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INEQUALITIES FOR COEFFICIENTS OF ALGEBRAIC POLYNOMIALS

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ABSTRACT. *Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree at most n and $\|P\|_{d\sigma} = (\int_{\mathbb{R}} |P(x)|^2 d\sigma(x))^{1/2}$, where $d\sigma(x)$ is a nonnegative measure on $\mathbb{R}$. We consider the best constant in the inequality $|a_\nu| \le C_{n,\nu}(d\sigma)\|P\|_{d\sigma}$, when $P \in \mathcal{P}_n$ and such that $P(\xi_k) = 0$ $(k = 1, 2, \ldots, m)$. The cases $C_{n,n}(d\sigma)$ and $C_{n,n-1}(d\sigma)$ were studied by Milovanović and Guessab [2] and for an arbitrary ν by Milovanović and Rančić [5], where they gave explicit expressions for some classical measures. In this paper we determine the best constants $C_{n,\nu}$ for the generalized Gegenbauer measure on $(-1, 1)$ and for the generalized Hermite measure on $(-\infty, +\infty)$.*

1. Introduction

Let $\mathcal{P}_n$ be the class of algebraic polynimials $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree of most n. Denote by

$$\|P\|_{d\sigma} = \sqrt{(P,P)} = \Big(\int_{\mathbb{R}} |P(x)|^2 d\sigma(x)\Big)^{1/2} \tag{1.1}$$

the norm of a polynomial $P \in \mathcal{P}_n$, where $d\sigma(x)$ is a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} x^k \, d\sigma(x)$, $k = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$. We will consider the problem of determining the best possible constants $C_{n,\nu}(d\sigma)$ such that the following inequalities

$$|a_\nu| \le C_{n,\nu}(d\sigma) \, \|P\|_{d\sigma} \qquad (0 \le \nu \le n), \tag{1.2}$$

are valid.

Polynomials in (1.2) belong to the restrictive class of polynomials

$$\mathcal{P}_n(\xi_1, \xi_2, \ldots, \xi_n) = \{P \in \mathcal{P}_n \mid P(\xi_k) = 0, \ \xi_k \in \mathbb{C}, \ k = 1, 2, \ldots, m\}.$$

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Let

$$\prod_{i=1}^{m}(x-\xi_i) \equiv x^m - s_1 x^{m-1} + \cdots + (-1)^{m-1} s_{m-1} x + (-1)^m s_m,$$

where s_k denotes the elementary symmetric functions of $\xi_1, \xi_2, \ldots, \xi_m$, i.e.,

$$s_k = \sum \xi_1 \xi_2 \cdots \xi_k \qquad (k = 1, 2, \ldots, m). \tag{1.3}$$

For $k = 0$ we have $s_0 = 1$ and $s_k = 0$ for $k > m$.

For the measure $d\sigma$ there exists an unique set of orthonormal polynomials $\pi_n(\cdot) = \pi_n(\cdot\,; d\sigma)$, $n = 0, 1, \ldots$, defined by

$$\pi_n(x) = b_n^{(n)}(d\sigma)x^n + b_{n-1}^{(n)}(d\sigma)x^{n-1} + \cdots + b_0^{(n)}(d\sigma) \qquad (b_n^{(n)}(d\sigma) > 0)$$

and

$$(\pi_n, \pi_m) = \delta_{nm} \qquad (n, m \geq 0),$$

where

$$(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\sigma(x) \qquad (f, g \in L^2(\mathbb{R})).$$

Denote by $d\hat{\sigma}(t)$ the weight

$$d\hat{\sigma}(x) = \prod_{k=1}^{m} |x - \xi_k|^2\, d\sigma(x). \tag{1.4}$$

The problem (1.2) was considered in [2] by Milovanović and Guessab (see also [4]). They have proved the following result:

Theorem A. *If $P \in \mathcal{P}_n(\xi_1, \xi_2, \ldots, \xi_m)$ and $\hat{b}_\nu^{(n)} = b_\nu^{(n)}(d\hat{\sigma})$, then inequalities*

$$|a_n| \leq \hat{b}_{n-m}^{(n-m)}\, \|P\|_{d\sigma} \tag{1.5}$$

and

$$|a_{n-1}| \leq \left(\left(\hat{b}_{n-m-1}^{(n-m)} - s_1\, \hat{b}_{n-m}^{(n-m)} \right)^2 + \left(\hat{b}_{n-m-1}^{(n-m-1)} \right)^2 \right)^{1/2} \|P\|_{d\sigma} \tag{1.6}$$

hold.

Inequality in (1.5) *and* (1.6) *are attained if and only if* $P(x)$ *is a constant multiple of*

$$\hat{\pi}_{n-m}(x\,;\,d\hat{\sigma})\prod_{k=1}^{m}|x-\xi_k|$$

and

$$\left(\left(\hat{b}_{n-m-1}^{(n-m)}-s_1\,\hat{b}_{n-m}^{(n-m)}\right)\hat{\pi}_{n-m}(x)+\hat{b}_{n-m-1}^{(n-m-1)}\,\hat{\pi}_{n-m-1}(x)\right)\prod_{k=1}^{m}|x-\xi_k|,$$

respectively.

Milovanović and Rančić in [5] considered the corresponding problem for arbitrary k and proved the following inequality

$$(1.7)\qquad |a_{n-k}|\le\left(\sum_{j=0}^{k}\left(\sum_{i=j}^{k}(-1)^{k-i}s_{k-i}\,\hat{b}_{n-m-i}^{(n-m-j)}\right)^2\right)^{1/2}\|P\|_{d\sigma},$$

with extremal polynomial

$$\left(\sum_{j=0}^{k}\hat{\pi}_{n-m-j}(x)\sum_{i=j}^{k}(-1)^{k-i}s_{k-i}\,\hat{b}_{n-m-i}^{(n-m-j)}\right)\prod_{k=1}^{m}(x-\xi_k).$$

In the above mentioned papers the authors have determined explicit constants $C_{n,\nu}(d\sigma)$ for some weights corresponding to the classical orthogonal polynomials. In this paper we are going to determine explicit constants $C_{n,\nu}(d\sigma)$ for weights that correspond to the generalized Gegenbauer and the generalized Hermite polynomials. This is significant because of the growing importance of these polynomials in many applications, particulary in numerical approximation (see for example [3]).

2. The Generalized Gegenbauer Case

At first we observe the generalized Gegenbauer case

$$d\sigma(x)=|x|^{\mu}(1-x^2)^{\alpha}dx\qquad(\mu,\alpha>-1).$$

Let $\beta=(\mu-1)/2$ and let $\{W_n^{(\alpha,\beta)}(x)\}$ be a sequence of the generalized Gegenbauer monic polynomials orthogonal with respect to the measure

$d\sigma(x)$ on $(-1,1)$ (which was introduced by Lascenov in [1]). For such polynomials we have the following recurrence relation

$$(2.1) \qquad W_{n+1}^{(\alpha,\beta)}(x) = xW_n^{(\alpha,\beta)} - \lambda_n W_{n-1}^{(\alpha,\beta)}(x), \qquad n = 0,1,\ldots,$$

with $W_{-1}^{(\alpha,\beta)}(x) = 0$ and $W_0^{(\alpha,\beta)}(x) = 1$, where

$$\lambda_{2n} = \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}$$

and

$$\lambda_{2n+1} = \frac{(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)},$$

for $n = 0,1,\ldots,$ except when $\alpha+\beta = -1$; then $\lambda_1 = (\beta+1)(\alpha+\beta+2)$.

Using the norm of $W_n^{(\alpha,\beta)}(x)$,

$$\|W_{2n}^{(\alpha,\beta)}\|^2 = \frac{n!}{(n+\alpha+\beta+1)_n} B(n+\alpha+1, n+\alpha+1),$$

$$\|W_{2n+1}^{(\alpha,\beta)}\|^2 = \frac{n!}{(n+\alpha+\beta+2)_n} B(n+\alpha+1, n+\beta+2),$$

where B is the beta function, we can obtain the loading coefficients

$$b_n^{(n;\,\alpha,\beta)} = b_n^{(n)}(d\sigma)$$

in the corresponding orthonormal polynomials $\hat{W}_n^{(\alpha,\beta)}(x)$

$$b_n^{(n;\,\alpha,\beta)} = \begin{cases} \left(\dfrac{(k+\alpha+\beta+1)_k}{k!B(k+\alpha+1,k+\beta+1)}\right)^{1/2} & (n = 2k), \\ \left(\dfrac{(k+\alpha+\beta+2)_k}{k!B(k+\alpha+1,k+\beta+2)}\right)^{1/2} & (n = 2k+1). \end{cases}$$

Using Theorem A and (1.7) we obtain:

Theorem 2.1. *Under restriction* $P^{(i)}(0) = 0$ $(i = 0,1,\ldots,k-1)$ *and* $P^{(i)}(-1) = P^{(i)}(1) = 0$ $(i = 0,1,\ldots,s-1)$, *where* $s = (m-k)/2 \in \mathbb{N}$ *for* $l = 0$ *and* $l = 1$, *we have that*

$$|a_{n-l}| \le b_{n-l-m}^{(n-l-m;\, m-k+\alpha,\, \beta+k)} \|P\|_{d\sigma}.$$

The equality is attained if and only if

$$P(x) = Ax^k(x^2-1)^s\,\hat{W}_{n-l-m}^{(m-k+\alpha,\,\beta+k)}(x) \qquad (A = \text{const}).$$

Proof. Since restrictions on polynomials are given only in the points $\xi_1 = \xi_2 = \cdots = \xi_k = 0$, $\xi_{k+1} = \xi_{k+2} = \cdots = \xi_s = -1$ and $\xi_{s+1} = \xi_{s+2} = \cdots = \xi_m = 1$, the new measure $d\hat{\sigma}(x)$ is again the generalized Gegenbauer measure

$$d\hat{\sigma}(x) = |x|^{2k+\mu}(1-x^2)^{m-k+\alpha}dx \qquad (\mu+2k,\ m-k+\alpha > -1).$$

Since the weight function is even, then according to (2.1) it is not difficult to prove that $b_\nu^{(n;\,\alpha,\beta)} = 0$ when $n-\nu = 2r+1$ ($r \in \mathrm{N}$, $\nu = 0,1,\ldots,n$). The required result can be directly obtained from Theorem A and (1.7), where, in our case, $s_1 = 0$. □

3. The Generalized Hermite Case

Consider now the generalized Hermite measure $d\sigma(x) = |x|^{2k}e^{-x^2}dx$ ($k > -1/2$) on $(-\infty,+\infty)$. With $H_n^{(k)}(x)$ we denote the generalized Hermite monic polynomial. For such polynomials the following differential equation is satisfied

$$xy'' + 2(k-x^2)y' + (2nx - \varepsilon x^{-1})y = 0, \tag{3.1}$$

where $\varepsilon = 0$, for n is even, and $\varepsilon = 2k$, for n is odd.

Using (3.1) we can obtain:

1° If n is even then

$$H_n^{(k)}(x) = (-1)^{\frac{n}{2}}\Gamma\Big(k+\frac{n}{2}+\frac{1}{2}\Big)\sum_{\nu=0}^{n/2}(-1)^\nu\binom{n/2}{\nu}\frac{x^{2\nu}}{\Gamma\Big(k+\nu+\dfrac{1}{2}\Big)}$$

and

$$\|H_n^{(k)}\|^2 = \Big(\frac{n}{2}\Big)!\,\Gamma\Big(k+\frac{n}{2}+\frac{1}{2}\Big).$$

2° If n is odd, then

$$H_n^{(k)}(x) = (-1)^{\frac{n-1}{2}}\Gamma\Big(k+\frac{n-1}{2}+\frac{3}{2}\Big)\sum_{\nu=0}^{(n-1)/2}(-1)^\nu\binom{\frac{n-1}{2}}{\nu}\frac{x^{2\nu+1}}{\Gamma\Big(k+\nu+\dfrac{3}{2}\Big)}$$

and

$$\|H_n^{(k)}\|^2 = \Big(\frac{n-1}{2}\Big)!\,\Gamma\Big(k+\frac{n-1}{2}+\frac{3}{2}\Big).$$

The coefficients $\hat{b}_\nu^{(n)}(d\sigma)$ $(\nu = 0,1,\dots,n)$ in the coresponding orthonormal polynomials $\hat{H}_n^{(k)}(x)$ are given by:

1° If n is even then

$$\hat{b}_\nu^{(n)}(d\sigma) = (-1)^{n/2+\nu}\,\frac{\binom{n/2}{\nu/2}\sqrt{\Gamma\Big(k+\frac{n}{2}+\frac{1}{2}\Big)}}{\Gamma\Big(k+\frac{\nu}{2}+\frac{1}{2}\Big)\sqrt{\Big(\frac{n}{2}\Big)!}}$$

if ν is even, and $\hat{b}_\nu^{(n)}(d\sigma) = 0$ if ν is odd.

2° If n is odd then

$$\hat{b}_\nu^{(n)}(d\sigma) = (-1)^{(n-1)/2+\nu}\,\frac{\binom{(n-1)/2}{(\nu-1)/2}\sqrt{\Gamma\Big(k+\frac{n}{2}+1\Big)}}{\Gamma\Big(k+\frac{\nu}{2}+1\Big)\sqrt{\Big(\frac{n-1}{2}\Big)!}}$$

if ν is odd, and $\hat{b}_\nu^{(n)}(d\sigma) = 0$ if ν is even.

Similarly, as in the previous section, we can prove the following theorem:

Theorem 3.1. *Let $P^{(i)}(0) = 0$ $(i = 0,1,\dots,m-1)$ and let the measure $d\hat{\sigma}(x) = x^{2m}d\sigma(x)$ and the norm $\|P\|_{d\sigma}$ be given by (1.1). Then*

$$\text{(3.2)}\qquad |a_{n-l}| \le \sqrt{A_{n,l}}\,\|P\|_{d\sigma} \qquad (l = 0,1,\dots,n),$$

where $A_{n,l} = 0$ for $n-l-m<0$, and otherwise

$$A_{n,l} = \frac{1}{K_1}\sum_{j=0}^{[l/2]}\binom{k+\frac{n-l+m}{2}-\frac{1}{2}+j}{j}\binom{\frac{n-l-m}{2}+j}{j} \quad (n-l-m \text{ is even}),$$

where $K_1 = \Big(\frac{n-l-m}{2}\Big)!\,\Gamma\Big(k+\frac{n-l+m}{2}+\frac{1}{2}\Big)$, and

$$A_{n,l} = \frac{1}{K_2}\sum_{j=0}^{[l/2]}\binom{k+\frac{n-l+m}{2}+j}{j}\binom{\frac{n-l-m-1}{2}+j}{j} \quad (n-l-m \text{ is odd}),$$

where $K_2 = \left(\frac{n-l-m-1}{2}\right)!\,\Gamma\left(k+\frac{n-l+m}{2}+1\right)$.

The inequality (3.2) *reduces to an equality if and only if*

$$P(x) = Ax^m \sum_{j=0}^{[l/2]} \hat{b}_{n-l-m}^{(n-l-m+2j)}\, \hat{H}_{n-l-m+2j}^{(k+m)} \qquad (A = \text{const}).$$

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