_1$, $\sin 2\beta = r\lambda_2$, $\sin 2\gamma = r\lambda_3$, $r \in R$, [1], *also*

$$\mu_1 = ka^2 2bc\cos\alpha = \lambda\frac{R^2}{2F}\sin 2\alpha, \qquad \textit{cycl.}$$

Theorem 2.

$$(8) \qquad \frac{a}{\sqrt{2b^2+2c^2-a^2}} + \frac{b}{\sqrt{2c^2+2a^2-b^2}} + \frac{c}{\sqrt{2a^2+2b^2-c^2}} \geq \sqrt{3},$$

equality holds if and only if the triangle is equilateral.

Proof. The inequality of J. Karamata [3]

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \geq \sqrt{3}$$

and the third formula in (4) for either an acute or a rectangular triangle induce given inequality. Let $A = \sqrt{2b^2+2c^2-a^2}$, cycl. and f — the left-hand side of (8). Then

$$f'_a = (b^2+c^2)A^{-3} - abB^{-3} - caC^{-3} = 0, \qquad \text{cycl.}$$

implies

$$a : b : c = A^{-3} : B^{-3} : C^{-3}.$$

Therefore,

$$a = b \qquad \text{or} \qquad 2(a^2+b^2+c^2) = 3\frac{a^{8/3} - b^{8/3}}{a^{2/3} - b^{2/3}}, \qquad \text{cycl.}$$

and either $a = b = c$ or e. g. $a = b$, $c = \left(\sqrt[3]{2} - 1\right)^{3/2}a$. Also $f > 2$ if $a = b + c$. □

References

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