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ITERATION METHOD FOR THE EQUATIONS OF I. N. VECUA TYPE OF HIGHER ORDER WITH ANALYTICAL COEFFICIENTS

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ABSTRACT. *We use two methods for the integration of Vecua equations: 1. the method of the areolar series, 2. the method of the iterations.*

We solve the following I. N. Vecua equation

$$\frac{\partial W}{\partial \overline{z}} = AW + B\overline{W} + F, \tag{1}$$

where $A(z,\overline{z})$, $B(z,\overline{z})$ and $F(z,\overline{z})$ are given analytical coefficients of two variables. We use two methods:

1° By the method of areolar series (used by B. Ilievski [1]).

$$A = \sum_{i,j=0}^{\infty} a_{ij} z^i \overline{z}^j, \quad B = \sum_{i,j=0}^{\infty} b_{ij} z^i \overline{z}^j, \quad F = \sum_{i,j=0}^{\infty} f_{ij} z^i \overline{z}^j, \tag{2}$$

$$W = \sum_{i,j=0}^{\infty} c_{ij} z^i \overline{z}^j, \tag{3}$$

we have a solution in the form of series of coefficients and of the integration element $\Phi(z)$ - an arbitrary analytic function in the role of integration "constant" :

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$$
(4)\quad
\begin{aligned}
W(z,\overline{z}) &= \Phi + \int A\Phi d\overline{z} + \int A d\overline{z} + \int A d\overline{z} \int A\Phi d\overline{z} \\
&+ \int A d\overline{z} \int A d\overline{z} \int A\Phi d\overline{z} + \int B\overline{\Phi} dz + \int B d\overline{z} \int \overline{B\Phi} d\overline{z} \\
&+ \int B d\overline{z} \int \overline{B} dz \int \overline{B}\Phi dz + \cdots + \int A d\overline{z} \int B\overline{\Phi} d\overline{z} + \int B d\overline{z} \int \overline{A\Phi} dz \\
&+ \int A d\overline{z} \int A dz \int B\overline{\Phi} d\overline{z} + \int B dz \int \overline{B} dz \int \overline{A\Phi} d\overline{z} + \cdots \\
&+ \int F d\overline{z} + \int A d\overline{z} \int F d\overline{z} + \int B d\overline{z} \int \overline{F} dz + \int A d\overline{z} \int A d\overline{z} \int F d\overline{z} \\
&+ \int B d\overline{z} \int \overline{B} dz \int F d\overline{z} + \cdots
\end{aligned}
$$

The series (4) is convergent in a finite closed domain G of the complex plane $z = x + iy$ and the coefficients are analytic functions of z and $\overline{z}$.

2° By the method of the Vecua integral equation (see [2,3]). With applications in the theory of iteration, we get the solution in the form

$$
(5)\quad
\begin{aligned}
W(z,\overline{z}) &= \frac{1}{\pi} \iint_G \frac{A(y)W(y) + B(y)\overline{W}(y)}{\zeta - z} d\xi d\eta \\
&+ \frac{1}{\pi} \iint_G \frac{F(\zeta) d\xi d\eta}{\zeta - z} + \psi(z)
\end{aligned}
$$

or for $F = 0$,

$$
(6)\qquad W = \Phi(z) e^{\omega(z)},
$$

where Φ is an analytic C function of z, and

$$
(7)\qquad \omega(z) = \frac{1}{\pi} \iint_G \left[A(\zeta) + B(\zeta) \frac{\overline{W(\zeta)}}{W(\zeta)} \right] \frac{d\xi d\eta}{\zeta - z}, \quad (\zeta = \xi + i\eta \in G).
$$

In what follows we consider the problem to apply a similar procedure for the Vecua equations with conjugations of higher order.

We start with the equation

$$
\frac{\partial^2 \omega}{\partial \overline{z}^2} + A(z) W = 0
$$

which is an ordinary areolar equation of the second order and is analogous to an ordinary differential equation

$$
y'' + \lambda y = 0
$$

with constant coefficients.

The equation

$$\frac{\partial^2 \omega}{\partial \overline{z}^2} + A(z,\overline{z})W = 0$$

is a complex analogy of the real equations of Hill, Lame and Mathie. Both of them are not Vecua equations of higher order.

The first equation which could be named a Vecua equation of higher order is

$$\frac{\partial^2 \omega}{\partial \overline{z}^2} + A(z,\overline{z})\overline{W} = 0 \tag{8}$$

and that contains the conjugation of a unknown function. Since

$$W = \rho e^{i\phi}, \qquad \overline{W} = We^{-2i\phi} = We^{-2i\,\mathrm{arctg}((W-\overline{W})/(i(W+\overline{W}))}$$

we have that equation (8) is not linear but trancendental, because the operation (the rotation of an argument while the modul remains the same) is such operation. Because of this, here we have an essential difference between areolar equations wich are almost completly analogie to ordinary differential equations and Vecua type equations, wich are specific in some sense.

Denote by $\hat{\int}$ the inverse operator of $\frac{\partial}{\partial \overline{z}}$. Then we have:

$$\frac{\partial}{\partial \overline{z}}\left(\frac{\partial W}{\partial \overline{z}}\right) = -A\overline{W}$$

$$\frac{\partial \omega}{\partial \overline{z}} = -\hat{\int} A\overline{W} = -\int A\overline{W}\,d\overline{z} + \Phi_2(z).$$

If we define the operator

$$T^1 W = \Phi_2(z) - \int A\overline{W}\,d\overline{z}, \tag{9}$$

then it is easy to prove that this is a contraction operator for every analytic coefficient $A(z,\overline{z})$.

Next, we have

$$\begin{aligned} W &= \hat{\int}\left[-\hat{\int} A\overline{W}\right] = \hat{\int}\left[\Phi_2 - \int A\overline{W}\,d\overline{z}\right] \\ &= \Phi_1(z) + \int \Phi_2(z)\,d\overline{z} - \int\left[\int A\overline{W}\,d\overline{z}\right]d\overline{z} \end{aligned} \tag{10}$$

It easy to check:

Theorem. *The operator*

$$T^2W = \Phi_1 + \Phi_2\overline{z} - \iint A(z,\overline{z})\overline{W}d\overline{z}^2 \tag{11}$$

is a contraction operator for every analytical choise $\Phi_1(z), \Phi_2(z), A(z.\overline{z})$. *If we substitute* W *by (10) in (11) we can define the sequence* $T^3W, \dots, T^nW$ *and than prove that* T^nW *is a contraction operator.*

Since, by the Cauchy theorem, the analytic equation has an analytic solution, the right side in T^nW is always continuous so the iteration method is valid for (8). Puting $W_1 = T^2W$ we have

$$\begin{aligned} W_2 &= \Phi_1 + \Phi_2\overline{z} - \iint A\left[\overline{\Phi_1 + \Phi_2\overline{z} - \iint A\overline{W}d\overline{z}^2}\right]d\overline{z}^2 \\ &= \Phi_1 + \Phi_2\overline{z} - \iint A\overline{\Phi}_1 d\overline{z} - \iint A\overline{\Phi}_2 z d\overline{z}^2 + \iint A d\overline{z}^2 \iint \overline{A}W dz^2. \end{aligned}$$

The last member has a role a remainder. Its estimation is of the order $|A|^2|W||z|^5/5!$.
Next, we have

$$\begin{aligned} W_3 = {} & \Phi_1 + \Phi_2\overline{z} - \iint A\overline{\Phi}_1 d\overline{z} - \iint A\Phi_2 z d\overline{z}^2 \\ & + \iint A d\overline{z}^2 \iint \overline{A}\Phi_1 dz^2 + \iint A d\overline{z}^2 \iint \overline{A}\Phi_2 dz^2 \\ & - \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A\overline{\Phi}_1 d\overline{z} - \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A\overline{\Phi}_2 d\overline{z}^2 + R_3, \end{aligned}$$

where the remainder R_3 is given by

$$R_3 = \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A}W d\overline{z}^2.$$

In the next step we give th 4^{th} approximation

(12)

$$\begin{aligned}
W_4 \simeq W(z,\bar{z}) &= \Phi_1(z) + \Phi_2(z)\bar{z} - \iint A\overline{\Phi}_1 d\bar{z}^2 - \iint A\overline{\Phi}_2 z d\bar{z}^2 \\
&+ \iint A d\bar{z}^2 \iint \overline{A}\Phi_1 dz + \iint A d\bar{z}^2 \iint \overline{A}\Phi_2 \bar{z} dz^2 \\
&- \iint A dz^2 \iint \overline{A} d\bar{z}^2 \iint A\overline{\Phi}_1 d\bar{z}^2 - \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A\overline{\Phi}_2 z\bar{z} dz^2 \\
&+ \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A}\Phi_1 dz^2 \\
&+ \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A}\Phi_2 \bar{z} dz^2 \\
&- \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A\overline{\Phi}_1 d\bar{z}^2 \\
&- \iint A dz^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A\overline{\Phi}_2 z d\bar{z}^2 \\
&+ \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A}\Phi_1 d\bar{z}^2 \\
&+ \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A}\Phi_2 \bar{z} dz^2 \\
&- \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} d\bar{z}^2 \iint \overline{A}\Phi_1 d\bar{z}^2 \\
&- \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A\Phi_2 z dz^2 + R_4,
\end{aligned}$$

where the remainder R_4 is

$$\begin{aligned}
R_4 = \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \iint \overline{A} d\bar{z}^2 \iint A d\bar{z}^2 \iint \overline{A} dz^2 \iint A d\bar{z}^2 \\
\iint \overline{A} W(z,\bar{z}) dz^2
\end{aligned}$$

and can be easily estimated. We may be satisfied by the second, third and if it is necessary by the fourth approximation.

Consequences. Definitions of new special functions. Conjugate exponent of the Vecua equation

$$\frac{\partial W}{\partial \bar{z}} + 1 \cdot \overline{W} = 0$$

defines a new exponent ${}^{\bar{z}}e$, as it was showed by Vecua. The Polozij [4] has tried to introduce p-exponential function denoted by ${}^{p}e^{z}$. A similar exponent can be defined by the equation

$$\frac{\partial W}{\partial \overline{z}} + A(z)\overline{W} = 0,$$

where $A(z)$ is an analytic function of z. We will denote this exponent by $\overset{\overline{z}}{_A}e^{z}$, the equation

$$\frac{\partial W}{\partial \overline{z}} + A(z,\overline{z})\overline{W} = 0$$

defines some other kinds of functions.

Conjugate areolar cosinus and sinus. If in (8) we take $A(z,\overline{z}) = 1$, then the equation

$$\frac{\partial^2 W}{\partial \overline{z}^2} + \overline{W} = 0$$

in a natural way defines $\sin\overline{z}$ and $\cos\overline{z}$. For $\Phi_1 \equiv 1$, $\Phi_2 \equiv 0$ we have by the definition:

$$\cos_A \overline{z} \equiv 1 - \frac{\overline{z}^2}{2!} + \frac{\overline{z}^2}{2!}\frac{z^2}{2!} - \cdots$$

Similarly, for $\Phi_1 \equiv 0$, $\Phi_2 \equiv 1$ we have

$$\sin_A \overline{z} \equiv z - z\frac{\overline{z}^2}{2!} + \cdots$$

Conjugate hyperbolic functions. If $A(z,\overline{z}) = 1$, $\Phi_1 \equiv 1$, $\Phi_2 \equiv 0$, we have

$$\mathrm{ch}_A \overline{z} \equiv 1 + \frac{\overline{z}^2}{2!} + \frac{\overline{z}^2}{2!}\frac{z^2}{2!} + \cdots$$

and for $A(z,\overline{z}) = 1$, $\Phi_1 \equiv 0$, $\Phi_2 \equiv 1$, we have

$$\mathrm{sh}_A \overline{z} \equiv \overline{z} + z\frac{z^3}{3!} + \frac{\overline{z}^2}{2!}\frac{\overline{z}^3}{3!} + \cdots$$

This suggests the following definition (loke the Euler - formula)

$$\overset{\overline{z}}{_A}e = \mathrm{ch}_A \overline{z} - \mathrm{sh}_A \overline{z}.$$

Conjugate Bessel's cylindrical and other functions. If we take $A(z,\overline{z}) = \overline{z}$ then by using the equation

$$\frac{\partial^2 W}{\partial \overline{z}^2} + \overline{z}\overline{W} = 0$$

and its solution (12) we can introduce variations of Bessel's functions. We also can make this if we choose $A(z,\overline{z}) \equiv z$, where now A has a role a constant with respect to operators $\frac{\partial}{\partial \overline{z}}$ and $\int d\overline{z}$: $\frac{\partial^2 W}{\partial \overline{z}^2} + z\overline{W} = 0$.
In this way we will have Bessel's functions and the corresponding functions of different classes and of different categories of transcendentality.
The same we can do for the general "constant" coefficient $A(z)$:

$$\frac{\partial^2 W}{\partial \overline{z}^2} + A(z)\overline{W} = 0$$

and hence we can have a new complex trigonometry.

If this way, the equation

$$\frac{\partial^2 W}{\partial \overline{z}^2} + B(\overline{z})A(z)\overline{W} = 0$$

can be regarded as a generatrise of some conjugate functions of Hille, Lame and Mathie of two complex variables z and $\overline{z}$.

So we can extend the spaces of elementary transcendental functions in the complex plane.

The Vecua equation of the second order with analytic coefficients. Consider the equation

$$\frac{\partial^2 W}{\partial \overline{z}^2} = A(z,\overline{z})W + B(z,\overline{z})\overline{W} + F(z,\overline{z}) \tag{13}$$

with analytic coefficients

$$A = \sum_{i,j=0}^{\infty} a_{ij} z^i \overline{z}^j, \quad B = \sum_{i,j=0}^{\infty} b_{ij} z^i \overline{z}^j, \quad F = \sum_{i,j=0}^{\infty} f_{ij} z^i \overline{z}^j.$$

By the method of areolar series

$$W(z,\overline{z}) = \sum_{i,j=0}^{\infty} c_{ij} z^i \overline{z}^j$$

or by the method of analytic change (in fact, by iterations) one get approximations of an analytic solution.

From

$$\frac{\partial W}{\partial \overline{z}} = \hat{\int} (AW + B\overline{W} + F)$$

one obtains the first integral

$$\frac{\partial W}{\partial \overline{z}} = \int (AW + B\overline{W} + F)d\overline{z}^2 + \Phi_1(z),$$

and also the second integral

$$W = \iint (AW + B\overline{W} + F)d\overline{z}^2 + \Phi_1(z)\overline{z} + \Phi_2(z);$$

using now iteration method we have a solution

$$\begin{aligned}
W = \Phi_2 &+ \iint A\Phi_2 d\overline{z}^2 + \iint B\overline{\Phi}_2 d\overline{z}^2 \\
&+ \iint A d\overline{z}^2 \iint A\Phi_2 d\overline{z}^2 + \iint A d\overline{z}^2 + \iint B\overline{\Phi}_2 d\overline{z}^2 \\
&+ \iint B d\overline{z}^2 \iint \overline{A\Phi}_2 d z^2 + \iint B d\overline{z}^2 \iint \overline{B}\Phi_2 d z^2 \\
&+ \Phi_1 \overline{z} + \iint \overline{z} A\Phi_1 d\overline{z}^2 + \iint z B\overline{\Phi}_1 d\overline{z}^2 \\
&+ \iint A d\overline{z}^2 \iint \overline{z} A\Phi_1 d\overline{z}^2 + \iint A d\overline{z}^2 \iint z B\Phi_1 d\overline{z}^2 \qquad (14)\\
&+ \iint B d\overline{z}^2 \iint z\overline{A\Phi}_1 d z^2 + \iint B d\overline{z}^2 \iint \overline{B}\Phi_1 \overline{z} d z^2 \\
&+ \iint F d\overline{z}^2 \iint A d\overline{z}^2 \iint F d\overline{z}^2 + \iint B d\overline{z}^2 \iint \overline{F} d z^2 \\
&+ \iint A d\overline{z}^2 \iint A d\overline{z}^2 \iint F d\overline{z}^2 + \iint A d\overline{z}^2 \iint B d\overline{z}^2 \iint \overline{F} d z^2 \\
&+ \iint B d\overline{z}^2 \iint \overline{A} d z^2 \iint \overline{F} d z^2 + \iint B d\overline{z}^2 \iint \overline{B} d z^2 + \iint F d\overline{z}^2 + R_2,
\end{aligned}$$

where the remainder R_2 has a similar form as R_4 for the solution (12).

Possibilities. Without diffculties this method can be extended to any linear (pseudolinear) equation of Vecua type of the order n with the conjugation of function

$$F\left(z, \overline{z}, W(z,\overline{z}), \overline{W(z,\overline{z})}, \frac{\partial W}{\partial \overline{z}}, \frac{\partial^2 W}{\partial \overline{z}^2}, \ldots, \frac{\partial^n W}{\partial \overline{z}^n}\right) = 0$$

which has anallytic coefficients.

The consequences are general solutions in the form of series of coefficients of systems of real partial equations.

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