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ON $\mathcal{M}$-HARMONIC SPACE $\mathcal{D}_p^s$

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ABSTRACT. *We show that the $\mathcal{M}$-harmonic Dirichlet space $\mathcal{D}_p^s$ is equal to the weighted Bergman space $\mathcal{A}_p^s$ for $0 < p < 1$ and $s > n$.*

1. Introduction

In [6, chapter 10] author considered the relationship between the weighted Bergman spaces $\mathcal{A}_p^s$ of $\mathcal{M}$-harmonic functions in the open unit ball B in $\mathbb{C}^n$ and the Dirichlet spaces $\mathcal{D}_p^s$. He showed that if $s > n$ and $1 \le p < \infty$, then $\mathcal{A}_p^s = \mathcal{D}_p^s$. In this note we show that also $\mathcal{A}_p^s = \mathcal{D}_p^s$ in the case $s > n$, $0 < p < 1$.

Let B be the open unit ball in $\mathbb{C}^n$ and $S = \partial B$ the unit sphere in $\mathbb{C}^n$. We denote by ν the normalized Lebesgue measure on B and by σ the rotation invariant probability mesure on S.

Let $\tilde{\Delta}$ be the invariant Laplacian on B. That is, $\tilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B, $\varphi_z \in \mathrm{Aut}(B)$, taking 0 to z (see [5]). The C^2-functions f that are anihilated by $\tilde{\Delta}$ are called $\mathcal{M}$-harmonic ($f \in \mathcal{M}$).

Definition 1.1. *For $0 < p < \infty$, and $s \in \mathbb{R}$, the weighted Bergman space $\mathcal{A}_p^s$ is defined as the space of $\mathcal{M}$-harmonic functions f on B for which*

$$\|f\|_{\mathcal{A}_p^s} = \left[\int_B (1 - |z|^2)^s |f(z)|^p d\lambda(z)\right]^{1/p} < \infty.$$

Here, $d\lambda(z) = (1 - |z|^2)^{-n-1} d\nu(z)$ is the measure on B that is invariant under the group $\mathrm{Aut}(B)$.

For $f \in C^1(B)$, $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{2n}})$, $z_k = x_{2k-1} + i x_{2k}$, $k = 1, 2, \dots, n$, denotes the real gradient of f and let $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant real gradient of f.

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Definition 1.2. *For $0 < p < \infty$, and $s \in \mathbb{R}$, the $\mathcal{M}$-harmonic Dirichlet space $\mathcal{D}_p^s$ is defined as the space of $\mathcal{M}$-harmonic functions f on B for which*

$$\int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^s d\lambda(z) < \infty.$$

For $f \in \mathcal{D}_p^s$, set

$$\|\,|f|\,\|_{p,s} = |f(0)| + \left(\int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^s d\lambda(z)\right)^{1/p}.$$

For the proof of our main result the following Theorem will be needed.

Theorem 1.3 ([4]). *Let $0 < p < \infty$, $s > n - p/2$ and $f \in \mathcal{M}$. Then following statements are equivalent:*

(i) $f \in \mathcal{D}_p^s$,

(ii) $\int_B |\nabla f(z)|^p (1-|z|^2)^{s+p} d\lambda(z) < \infty$,

(iii) $\int_B (1-|z|^2)^{s+p} \big(|Rf(z)| + |R\overline{f}(z)|\big)^p d\lambda(z) < \infty$.

As usual, $Rf(z) = \sum_{j=1}^n z_j \dfrac{\partial f}{\partial z_j}$ is the radial derivative of f.

Theorem 1.4. *Let h be $\mathcal{M}$-harmonic on B.*

(i) *For all p, $0 < p < \infty$, and $s \in \mathbb{R}$, there exists a constant C, independent of h, such that*

$$\int_B (1-|z|^2)^s |\widetilde{\nabla} h(z)|^p d\lambda(z) \le C \int_B (1-|z|^2)^s |h(z)|^p d\lambda(z).$$

(ii) *For all p, $0 < p < \infty$, and $s > n$, there exists a positive constant C, independent of h, such that*

(1.1)
$$\int_B (1-|z|^2)^s |h(z)|^p d\lambda(z) \le C\Big(|h(0)|^p + \int_B (1-|z|^2)^{s+p} |\nabla h(z)|^p d\lambda(z)\Big).$$

Item (i) was proved in [6], Theorem 10.10. If $1 \le p < \infty$, then the second part follows from Theorem 1.3 and Theorem 10.10 [6]. So it remains to show that (1.1) holds for $0 < p < 1$. The proof will be given in section 2.

Corollary 1.5. *For all p, $0 < p < \infty$, and $s > n$, we have $\mathcal{A}_p^s = \mathcal{D}_p^s$.*

Next, we consider the relationship between the $\mathcal{M}$-harmonic Hardy space $\mathcal{H}^p$ and the spaces $\mathcal{D}_p^n$. For $0 < p < \infty$, $\mathcal{H}^p$ denotes the set of $\mathcal{M}$-harmonic functions f on B for which

$$\|f\|_p^p = \int_S [M_\alpha f(\xi)]^p d\sigma(\xi) < \infty, \text{ for some (any) } \alpha > 1.$$

Here $M_\alpha f(\xi) = \sup_{z \in D_\alpha(\xi)} |f(z)|$, $\xi \in S$, where $D_\alpha(\xi) = \{z \in B : |1 - \langle z, \xi\rangle| < \frac{\alpha}{2}(1 - |z|^2)\}$, $\alpha > 1$, denotes the Koranyi admissible approach regions.

By Theorem 6.18 ([6]) for $1 < p < \infty$, $f \in \mathcal{H}^p$ if and only if

$$\int_B (1 - |z|^2)^n |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 d\lambda(z) < \infty.$$

Thus when $p = 2$, $\mathcal{H}^2 = \mathcal{D}_2^n$.

For all p, $2 \le p < \infty$, $\mathcal{H}^p \subset \mathcal{D}_p^n$, with $\| \, |f| \, \|_{p,n} \le C_{n,p} \|f\|_p$, for all $f \in \mathcal{H}^p$, where $C_{n,p}$ is a constant depending only on n and p (see [3], [6]).

For all p, $0 < p \le 2$, $\mathcal{D}_p^n \subset \mathcal{H}^p$.

For $\alpha > 1$, $\xi \in S$, let

$$S_\alpha f(\xi) = \left(\int_{D_\alpha(\xi)} |\widetilde{\nabla} f(z)|^2 d\lambda(z) \right)^{1/2}$$

denote the area integral of f. In [1] it is shown that if $f \in \mathcal{M}$ then $f \in \mathcal{H}^p$, $0 < p < \infty$, if and only if $S_\alpha f \in L^p(\sigma)$. From this and the inequality

$$\int_S [S_\alpha f(\xi)]^p d\sigma(\xi) \le C \int_B (1 - |w|^2)^n |\widetilde{\nabla} f(w)|^p d\lambda(w),$$

where $f \in \mathcal{M}$ and $0 < p \le 2$ (see [6]), it follows that $\mathcal{D}_p^n \subset \mathcal{H}^p$, $0 < p \le 2$. We note that this inclusion was proved in [6] for $1 < p \le 2$.

In this note we follow the custom of using the letter C to stand for a positive constant which changes its value from one appearance to another while remaining independent of the important variables.

2. Proof of (1.1), case $0 < p < 1$

If $0 < r < 1$, we set $E_r(z) = \{w \in B : |\varphi_z(w)| < r\} = \varphi_z(rB)$. $E_r(z)$ is an ellipsoid and its volume is given by $\nu(E_r(z)) = \dfrac{r^{2n}(1 - |z|^2)^{n+1}}{(1 - r|z|)^{n+1}}$ (see [5], p.30).

For the proof of (1.1), $0 < p < 1$, the following lemmas will be needed.

Lemma 2.1. *If $s > 1$, then*

$$\int_0^1 \frac{dt}{|1 - t\langle z, w\rangle|^s} \leq \frac{C}{|1 - \langle z, w\rangle|^{s-1}}, \quad z, w \in B.$$

Lemma 2.2 ([4]). *Let $0 < r < 1$ and $0 < p < \infty$. There is a constant $C > 0$ such that if $f \in \mathcal{M}$ then*

$$\left(\frac{|\nabla f(w)|}{|1 - \langle z, w\rangle|}\right)^p \leq C \int_{E_r(w)} \left(\frac{|\nabla f(\xi)|}{|1 - \langle z, \xi\rangle|}\right)^p d\lambda(\xi), \; z, w \in B.$$

Lemma 2.3 ([2]). *For $1 < p < r < \infty$, $0 < q < \infty$ and a measurable $f \in L^{p,q-1}$ ($\|f\|^p_{p,q-1} = \int_B |f(z)|^p (1 - |z|^2)^{q-1} d\nu(z) < \infty$) we have*

$$\left(\int_B \left(\int_B \frac{|f(w)|(1 - |w|^2)^{q-1}}{|1 - \langle z, w\rangle|^{n+q}} d\nu(w)\right)^r (1 - |z|^2)^{r(\frac{n+q}{p} - \frac{n}{r}) - 1} d\nu(z)\right)^{1/r}$$
$$\leq C\|f\|_{p,q-1}.$$

Lemma 2.4 ([5], p.17). *If $\alpha > 0$, then*

$$\int_S \frac{d\sigma(\xi)}{|1 - \langle \xi, z\rangle|^{n+\alpha}} = O\left(\frac{1}{(1 - |z|)^\alpha}\right), \quad z \in B.$$

Lemma 2.5. *For $0 < s < t$ we have*

$$\int_0^1 \frac{(1 - r)^{s-1} dr}{(1 - r\rho)^t} \leq C(1 - \rho)^{s-t}, \quad 0 \leq \rho < 1.$$

Assume now that $s > n$, $0 < p < 1$ and $\int_B (1-|z|^2)^{s+p} |\nabla h(z)|^p d\lambda(z) < \infty$. Since $|\nabla h(z)|$ has $\mathcal{M}$-subharmonic behavior, i.e. $|\nabla h(w)| \leq C \int_{E_r(w)} |\nabla h(z)| d\lambda(z)$, $w \in B$, for some $0 < r < 1$, we have for any $a > 0$

$$\begin{aligned}
|h(z)|^p &\leq C\left(|h(0)|^p + \left(\int_0^1 \int_{E_r(tz)} |\nabla h(w)| d\lambda(w) dt\right)^p\right) \\
&\leq C\left(|h(0)|^p + \left(\int_0^1 \int_B \frac{|\nabla h(w)|(1 - |w|^2)^a}{|1 - t\langle z, w\rangle|^{n+a+1}} d\nu(w)\, dt\right)^p\right) \\
&= C\left(|h(0)|^p + \left(\int_B |\nabla h(w)|(1 - |w|^2)^a d\nu(w) \int_0^1 \frac{dt}{|1 - t\langle z, w\rangle|^{n+a+1}}\right)^p\right) \\
&\leq C\left(|h(0)|^p + \left(\int_B \frac{|\nabla h(w)|(1 - |w|^2)^a}{|1 - \langle z, w\rangle|^{n+a}} d\nu(w)\right)^p\right),
\end{aligned}$$

by Lemma 2.1.

Applying Lemma 2.3 to the function $F(w) = \left(|\nabla h(w)|\,|1-\langle z,w\rangle|^{-n-a}\right)^{p/2}$, $w \in B$ ($z \in B$-fixed) and replacing p, r, q by $2, 2/p, p(a+n+1)-n$ respectively and using Lemma 2.2 we find that

$$\begin{aligned}
&\int_B \frac{|\nabla h(w)|(1-|w|^2)^a}{|1-\langle z,w\rangle|^{n+a}}\,d\nu(w)\\
&\leq C\int_B\left(\int_{E_r(w)} \frac{F(\xi)(1-|\xi|^2)^{p(a+n+1)-n-1}d\nu(\xi)}{|1-\langle w,\xi\rangle|^{p(a+n+1)}}\right)^{2/p}(1-|w|^2)^a d\nu(w)\\
&\leq C\left(\int_B\left(\int_B \frac{F(\xi)(1-|\xi|^2)^{p(a+n+1)-n-1}}{|1-\langle w,\xi\rangle|^{p(a+n+1)}}d\nu(\xi)\right)^{2/p}(1-|w|^2)^a d\nu(w)\right.\\
&\leq C\left(\int_B \frac{|\nabla h(w)|^p(1-|w|^2)^{p(a+n+1)-n-1}}{|1-\langle z,w\rangle|^{p(n+a)}}d\nu(w)\right)^{1/p},
\end{aligned}$$

we may assume that $a > \frac{s}{p} - n$.
Thus, by using Fubini's theorem, Lemma 2.4 and Lemma 2.5 we obtain

$$\begin{aligned}
&\int_B (1-|z|^2)^s |h(z)|^p d\lambda(z) \leq C\left[|h(0)|^p + \int_B (1-|z|^2)^{s-n-1}d\nu(z)\times\right.\\
&\left.\int_B \frac{|\nabla h(w)|^p(1-|w|^2)^{p(a+n+1)-n-1}}{|1-\langle z,w\rangle|^{p(n+a)}}d\nu(w)\right] = C\left[|h(0)|^p\right.\\
&\left.+\int_B |\nabla h(w)|^p(1-|w|^2)^{p(a+n+1)-n-1}d\nu(w)\int_B \frac{(1-|z|^2)^{s-n-1}d\nu(z)}{|1-\langle z,w\rangle|^{p(a+n)}}\right]\\
&\leq C\left[|h(0)|^p + \int_B |\nabla h(w)|^p(1-|w|^2)^{s+p-n-1}d\nu(w)\right].
\end{aligned}$$

This finishes the proof of Theorem 1.4.

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