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PAWLEY MULTIPLE ANTISYMMETRY THREE-DIMENSIONAL SPACE GROUPS $G_3^{l,p'}$ I. SYMMORPHIC GROUPS

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ABSTRACT. *By use of antisymmetric characteristic method, Pawley multiple antisymmetry three-dimensional space groups* $G_3^{l,p'}$ $(p = 3, 4, 6)$, *are derived.*

Crystallographic (p')-symmetry three-dimensional space groups (or Pawley colored antisymmetry groups) $G_3^{p'}$ $(p = 3, 4, 6)$ are derived by A. F. Palistrant [1,2,3,4]. From 73 symmorphic space groups G_3 are derived 670 junior $G_3^{p'}$ (96 $G_3^{3'}$ + 266 $G_3^{4'}$ + 308 $G_3^{6'}$), from 54 hemisymmorphic G_3 are derived 562 junior $G_3^{p'}$ (75 $G_3^{3'}$ + 252 $G_3^{4'}$ + 235 $G_3^{6'}$), and from 103 asymmorphic G_3 are derived 980 junior $G_3^{p'}$ (138 $G_3^{3'}$ + 432 $G_3^{4'}$ + 410 $G_3^{6'}$); this means, the category $G_3^{p'}$ $(p = 3, 4, 6)$ consists of 2212 junior groups (309 $G_3^{3'}$ + 950 $G_3^{4'}$ + 953 $G_3^{6'}$).

By the use of the generalized antisymmetric characteristic method (*AC*-method) [5,6,7], we will derive all crystallographic $(p', 2^l)$-symmetry three-dimensional space groups $G_3^{l,p'}$ $(p = 3, 4, 6)$.

1. Some General Remarks on (p')- and $(p', 2^l)$-symmetry

Pawley (p')-symmetry is a particular case of the general P- symmetry with $P = D_{p(2p)}$, where $D_{p(2p)}$ is the regular dihedral permutation group, generated by the permutations $e_1 = (1...p)(2p...p+1)$ and $e_2 = (1\ p+1)(2\ p+2)...(p\ 2p)$, $(p \geq 2)$ satisfying the relations:

$$e_1^p = e_2^2 = (e_1 e_2)^2 = E.$$

For every p the group $D_{p(2p)}$ is irreducible.

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By introducing l antiidentity transformations $e_3, ..., e_{l+2}$ [8,9] ($l \in N$) commuting between themeselves and with e_1, e_2, we have $(p', 2^l)$-symmetry, with the group $P = D_{p(2p)} \times C_2^l$.

In this work only junior groups of complete $(p', 2^l)$-symmetry will be considered. Every junior (p')-symmetry group $G^{p'}$ is derived from certain generating symmetry group G, as well as every junior $(p', 2^l)$-symmetry group $G^{l,p'}$ is derived from certain junior (p')-symmetry group [1,2,8].

Theorem 1. *a) A $(p', 2^l)$-symmetry group $G^{l,p'}$ is the junior $(p', 2^l)$-symmetry group if all relations given in the presentation of its generating symmetry group G remain satisfied after replacing the generators of the group G by the corresponding $(p', 2^l)$-symmetry group generators;*

b) a junior $(p', 2^l)$-symmetry group is called the M^m-type $(p', 2^l)$- symmetry group, if it is a M^m-type group regarded as a l- multiple antisymmetry group;

c) a junior M^m-type $(p', 2^l)$-symmetry group $G^{l,p'}$ is a group of the complete $(p', 2^l)$-symmetry, if for every i $(i = 1, ..., l+2)$ e_i-transformation can be obtained in the group $G^{l,p'}$ as an independent $(p', 2^l)$-symmetry transformation.

If only the condition c) it is not satisfied, such a group $G^{l,p'}$ is the uncomplete junior $(p', 2^l)$-symmetry group of the M^m-type.

Because the derivation of $(3', 2^l)$-symmetry groups coincides to the derivation of $(32, 2^l)$-symmetry groups [10], as the basis for the derivation of all crystallographic $(p', 2^l)$-symmetry groups $(p = 3, 4, 6)$, (4')- and (6')-symmetry groups will be sufficient. The derivation will be realised by the use of generalized AC:

Definition 1. *Let all the products of (p')-symmetry generators of a group $G^{p'}$, within which every generator participates once at the most, be formed, and then subsets of transformations equivalent with regard to (p')-symmetry, be separated. The resulting system is called the antisymmetric characteristic of the group $G^{p'}$ and denoted by $AC(G^{p'})$ [5,6,7,10].*

Theorem 2. *Two $(p', 2^l)$-symmetry groups of the M^m-type derived from the same (p')-symmetry group for m fixed $(m = 1, ..., l)$ are equal iff they possess equal antisymmetric characteristics.*

The problem of differing between complete and uncomplete $(p', 2^l)$-symmetry junior M^m-type groups can be solved by the use of the homomorphism of the subgroup $C_p = \{e_1\}$ of the group $D_{p(2p)}$ to the group C_2 at $p = 0 (\mathrm{mod}\ 2)$:

$$e_1^{2k-1} \to e_1, \quad e_1^{2k} \to E \quad (1 \leq k \leq (p+1)/2) \quad [5,10].$$

2. Symmorphic $(p', 2^l)$-symmetry Three-dimensional Space Groups $G_3^{l,p'}$ $(p = 3,4,6)$

For denoting space symmetry groups the original Fedorov symbols [1,2,8], Zamorzaev notation and International symbols [11] are used, where p'-symmetry transformations e_1^q, e_2, e_1e_2 $(p = 3,4,6;\ q|p)$ and $e_1^{p/2}e_2$ $(p = 4,6)$ are denoted by (p/q, '), ") and (2' respectively.

The application of the theoretical assumptions given above will be illustrated by example of complete $(p', 2^l)$-symmetry junior three-dimensional space groups of the M^m-type $(p = 3,4,6)$ derived in the family with the common generating symmetry group G =**7s** (P2/m), $\{a, b, c\}(2 : m)$ with the AC: $\{m, cm\}\{2, 2a, 2b, 2ab\}$ belonging to the AC- equivalency class VII [6, Tab.1]. At $p = 3$ we have two junior (3')-symmetry groups:

1) $\{a, b, c^{(3}\}(2 : m^{')})$,

2) $\{a^{(3}, b, c\}(2^{')} : m)$.

Because of the e_2-transformation $m^{')}$, the AC of the first group is of the form $\{e_2, e_2\}\{E, E, E, E\}$ and of the type $(2)(5)^1$, and the AC of the second is of the form $\{E, E\}\{e_2, e_2, e_2, e_2\}$ and of the same type $(3)(5)^1$. Hence, for the both of them $N_1 = 7$, $N_2 = 64$, $N_3 = 700$, $N_4 = 6720$ [6,10].

So, we have the following complete $(3', 2)$-symmetry groups:

$\{{}^*a, b, c^{(3}\}(2 : m^{')})$, $\{a, b, {}^*c^{(3}\}(2 : m^{')})$, $\{a, b, c^{(3}\}({}^*2 : m^{')})$,

$\{{}^*a, b, {}^*c^{(3}\}(2 : m^{')})$, $\{{}^*a, b, c^{(3}\}(2 :{}^* m^{')})$, $\{a, b, {}^*c^{(3}\}({}^*2 : m^{')})$,

$\{a, b, c^{(3}\}({}^*2 :{}^* m^{')})$, $\{{}^*a^{(3}, b, c\}(2^{')} : m)$, $\{a^{(3}, b, {}^*c\}(2^{')} : m)$,

$\{a^{(3}, b, c\}(2^{')} :{}^* m)$, $\{{}^*a^{(3}, b, {}^*c\}(2^{')} : m)$, $\{{}^*a^{(3}, b, c\}(2^{')} :{}^* m)$,

$\{a^{(3}, b, {}^*c\}({}^*2^{')} : m)$ and $\{a^{(3}, b, c\}({}^*2^{')} :{}^* m)$,

where the antisymmetries are denoted by an asterisk.

At $p = 0 (\mathrm{mod}\ 2)$, the form and, consequently, the type of $AC(G^{p'})$ is obtained by the use of the homomorphism mentioned in Chapter 1. By treating in this way the six (4')-symmetry groups belonging to this family, we have the following results: the three of them, $\{a^{(4}, b, c\}(2^{')} : m)$, $\{a^{(4}, b, c\}(2^{')} : m^{(2})$ and $\{a^{(4}, b, c^{(2}\}(2^{')} : m)$, possess the AC of the form $\{E, E\}\{e_2, e_2, e_1e_2, e_1e_2\}$ and of the type $(3)(\underline{9})$, where by $(\underline{9})$ is denoted the type of the term $\{e_2, e_2, e_1e_2, e_1e_2\}$ which contains e_2- and e_1e_2-transformations. These transformations are nonequivalent in the sense multiple antisymmetry, so according to the multiple antisymmetry the type of the term mentioned is (9). Contrariwise, they are equivalent in the sense of (p')-symmetry, so the type of this term is denoted by $(\underline{9})$. This is the reason why the derivation of multiple antisymmetry groups from the (p')- symmetry groups with such antisy mmetric cha racteristics cannot be simply reduced on the theory of multiple antisymmetry, this means, on the derivation of multiple antisymmetry groups of the M^{m+2}-type from the M^2-type groups, as it has been done in the case of $(p2,2^l)$-symmetry groups. From the first group $\{a^{(4},b,c\}(2^{')}:m)$ we derive $N_1(\{a^{(4},b,c\}(2^{')}:m))=9$ junior complete (4',2)-symmetry groups of the type M^1:

$\{a^{(4},b,c\}(2^{')}:^* m)$ with the AC: $\{e_3,e_3\}\{e_2,e_2,e_1e_2,e_1e_2\}$ of the type $(3)(\underline{9})^3$;

$\{a^{(4},b,c\}(^*2^{')}:^* m)$ with the AC: $\{e_3,e_3\}\{e_2e_3,e_2e_3,e_1e_2e_3,e_1e_2e_3\}$ of the type $(3)(\underline{9})^3$;

$\{^*a^{(4},b,c\}(2^{')}:^* m)$ with the AC: $\{e_3,e_3\}\{e_2,e_2,e_1e_2e_3,e_1e_2e_3\}$ of the type $(3)(9)^3$;

$\{a^{(4},b,^*c\}(2^{')}:m)$ with the AC: $\{E,e_3\}\{e_2,e_2,e_1e_2,e_1e_2\}$ of the type $(4)(\underline{9})^3$;

$\{a^{(4},b,^*c\}(^*2^{')}:m)$ with the AC: $\{E,e_3\}\{e_2e_3,e_2e_3,e_1e_2e_3,e_1e_2e_3\}$ of the type $(4)(\underline{9})^3$;

$\{^*a^{(4},b,^*c\}(2^{')}:m)$ with the AC:$\{E,e_3\}\{e_2,e_2,e_1e_2e_3,e_1e_2e_3\}$ of the type $(4)\ (9)^3$;

$\{a^{(4},^*b,c\}(2^{')}:m)$ with the AC: $\{e_3,e_3\}\{e_2,e_1e_2,e_2e_3,e_1e_2e_3\}$ of the type $(3)(\underline{16})^3$;

$\{a^{(4},^*b,c\}(2^{')}:^* m)$ with the AC: $\{e_3,e_3\}\{e_2,e_1e_2,e_2e_3,e_1e_2e_3\}$ of the type (3)
$(\underline{16})^3$;

$\{a^{(4},^*b,c\}(2^{')}:m)$ with the AC: $\{E,e_3\}\{e_2,e_1e_2,e_1e_2e_3,e_2e_3\}$ of the type $(4)(\underline{16})^3$.

From the groups with the AC of the type $(3)(\underline{9})^3$ can be derived the 6 M^2-type groups: 2 of the type $(4)(\underline{9})^4$, 1 of the type $(4)(9)^4$, 2 of the type $(3)(\underline{16})^4$ and 1 of the type $(4)(\underline{16})^4$; from the group with the AC of the type $(3)(9)^3$ the 7 M^2-type groups: 4 of the type $(4)(9)^4$, 2 of the type $(3)(16)^4$ and 1 of the type $(4)(16)^4$; from the groups with the AC of the type $(4)(\underline{9})^3$ the 10 M^2-type groups: 4 of the type $(4)(\underline{9})^4$, 2 of the type $(4)(9)^4$ and 4 of the type $(4)(\underline{16})^4$; from the group with the AC of the type $(4)(9)^3$ the 12 M^2-type groups: 8 of the type $(4)(9)^4$ and 4 of the type $(4)(16)^4$; from the group with the AC of the type $(3)(\underline{16})^3$ the 12 M^2-type groups: 4 of the type $(3)(\underline{16})^4$, 2 of the type $(3)(16)^4$, 4 of the type $(4)(\underline{16})^4$ and 2 of the type $(4)(16)^4$; from the group with the AC of t he type $(4)(\underline{16})^3$ the 18 M^2-type groups: 12 of the type $(4)(\underline{16})^4$ and 6 of the type $(4)(16)^4$. Hence, $N_2(\{a^{(4},b,c\}(2^{')}:m))=93$. Because from the groups of the types $(4)(\underline{9})^4$ and $(4)(9)^4$ can be derived 4 M^3-type groups, from the groups of the type

$(3)(\underline{16})^4$ 6, from the groups of the type $(3)(16)^5$ 8, from the groups of the type $(4)(\underline{16})^4$ 12 and from the groups of the type $(3)(\underline{16})^4$ 16 M^3-type groups, $N_3(\{a^{(4}, b, c\}(2^{')} : m)) = 840$.

The remaining three (4')-symmetry groups $\{a, b, c^{(4}\}(2 : m^{'})$, $\{a, b, c^{(4}\}(2^{(2} : m^{'})$ and $\{a^{(2}, b, c^{(4}\}(2 : m^{'})$ possess the AC of the form $\{e_2, e_1e_2\}\{E, E, E, E\}$ and of the type $(\underline{4})(5)^2$, where by $(\underline{4})$ is denoted the type of the term $\{e_2, e_1e_2\}$. In the case of (p')-symmetry groups with the AC in which the term $\{e_2, e_1e_2\}$ occurs once and only once, the series of the numbers $N_m^{p'}$ can be simply computed using the following theorem:

Theorem 3. *Let in the $AC(G^{p'})$ the term $\{e_2, e_1e_2\}$ occurs once and only once. If by N_m is denoted the number of the junior M^{m+2}-type multiple antisymmetry groups derived from the $AC(G^{p'})$ treated as the AC of a 2-multiple antisymmetry group, then $N_m(G^{p'}) = (2^m + 1)N_m/2^{m+1}$ $(m = 1, ..., l)$.*

Proof: Because the term $\{e_2, e_1e_2\}$ occurs once and only once in the $AC(G^{p'})$ it is independent from the other part of the AC. For $m = 1$ it is transformed into the four terms different in the sense of 3-multiple antisymmetry: $\{e_2, e_1e_2\}$, $\{e_2e_3, e_1e_2\}$, $\{e_2, e_1e_2e_3\}$, $\{e_2e_3, e_1e_2e_3\}$, resulting in the three terms different in the sense of $(p', 2)$-symmetry: $\{e_2, e_1e_2\}$, $\{e_2e_3, e_1e_2\} = \{e_2, e_1e_2e_3\}$, $\{e_2e_3, e_1e_2e_3\}$. Hence, $N_1(G^{p'}) = 3N_1/4$. Proceeding in the same way, for every m $(m = 2, ..., l)$ it is transformed into the 2^{m+1} terms different in the sense of $(m+2)$-multiple antisymmetry, resulting in the $2^m + 1$ terms different in the sense of $(p', 2^l)$-symmetry, so $N_m(G^{p'}) = (2^m + 1)N_m/2^{m+1}$.

Treated as the AC of a 2-multiple antisymmetry group, the AC of the form $\{e_2, e_1e_2\}\{E, E, E, E\}$ and of the type $(4)(5)^2$ gives $N_1 = 8$, $N_2 = 64$, $N_3 = 448$, so for the (4')- symmetry group $G^{4'} = \{a, b, c^{(4}\}(2 : m^{'})$ with the same AC of the type $(\underline{4})(5)^2$, $N_1(G^{4'}) = 6$, $N_2(G^{4'}) = 40$, $N_3(G^{4'}) = 252$. The same holds for the other two (4')-symmetry groups $\{a^{(4}, b, c\}(2^{(2} : m^{'})$, $\{a^{(2}, b, c^{(4}\}(2 : m^{'})$ with the identical AC. Hence, for the symmetry group **7s** (P2/m), $N_1^{4'}(\mathbf{7s}) = 45$, $N_2^{4'}(\mathbf{7s}) = 399$, $N_3^{4'}(\mathbf{7s}) = 3276$.

From the ten (6')-symmetry groups of the same family, the two of them, $\{a, b, c^{(3}\}$ $(2^{(2} : m^{'})$ and $\{a^{(3}, b, c\}(2^{')} : m^{(2)})$ possess the AC of the type $(3)(5)^2$ giving $N_1^{6'} = 5$, $N_2^{6'} = 34$, $N_3^{6'} = 234$; the one of them, $\{a^{(2}, b, c^{(3}\}(2 : m^{'})$ the AC of the type $(3)(9)^2$ giving $N_1^{6'} = 11$, $N_2^{6'} = 132$, $N_3^{6'} = 1344$; the one of them, $\{a^{(3}, b, c^{(2}\}(2^{')} : m)$ the AC of the type $(4)(5)^2$ giving $N_1^{6'} = 8$, $N_2^{6'} = 64$, $N_3^{6'} = 448$; the two of them, $\{a^{(6}, b, c\}(2^{')} : m)$ and $\{a^{(6}, b, c\}(2^{')} : m^{(2)})$ the AC of the type $(3)(\underline{9})^2$ giving $N_1^{6'} = 9$, $N_2^{6'} = 93$,

$N_3^{6'} = 840$; the three of them, $\{a,b,c^{(6}\}(2:m^{'})$, $\{a,b,c^{(6}\}(2^{(2}:m^{'})$ and $\{a^{(2},b,c^{(6}\}(2:m^{'})$ the AC of the type $(\underline{4})(9)^2$ giving $N_1^{6'} = 12$, $N_2^{6'} = 150$, $N_3^{6'} = 1512$; and the one of them, $\{a^{(6},b,c^{(2}\}(2^{'}):m)$ the AC of the type $(4)(\underline{9})^2$ giving $N_1^{6'} = 13$, $N_2^{6'} = 168$, $N_3^{6'} = 1680$. Hence, $N_1^{6'}(\mathbf{7s}) = 84$, $N_2^{6'}(\mathbf{7s}) = 848$, $N_3^{6'}(\mathbf{7s}) = 7616$.

The possible applications of the generalized colored symmetry groups are considered by V.A.Koptsik [12].

3. Partial Catalogue of Symmorphic $(p',2^l)$-symmetry Three-dimensional Space Groups $G_3^{l,p'}$ $(p=3,4,6)$

In the same manner, the partial catalogue of all complete $(p',2^l)$-symmetry junior symmorphic three-dimensional space groups of the M^m-type $G_3^{l,p'}$ $(p=3,4,6)$, is realized. According to the work [6], this partial catalogue gives the possibility for their complete cataloguation.

The complete results are given only for the first ten symmetry groups $1s-10s$. The remaining tables of this partial catalogue can be ordered from the author.

$2s$ $(P1)$ $\{a,b,c\}(2)$, $AC:\{2,2a,2b,2c,2ab,2ac,2bc,2abc\}$, II

1) $\{a^{(3},b,c\}(2'))$,	$(9)^1$,	$N1=1$ $N2=1$ $N3=1$
2) $\{a^{(4},b,c\}(2'))$,	$(25)^2$,	$N1=1$ $N2=1$
3) $\{a^{(6},b,c\}(2'))$,	$(25)^2$,	$N1=1$ $N2=1$

$3s$ $(P2)$, $\{a,b,c\}(2)$, $AC:\{c\}\{2,2a,2b,2ab\}$, III

1) $\{a^{(3},b,c\}(2^{'}))$,	$(2)(5)^1$, $N_1=4$ $N_2=16$ $N_3=56$
2) $\{a^{(4},b,c\}(2^{'}))$,	$(2)(9)^2$, $N_1=5$ $N_2=18$
3) $\{a^{(4},b,c(2\}(2^{'}))$,	$(2)(9)^2$, $N_1=5$ $N_2=18$
4) $\{a^{(6},b,c\}(2^{'}))$,	$(2)(9)^2$, $N_1=5$ $N_2=18$
5) $\{a^{(3},b,c(2\}(2^{'}))$,	$(2)(5)^2$, $N_1=2$ $N_2=4$
6) $\{a^{(6},b,c(2\}(2^{'}))$,	$(2)(9)^2$, $N_1=5$ $N_2=18$

$4s$ $(B2)$ $\{a,b,(a+c)/2\}(2)$, $AC:\{2,2b\}\{2(a+c)/2,2b(a+c)/2\}$, IV

1) $\{a,b^{(3},(a+c)/2\}(2^{'}))$,	$(3)(3)^1$, $N_1=3$ $N_2=6$
2) $\{a^{(2},b,(a+c)^{'})/2)\}(2^{"}))$,	$(3)(3)^2$, $N_1=1$
3) $\{a,b^{(4},(a+c)/2\}(2^{'}))$,	$(4)(4)^2$, $N_1=3$
4) $\{a^{(2},b,(a+c)^{(4}/2\}(2^{'}))$,	$(3)(3)^2$, $N_1=1$
5) $\{a,b^{(6},(a+c)/2\}(2^{'}))$,	$(4)(4)^2$, $N_1=3$
6) $\{a^{(3},b,(a+c)^{(6}/2\}(2^{'}))$,	$(3)(3)^2$, $N_1=1$

$5s$ (Pm) $\{a,b,c\}(m)$, $AC:\{a,b,ab\}\{m,mc\}$, V

1) $\{a,b,c^{(3}\}(m^{'})$, $(4)(3)^1$, $N_1=4$ $N_2=22$ $N_3=112$
2) $\{a,b,c^{(4}\}(m^{'})$, $(4)(4)^2$, $N_1=3$ $N_2=10$
3) $\{a^{(2},b,c^{(4}\}(m^{'})$, $(4)(4)^2$, $N_1=3$ $N_2=10$
4) $\{a,b,c^{(6}\}(m^{'})$, $(4)(4)^2$, $N_1=3$ $N_2=10$
5) $\{a^{(2},b,c^{(3}\}(m^{'})$, $(6)(3)^2$, $N_1=5$ $N_2=24$
6) $\{a^{(2},b,c^{(6}\}(m^{'})$, $(6)(4)^2$, $N_1=6$ $N_2=30$

$6s$ (Bm) $\{a,b,(a+c)/2\}(m)$, $AC:\{m\}\{(a+c)/2,b(a+c)/2\}$, VI

1) $\{a,b,(a+c)^{(3}/2\}(m^{'})$, $(2)(3)^1$, $N_1=4$ $N_2=12$
2) $\{a^{(2},b,(a+c)^{')}/2)\}(m^{"})$, $(2)(3)^2$, $N_1=2$
3) $\{a,b,(a+c)^{(4}/2\}(m^{'})$, $(2)(3)^2$, $N_1=2$
4) $\{a,b^{(2},(a+c)^{(4}/2\}(m^{'})$, $(2)(3)^2$, $N_1=2$
5) $\{a,b,(a+c)^{(6}/2\}(m^{'})$, $(2)(3)^2$, $N_1=2$
6) $\{a,b^{(2},(a+c)^{(3}/2\}(m'))$, $(2)(4)^2$, $N_1=4$

$7s$ $(P2/m)$ $\{a,b,c\}(2:m)$, $AC:\{m,cm\}\{2,2a,2b,2ab\}$, VII

1) $\{a,b,c^{(3}\}(2:m^{'})$, $N_4=6720$ $(3)(5)^1$, $N_1=7$ $N_2=64$ $N_3=700$
2) $\{a^{(3},b,c\}(2^{'}):m)$, $N_4=6720$ $(3)(5)^1$, $N_1=7$ $N_2=64$ $N_3=700$
3) $\{a,b,c^{(4}\}(2:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
4) $\{a,b,c^{(4}\}(2(2:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
5) $\{a^{(4},b,c\}(2^{'}):m)$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
6) $\{a^{(4},b,c\}(2^{'}):m^{(2})$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
7) $\{a^{(2},b,c^{(4}\}(2:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
8) $\{a^{(4},b,c^{(2}\}(2^{'}):m)$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
9) $\{a,b,c^{(3}\}(2^{(2}:m^{'})$, $(3)(5)^2$, $N_1=5$ $N_2=34$ $N_3=224$
10) $\{a^{(3},b,c\}(2^{'}):m^{(2})$, $(3)(5)^2$, $N_1=5$ $N_2=34$ $N_3=224$
11) $\{a,b,c^{(6}\}(2:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
12) $\{a,b,c^{(6}\}(2^{(2}:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
13) $\{a^{(6},b,c\}(2^{'}):m)$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
14) $\{a^{(6},b,c\}(2^{'}):m^{(2})$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
15) $\{a^{(3},b,c^{(2}\}(2^{'}):m)$, $(4)(5)^2$, $N_1=8$ $N_2=64$ $N_3=448$
16) $\{a^{(2},b,c^{(3}\}(2:m^{'})$, $(3)(9)^2$, $N_1=11$ $N_2=132$ $N_3=1344$
17) $\{a^{(2},b,c^{(6}\}(2:m^{'})$, $(4)(9)^2$, $N_1=12$ $N_2=150$ $N_3=1512$
18) $\{a^{(6},b,c^{(2}\}(2^{'}):m)$, $(4)(9)^2$, $N_1=13$ $N_2=168$ $N_3=1680$

$8s$ $(B2/m)$ $\{a,b,(a+c)/2\}(2:m)$, $AC:\{m\}\{2,2b\}\{(a+c)/2,b(a+c)/2\}$, $VIII$

1) $\{a,b,(a+c)^{(3}/2\}(2:m^{'})$, $(2)(3)(3)^1$, $N_1=8$ $N_2=60$ $N_3=336$
2) $\{a,b^{(3},(a+c)/2\}(2^{'}):m)$, $(2)(3)(3)^1$, $N_1=8$ $N_2=60$ $N_3=336$

3) $\{a^{(2}, b, (a+c)'/2)\}(2^{"}) : m^{"})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
4) $\{a, b, (a+c)^{(4}/2\}(2 : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
5) $\{a, b, (a+c)^{(4}/2\}(2^{(2} : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
6) $\{a, b^{(4}, (a+c)/2\}(2^{'}) : m)$, $(2)(4)(4)^2$, $N_1 = 9$ $N_2 = 60$
7) $\{a, b^{(4}, (a+c)/2\}(2^{'}) : m^{(2})$, $(2)(4)(4)^2$, $N_1 = 9$ $N_2 = 60$
8) $\{a, b^{(2}, (a+c)^{(4}/2\}(2 : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
9) $\{a^{(2}, b, (a+c)^{(4}/2\}(2^{'}) : m)$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
10) $\{a^{(2}, b, (a+c)^{(4}/2\}(2^{'}) : m^{(2})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
11) $\{a, b, (a+c)^{(3}/2\}(2^{(2} : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
12) $\{a, b^{(3}, (a+c)/2\}(2^{'}) : m^{(2})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
13) $\{a, b, (a+c)^{(6}/2\}(2 : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
14) $\{a, b, (a+c)^{(6}/2\}(2^{(2} : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
15) $\{a, b^{(2}, (a+c)^{(3}/2\}(2 : m^{'})$, $(2)(4)(4)^2$, $N_1 = 12$ $N_2 = 96$
16) $\{a, b^{(6}, (a+c)/2\}(2^{'}) : m)$, $(2)(4)(4)^2$, $N_1 = 9$ $N_2 = 60$
17) $\{a, b^{(6}, (a+c)/2\}(2^{'}) : m^{(2})$, $(2)(4)(4)^2$, $N_1 = 9$ $N_2 = 60$
18) $\{a^{(3}, b, (a+c)^{(6}/2\}(2^{'}) : m)$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
19) $\{a^{(3}, b, (a+c)^{(6}/2\}(2^{'}) : m^{(2})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$

$9s$ $(P222)$ $a, b, c(2 : 2')$, $AC : \{\{c\}\{2, 2a, 2b, 2ab\}, \{b\}\{2', 2'a, 2'c, 2'ac\}, \{a\}$ $\{22', 22'b, 22'c, 22'bc\}\}\{\{2, 2', 22'\}, \{2a, 2'a, 22'\}, \{2', 2b, 22'b\}, \{2'a, 2ab, 22'b\},$ $\{2, 2'c, 22'c\}, \{2a, 2'ac, 22'c\}, \{2b, 2'c, 22'bc\}, \{2ab, 2'ac, 22'bc\}\}$, IX

1) $\{a, b, c(3\}(2 : 2'))$, $(4)(5,(5,5))^1$, $N_1 = 8$ $N_2 = 96$ $N_3 = 1516$
$N_4 = 20160$
2) $\{a, b, c^{(4}\}(2 : 2^{'})$, $(6)(5,(9,9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
3) $\{a, b, c^{(4}\}(2^{(2} : 2^{'})$, $(6)(5,(9,9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
4) $\{a^{(2}, b, c^{(4}\}(2 : 2^{'})$, $(6)(5,(9,9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
5) $\{a^{(2}, b^{(2}, c^{(4}\}(2 : 2^{'})$, $(6)(5,(9,9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
6) $\{a, b, c^{(3}\}(2^{(2} : 2^{'})$, $(4)(5,(5,5))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
7) $\{a, b, c^{(6}\}(2 : 2^{'})$, $(6)(5,(9,9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
8) $\{a, b, c^{(6}\}(2^{(2} : 2^{'})$, $(6)(5,(9,9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
9) $\{a^{(2}, b, c^{(6}\}(2 : 2^{'})$, $(6)(9,9,9)^2$, $N_1 = 20$ $N_2 = 384$ $N_3 = 5376$
10) $\{a^{(2}, b^{(2}, c^{(6}\}(2 : 2^{'})$, $(4)(9,(9,9))^2$, $N_1 = 9$ $N_2 = 156$ $N_3 = 1680$
11) $\{a^{(2}, b, c^{(3}\}(2 : 2^{'})$, $(6)(5,9,9)^2$, $N_1 = 13$ $N_2 = 196$ $N_3 = 1680$
12) $\{a^{(2}, b^{(2}, c^{(3}\}(2 : 2^{'})$, $(6)(9,(9,9))^2$, $N_1 = 11$ $N_2 = 172$ $N_3 = 1680$

$10s$ $(C222)$ $\{a, (a+b)/2, c\}(2 : 2')$, $AC : \{(a+b)/2\}\{(2', 22'), (2'c, 22'c)\}$, $VIII$

1) $\{a, (a+b)^{(3}/2, c\}(2^{'}) : 2^{'})$, $(2)(4,4)^1$, $N_1 = 14$ $N_2 = 168$ $N_3 = 1344$
2) $\{a, (a+b)/2, c^{(3}\}(2 : 2^{'})$, $(2)(3,3)^1$, $N_1 = 8$ $N_2 = 60$ $N_3 = 336$
3) $\{a^{(2}, (a+b)'/2), c\}(2 : 2^{"})$, $(2)(3,3)^2$, $N_1 = 6$ $N_2 = 24$
4) $\{a^{(2}, (a+b)'/2), c^{(2}\}(2 : 2^{"})$, $(2)(3,3)^2$, $N_1 = 6$ $N_2 = 24$

5) $\{a, (a+b)/2, c^{(4}\}(2:2^{'})$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$
6) $\{a, (a+b)/2, c^{(4}\}(2^{(2}:2^{'})$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$
7) $\{a, (a+b)^{(2}/2, c^{(4}\}(2:2^{'})$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$
8) $\{a, (a+b)^{(2}/2, c^{(4}\}(2^{(2}:2^{'})$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$
9) $\{a, (a+b)^{(4}/2, c\}(2^{')}:2^{'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$
10) $\{a, (a+b)^{(4}/2, c\}(2^{')}:2^{(2'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$
11) $\{a, (a+b)^{(4}/2, c^{(2}\}(2^{')}:2^{'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$
12) $\{a, (a+b)/2, c^{(3}\}(2^{(2}:2^{'})$, $(2)(4,4)^2$, $N_1 = 7$ $N_2 = 36$
13) $\{a, (a+b)^{(3}/2, c\}(2^{')}:2^{(2'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$
14) $\{a, (a+b)/2, c^{(6}\}(2:2^{'})$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$
15) $\{a, (a+b)/2, c^{(6}\}(2^{(2}:2^{'})$, $(2)(4,4)^2$, $N_1 = 7$ $N_2 = 36$
16) $\{a, (a+b)^{(2}/2, c^{(3}\}(2:2^{'})$, $(2)(3,3)^2$, $N_1 = 6$ $N_2 = 24$
17) $\{a, (a+b)^{(2}/2, c^{(3}\}(2^{(2}:2^{'})$, $(2)(4,4)^2$, $N_1 = 7$ $N_2 = 36$
18) $\{a, (a+b)^{(2}/2, c^{(6}\}(2:2^{'})$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$
19) $\{a, (a+b)^{(2}/2, c^{(6}\}(2^{(2}:2^{'})$, $(2)(4,4)^2$, $N_1 = 7$ $N_2 = 36$
20) $\{a, (a+b)^{(6}/2, c\}(2^{')}:2^{'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$
21) $\{a, (a+b)^{(6}/2, c\}(2^{')}:2^{(2'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$
22) $\{a, (a+b)^{(3}/2, c^{(2}\}(2^{')}:2^{'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$
23) $\{a, (a+b)^{(6}/2, c^{(2}\}(2^{')}:2^{'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

The complete results are given in Table 1.

Table 1.

	(3')	(4')	(6')								
2s	1	1	1	24s		8		48s	3	2	7
3s	1	2	3	25s		7		49s	1		1
4s	1	3	2	26s	1	4	3	50s	2		6
5s	1	2	3	27s	1	2	1	51s	3	1	3
6s	1	3	2	28s	1	6	5	52s	1	1	1
7s	2	6	10	29s	1	7	3	53s	2	2	5
8s	2	8	9	30s	1	13	5	54s	5	2	14
9s	1	4	7	31s	1	10	3	55s	3	2	7
10s	2	9	12	32s	1	10	5	56s	4	2	11
11s	1	3	3	33s	1	11	5	57s	2	2	5
12s	1	3	4	34s	1	7	3	58s	3	4	17
13s	1	6	11	35s	1	7	3	63s		1	
14s	1	6	7	36s	1	18	11	65s	1		1
15s	2	10	12	37s	1	16	7	66s	1		1
16s	1	4	3	40s	2		2	67s	1		
17s	1	4	7	41s	1		1	68s	1		1
18s	1	6	11	42s	1		1	69s	1	1	1
19s	2	15	22	43s	1	1	1	70s	1		

$20s$	1	6	5	$44s$	4	1	4	$71s$	1		2
$21s$	1	6	9	$45s$	6	1	6	$72s$	1	2	3
$22s$		2		$46s$	4	1	4	$73s$	1		1
$23s$		3		$47s$	2	2	5				

	$(3',2)$	$(4',2)$	$(6',2)$	$(3',2^2)$	$(4',2^2)$	$(6',2^2)$
$2s$	1	1	1	1	1	1
$3s$	4	10	12	16	36	40
$4s$	3	5	4	6		
$5s$	4	6	14	22	20	64
$6s$	4	6	6	12		
$7s$	14	45	84	128	399	848
$8s$	16	54	66	120	264	360
$9s$	8	28	81	96	352	1260
$10s$	22	64	88	228	360	520
$11s$	4	6	8	12		
$12s$	5	17	23	42	108	156
$13s$	16	90	184	300	1440	3216
$14s$	14	64	84	168	480	672
$15s$	20	84	108	192	528	720
$16s$	6	12	12	24		
$17s$	14	48	84	168	384	672
$18s$	16	90	274	450	2340	9636
$19s$	38	252	420	804	4392	8016
$20s$	10	44	48	96	256	336
$21s$	18	168	228	432	4032	5376
$22s$		8				
$24s$		80			576	
$25s$		28				
$26s$	3	10	7	6		
$27s$	1					
$28s$	7	42	34	54	234	192
$29s$	6	28	12	24		
$30s$	7	80	34	54	420	192
$31s$	6	40	12	24		
$32s$	8	69	34	60	372	168
$33s$	7	70	34	54	378	192
$34s$	6	28	12	24		
$35s$	4	22	8	12		
$36s$	16	294	202	300	5040	3720
$37s$	14	192	84	168	1536	672
$40s$	4					
$41s$	2					
$42s$	2					
$43s$	1					

44s	4					
45s	6					
46s	4					
47s	8	6	13	24		
48s	12	6	20	36		
49s	2					
50s	12		24	48		
51s	3					
52s	1					
53s	8	6	14	24		
54s	20	6	38	60		
55s	12	6	19	36		
56s	16	6	29	48		
57s	8	6	13	24		
58s	30	36	160	288	240	1104
65s	2					
66s	2					
68s	1					
69s	2					
71s	4					
72s	6	8	12	24		
73s	2					

	$(3',2^3)$	$(4',2^3)$	$(6',2^3)$	$(3',2^4)$	$(4',2^4)$	$(6',2^4)$	$(3',2^5)$
2s	1						
3s	56						
5s	112						
7s	1400	3276	7616	13440			
8s	672						
9s	1516	3360	13776	20160			
10s	1680						
12s	336						
13s	5712	18144	43008	80640			
14s	1344						
15s	2688						
17s	1344						
18s	17220	77112	364224	685440	2056320	10321920	19998720
19s	16464	49392	106848	241920			
20s	672						
21s	10080	64512	86016	161280			
28s	336						
30s	336						
32s	336						
33s	336						
36s	5712	57456	45024	80640			

37s 1344
58s 2016

For the complete $(p', 2^l)$-symmetry junior symmorphic three-dimensional space groups of the M^m-type the numbers $N_m^{p'}$ $(p = 3, 4, 6)$ are the following:

$N_0^{p'} = 96G_3^{3'} + 266G_3^{4'} + 308G_3^{6'} = 670$

$N_1^{p'} = 496G_3^{1,3'} + 2171G_3^{1,4'} + 2644G_3^{1,6'} = 5311$

$N_2^{p'} = 4709G_3^{2,3'} + 24088G_3^{2,4'} + 38133G_3^{2,6'} = 66930$

$N_3^{p'} = 71713G_3^{3,3'} + 273252G_3^{3,4'} + 666512G_3^{3,6'} = 1011477$

$N_4^{p'} = 1283520G_3^{4,3'} + 2056320G_3^{4,4'} + 10321920G_3^{4,6'} = 13661760$

$N_5^{p'} = 19998720G_3^{5,3'} = 19998720$

References

[1] Zamorzaev A.M., Gelyarskij E.I., Palistrant A.F., *Tsvetnaya simmetriya, eyo obobscheniya i prilozeniya*, Shtiintsa, Kishinev, 1978.

[2] Zamorzaev A.M., Karpova Yu.S., Lungu A.P., Palistrant A.F., *P-simmetriya i eyo dal'nejshee razvitie*, Shtiintsa, Kishinev, 1986.

[3] Palistrant A.F., *Prostranstvennye gruppy (p')-simmetrii (polievy) i ih primenenie k vyvodu pyatimernyh kristallograficheskih grupp simmetrii*, DAN SSSR 254, 5 (1980), 1126-1130.

[4] Palistrant A.F., *Gruppy tsvetnoi simmetrii, ih obobscheniya i prilozeniya*, Avtoref. dis. ... dr-a fiz.-mat. nauk, KGU, Kishinev, 1981.

[5] Jablan S.V., *Plane and Layer $(p', 2^l)$-symmetry Groups*, Mat. Vesnik 42 (1990), 87-95.

[6] Jablan S.V., *A New Method of Deriving and Cataloguing Simple and Multiple Antisymmetry G_3^l Space Groups*, Acta Cryst., A43 (1987), 326-337.

[7] Jablan S.V., *Algebra of Antisymmetric Characteristics*, Publ. Inst. Math. 47 (61), 1990, 39-55.

[8] Zamorzaev A.M., *Teoriya prostoj i kratnoj antisimmetrii*, Shtiintsa, Kishinev, 1976.

[9] Zamorzaev A.M., Palistrant A.F., *Antisymmetry, its Generalizations and Geometrical Applications*, Z.Kristall., 151 (1980), 231-248.

[10] Jablan S.V., *$(p2, 2^l)$-symmetry Space Groups $G_3^{l,p2}$*, Acta Crystall. A48 (1992) (to appear).

[11] Hahn Th. (Editor), *International Tables for Crystallography*, Vol. A, D. Reidel Publ. Co., Dordecht, 1983, 1987.

[12] Koptsik V.A., *Generalized Symmetry in Crystal Physics*, Comput. Math. Applic. 16, 5-8 (1988), 407-424.

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