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A NOTE ON CERTAIN CLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT. *We give some results on $f'(z)$, $f(z)/z$ and $zf'(z)/f(z)$ for certain classes of univalent functions in the unit disc $|z| < 1$.*

1. Introduction and preliminaries

Let A denote the class of functions f analytic in the unit disc $U = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$.

Ozaki [4] proved that if $f \in A$ and

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} < \frac{3}{2}, \; z \in U, \tag{1}$$

then f is univalent in U. Later, Umezawa [7] showed that if $f \in A$ satisfies the condition (1), then f is univalent and convex in one direction. Sakaguchi [5] proved that if $f \in A$ satisfies (1), then $|\arg f'(z)| < \pi/2$, $z \in U$, i.e. f is close-to-convex function. Finally, R. Singh and S. Singh [6] proved that the same class is the subclass of starlike functions in U.

In his paper [3] Nunokawa considered the class of functions $f \in A$ such that

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} < 1 + \frac{\alpha}{2}, \; z \in U, \tag{2}$$

for some $0 < \alpha \leq 1$.

He proved that for such class

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha, \; z \in U.$$

It is evident that for $\alpha = 1$ in (2) we have class defined by (1) and the classes defined by (2) are the subclasses of the class defined by (1).

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In this note we consider the values $zf'(z)/f(z)$, $f'(z)$ and $f(z)/z$ for the classes defined by (2).

We need the following definition and lemmas.

Let f and g be analytic in the unit disc U. We say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there exists an analytic function ω in U satisfying $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in U$, and $f(z) = g(\omega(z))$. In patrticular, if g is univalent in U, then f is subordinate to g if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma A ([2]). *Let ω be nonconstant and analytic in U with $\omega(0) = 0$. If $|\omega|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0\omega'(z_0) = k\omega(z_0)$, $k \geq 1$.*

Lemma B ([1]). *Let g be a convex function in U and let γ be a complex number with $\text{Re}\{\gamma\} > 0$. If f is analytic in U and $f \prec g$, then*

$$z^{-\gamma} \int_0^z f(w)w^{\gamma-1}dw \prec z^{-\gamma} \int_0^z g(w)w^{\gamma-1}dw.$$

We note the that we can find more details of the classes of the functions we mentioned above in any standard book on univalent functions

2. Results and consequences

We start with the following

Theorem 1. *Let f satisfy the condition (2). Then*

$$z\frac{f'(z)}{f(z)} \prec (1-z)\left(1 - \frac{1}{\alpha+1}z\right)^{-1}. \tag{3}$$

Proof. Let's put $a = 1/(\alpha + 1)$ and

$$z\frac{f'(z)}{f(z)} = \frac{1-\omega(z)}{1-a\omega(z)}. \tag{4}$$

Evidently $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$, $z \in U$. From (4), after taking logarithmical differentiation, we get

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1-\omega(z)}{1-a\omega(z)} - \frac{z\omega'(z)}{1-\omega(z)} + \frac{az\omega'(z)}{1-a\omega(z)}. \tag{5}$$

If it is not $|\omega(z)| < 1$, then by Lemma A, there exists a z_0, $|z_0| < 1$. such that $z_0\omega'(z_0) = k\omega(z_0)$ and $|\omega(z_0)| = 1$, $k \geq 1$. If we put $\omega(z_0) = e^{i\theta}$, then for such z_0 from (5) we have

$$\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\bigg|_{z=z_0}\right\}=\frac{1+a}{2a}+\frac{(1-a^2)(2a-1)}{2a(1-2a\cos\theta+a^2)}$$
$$+(k-1)\frac{1-a^2}{2(1-2a\cos\theta+a^2)}\geq\frac{1+a}{2a}$$
$$+\frac{(1-a^2)(2a-1)}{2a(1+a)^2}=1+\frac{\alpha}{2}+\frac{\alpha(1-\alpha)}{2(2+\alpha)}\geq 1+\frac{\alpha}{2},$$

which is a contradiction to (2). Therefore, $|\omega(z)|<1$, $z\in U$, and from (4) we finally get the relation (3).

We note that the function on the right side of (3) is univalent and maps the unit disc U onto the disc with the diameter end points 0 and $2(\alpha+1)/(\alpha+2)$.

Remark 1. *For $\alpha=1$ in Theorem 1 and the previously cited result of Nunokawa we have that the image of U under $zf'(z)/f(z)$, where $f\in A$ satisfies (2), lies in the intersection of the angle $\{w:|\arg\omega|<\alpha\pi/2\}$ and the disc which is the image of U under the function $w=(1-z)/(1-z/(\alpha+1))$.*

Also we have

Theorem 2. *Let $f\in A$ satisfy the condition* (2). *Then*

(6) a) $f'(z)\prec(1+z)^{\alpha}$;

b) $\dfrac{f(z)}{z}\prec\dfrac{(1+z)^{\alpha+1}-1}{(\alpha+1)z}$.

Proof. a) From the condition (2) we conclude that f' has no zero in U. Let's put

$$(f'(z))^{1/\alpha}=1+\omega(z) \tag{7}$$

(where we take the principal value). Evidently $\omega(0)=0$. We want to prove that $|\omega(z)|<1$, $z\in U$. From (7) after some transformations, we have

$$1+z\frac{f''(z)}{f'(z)}=1+\alpha\frac{z\omega'(z)}{1+\omega(z)}. \tag{8}$$

If it is not $|\omega(z)|<1$, $z\in U$, then by Lemma A there exists a z_0, $|z_0|<1$, such that $z_0\omega'(z_0)=k\omega(z_0)$ and $|\omega(z_0)|=1$, $k\geq1$. If we put $\omega(z_0)=e^{i\theta}$, from (8), we get

$$\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\bigg|_{z=z_0}\right\}=1+\frac{\alpha k}{2}\geq1+\frac{\alpha}{2},$$

which is a contradiction to (2). Therefore, $|\omega(z)|<1$, $z\in U$, and from (7) we conclude that the relation (6) is true.

b) Since the function $(1+z)^{\alpha}$, $0<\alpha\le 1$, is convex, then the result follows directly from the result of Lemma B, for $\gamma=1$. $\square$

From Theorem 2, for $\alpha=1$, we easily obtain

Corollary 1. *If $f\in A$ satisfies* (1), *then*

a) $\mathrm{Re}\{f'(z)\}>0$, $z\in U$,

(*which is the earlier result given in* [5]);

b) *f is bounded in U and $|f(z)|<3|z|/2$, $z\in U$.*

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SOME THEOREMS ABOUT PRIMARY COIDEALS

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ABSTRACT. *In this short note we give some theorems about primary coideals of commutative ring with an apartness.*

Throughout this paper R will denote a commutative ring with an apartness in sense of the book [1] and paper [2]. A subset S of R is a *coideal* ([2]) of R iff $\varnothing \# S$, $-a \in S \Rightarrow a \in S$, $a + b \in S \Rightarrow a \in S \vee b \in S$, $ab \in S \Rightarrow a \in S \wedge b \in S$. The coideal S of R is a strongly extensional subset of R and $S \neq \varnothing \Rightarrow 1 \in S$ holds. The coideal S of R is a *prime coideal* iff $a \in S \wedge b \in S \Rightarrow ab \in S$. If S is a coideal of R, then the set $c(S) = \{b \in R : (\forall n \in \mathbb{N})(b^n \in S\}$ is a coideal of R under S, called *coradical* of S. The coideal Q of R is a *primary coideal* of R iff $c(Q) \subset Q$. If Q is a primary coideal of R, the the coradical $c(Q)$ of Q is a prime coideal. In this case, we say that the primary coideal Q belonging to the prime coideal $c(Q)$. Let S be a coideal of R and let X be a subset of R. Then the set $[S : X] = \{b \in R : (\exists x \in X)(bx \in S\}$ is a coideal of R called *quotient coideal* of Q by the subset X. It is clear that $[S : X] \subset S$ and $X \cap S = \varnothing \Rightarrow [S : X] = \varnothing$.

First, we shall give a description of irreducibility of a primary coideal Q as the union of the coradical $c(Q)$ and of one coideal S under Q.

Theorem 1. *Let Q be a primary coideal of R.*

(1) If $c(Q) \subsetneqq Q$, then it does not exist a coideal S of R under Q such that $Q = c(Q) \cup S$.

(2) If $Q = c(Q) \cup S$, where S is a coideal of R under Q such that $S \subsetneqq Q$, then $Q = c(Q)$.

Proof. (1) There exists an elemenat b in Q such that $b \# c(Q)$. Suppose that S is a coideal of R under Q such that $Q = c(Q) \cup S$. Let z be an arbitrary

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element of Q. Then $z \in c(Q)$ or $z \in S$. If $z \in c(Q)$, then $zb \in c(Q) \subset Q = c(Q) \cup S$, and

$$z \in [c(Q) \cup S : b] = [c(Q) : b] \cup [S : b] = \emptyset \cup [S : b] \subset S.$$

Therefore $Q \subset S$. It is a contradiction.

(2) Suppose that there is a coideal S of R under Q such that $S \subsetneqq Q$ and $Q = c(Q) \cup S$. Then there exists an element b in Q such that $b \# S$. Thus $b \in Q = c(Q) \cup S$ and $b \neq S$ implies that $b \in c(Q)$. Therefore

$$\begin{aligned} z \in Q &\Rightarrow zb \in Q\, c(Q) \subset Q = c(Q) \cup S \\ &\Rightarrow z \in [c(Q) \cup S : b] = [c(Q) : b] \cup [S : b] = [c(Q) : b] \subset c(Q). \quad \Box \end{aligned}$$

Let S be a subset of R. We say that S is a *stable* subset of R iff $(\forall x \in R)(\neg\neg(x \in S) \Rightarrow x \in S)$. If S is a stable coideal of R and if X is a multiplicative subset of R, then the set $\langle S : X \rangle = \{b \in R : bX \subset S\}$ is a coideal of R. $\langle S : X \rangle \subset S$. In the next theorem we shall give a construction of a primary coideal $\langle Q : c(Q) \rangle$, where Q is a stable coideal of R such that $c(Q)$ is a prime coideal of R.

Theorem 2. *Let Q be a stable coideal of R such that $c(Q)$ is a prime coideal of R. Then the set $\langle Q : c(Q) \rangle$ is a primary coideal of R belonging to $c(Q)$.*

Proof. Let Q be a stable coideal of R such that $c(Q)$ is a prime coideal of R. Then the set $\langle Q : c(Q) \rangle$ is a coideal of R and it holds

$$\begin{aligned} c(Q) \subset Q &\Rightarrow c(Q) = \langle c(Q) : c(Q) \rangle \subset \langle Q : c(Q) \rangle \subset Q \\ &\Rightarrow c(Q) = c(c(Q)) \subset c(\langle Q : c(Q) \rangle) \subset c(Q) \end{aligned}$$

Further, we have

$$\begin{aligned} & a \in \langle Q : c(Q) \rangle \ \& \ b \in c(Q) = \langle c(Q) : c(Q) \rangle \Longleftrightarrow \\ & a\, c(Q) \subset Q \ \& \ b\, c(Q) \subset c(Q) \Rightarrow \\ & ab\, c(Q) \subset a\, c(Q) \subset Q \Rightarrow \\ & ab \in \langle Q : c(Q) \rangle. \quad \Box \end{aligned}$$

Let $\mathcal{F} = (P_j)_{j \in J}$ be a family of coideals of a ring R. We say that a coideal P of R is an *isolated coideal* from the family $\mathcal{F}$ iff $(\exists p \in P)(p \# \cup P_j)$. Let S be a stable coideal of R. If $c(S)$ is the union of prime coideals under S $c(S) = \cup_{j \in J} P_j$, where J is a discrete set, and if P_i is an isolated coideal from the family $\mathcal{F} = (P_j)_{j \in J \setminus \{i\}}$, then the set $Q_i = \langle S : P_i \rangle$ is a primary coideal of R belonging to P_i. Therefore, the stable coideal S of R such that $c(S) = \cup_{j \in J} P_j$ contains an union of primary coideals $Q_i = \langle S : P_i \rangle$ where P_i are isolated prime coideals of R under S.

Theorem 3. *Let S be a stable coideal of R such that $c(S) = \cup_{j\in J}P_j$, where the P_j 's are prime coideals of R under S, and the set J is discrete. If P_i is an isolated prime coideal from the family $(P_j)_{j\in J\setminus\{i\}}$, then the coideal $Q_i = \langle S : P_i\rangle$ is a primary coideal of R belonging to P_i.*

Proof. Without difficultes one can verify that the set Q_i is a coideal of R under S. On the other hand, we have $c(Q_i) = c(\langle S : P_i\rangle) = \langle c(S) : P_i\rangle = \langle \cup_{j\in J}P_j : P_i\rangle$. Suppose that b is an arbitrary element of $c(Q_i)$. Then $b\,P_i \subset \cup_{j\in J}P_j$. As P_i is an isolated prime coideal from the family $(P_j)_{j\in J\setminus\{i\}}$, there exists an element p_i in P_i such that $p_i \# \cup\{P_j : j \in J \;\;\&\;\; j \neq i\}$. Now, we have

$$
\begin{aligned}
b \in \left\langle \cup_{j\in J}P_j : P_i\right\rangle &\iff bP_i \subset \cup_{j\in J}P_j = \cup_{j\neq i}P_j \cup P_i \\
&\iff (\forall p \in P_i)(bp \cup_{j\neq i} P_j \cup P_i) \\
&\implies bp_i \cup_{j\neq i} P_j \cup P_i \\
&\iff b \in \left[\cup_{j\neq i}P_j \cup P_i : p_i\right] = \left[\cup_{j\neq i}P_j : p_i\right] \cup \left[P_i : p_i\right] \\
&\iff b \in \varnothing \cup P_i \\
&\iff b \in P_i.
\end{aligned}
$$

Therefore $c(Q_i) = P_i$. Thus

$$
\begin{aligned}
&a \in Q_i = \langle S : P_i\rangle \;\;\&\;\; b \in P_i = \langle P_i : p_i\rangle \iff \\
&aP_i \subset S \;\;\&\;\; bP_i \subset P_i \implies \\
&abP_i \subset aP_i \subset S \implies \\
&ab \in \langle S : P_i\rangle = Q_i. \quad \square
\end{aligned}
$$

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