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SOME CARDINAL FUNCTIONS ON URYSOHN SPACES

Ljubiša D. Kočinac

ABSTRACT. *We give some results on the cardinality of Urysohn H-closed topological spaces involving a new cardinal function denoted by $sqL_\theta(X)$.*

1. Introduction

In [7], the following cardinal function was introduced. For a space X, $sqL(X)$ is the smallest infinite cardinal τ such that there exists a subset A in X of cardinality $\leq 2^\tau$ satisfying: for every family $\mathcal{U}$ of open subsets of X there exist a subfamily $\mathcal{V}$ of $\mathcal{U}$ and a subset B of A such that $|\mathcal{V}| \leq \tau$, $|B| \leq \tau$ and $\cup\mathcal{U} \subset \overline{B} \cup (\cup\mathcal{V})$. In [10], this cardinal function was studied in some details. In a similar way we define here another cardinal function, denoted by $sqL_\theta(X)$, and prove some results on the cardinality of Urysohn spaces involving this function. These results improve some results from [6] and [10].

2. Notation and terminology. Definitions

Notations and terminology in this paper are standard as in [2], [4], [5]. Unless otherwise indicated, all spaces are assumed to be at least T_1 and infinite. α, β, γ, δ are ordinal numbers, while τ, λ denote infinite cardinals; τ^+ is the successor cardinal of τ. As usual, cardinals are assumed to be initial ordinals. If S is a set, then $[S]^{\leq\tau}$ denote the collection of all subsets of X having cardinality $\leq \tau$.

We recall some definitions that we need.

2.1. A space X is *Urysohn* if for every two distinct points x and y in X there are open sets U and V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.

2.2. If X is a space and A a subset of X, then we put

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$$Cl_\theta A = \{x \in X : \overline{U} \cap A \neq \emptyset \text{ for every neighbourhood } U \text{ of } x\}.$$
The set $Cl_\theta A$ is called the *θ-closure* of A. A is *θ-closed* if $Cl_\theta A = A$.

2.3. A Hausdorff space X is called *H-closed* if every open cover $\mathcal{U}$ of X has a finite subcollection $\mathcal{V}$ whose union is dense in X.

2.4. ([1]) The *θ-bitightness* of a space X, denoted by $bt_\theta(X)$, is the smallest cardinal τ such that for each non-θ-closed set $A \subset X$ there exist a point $x \in X \setminus A$ and a collection $\mathcal{S} \in [[A]^{\leq\tau}]^{\leq\tau}$ such that $\{x\} = \cap\{Cl_\theta S : S \in \mathcal{S}\}$.

2.5. ([6]) Call a subset A of a space X *θ-dense* in X if $Cl_\theta A = X$, i.e. if for every open set $U \subset X$, $\overline{U} \cap A \neq \emptyset$. The *θ-density* of X is
$$d_\theta(X) = \omega \cdot \min\{|A| : A \text{ is a } \theta\text{-dense subset of } X\}.$$
Clearly, for every space X, $d_\theta(X) \leq d(X)$. There are spaces X for which $d_\theta(X) < d(X)$ holds.

2.6. ([9]) The *θ-spread* $s_\theta(X)$ of a space X is the supremum of the cardinalities of subsets D of X such that for every $x \in D$ there exists a neighbourhood U of x with $\overline{U} \cap D = \{x\}$. The inequality $s_\theta(X) < s(X)$ is possible.

2.7. A Hausdorff space X is said to be of *closed pseudocharacter* τ, denoted by $\psi_c(X) = \tau$, if τ is the smallest cardinal such that for each point $x \in X$ there exists a family $\{U_\alpha : \alpha \in \tau\}$ of neighbourhoods of x with $\{x\} = \cap\{\overline{U}_\alpha : \alpha \in \tau\}$.

3. Results

In [9], the following lemma is proved.

Lemma 3.1. *Let X be a topological space and $s_\theta(X) = \tau$. If $\mathcal{U}$ is a family of open subsets of X, then there exist $A \in [\cup\mathcal{U}]^{\leq\tau}$ and $\mathcal{V} \in [\mathcal{U}]^{\leq\tau}$ such that $\cup\mathcal{U} \subset Cl_\theta A \cup \cup\{\overline{V} : V \in \mathcal{V}\}$.* □

After this lemma and the definition of $sqL(X)$ it is reasonable to introduce:

Definition 3.2. *Let X be a space. Then $sqL_\theta(X)$ is defined to be the smallest cardinal τ such that there exists a subset A in X of cardinality $\leq 2^\tau$ satisfying: for every family $\mathcal{U}$ of open subsets of X there exist $\mathcal{V} \in [\mathcal{U}]^{\leq\tau}$ and $B \in [A]^{\leq\tau}$ such that $\cup\mathcal{U} \subset Cl_\theta B \cup (\cup\overline{\mathcal{V}})$.* □

Fact 1. $sqL_\theta(X) \leq sqL(X) \leq d(X)$.
Fact 2. $sqL_\theta(X) \leq d_\theta(X)$.

We shall also need the following lemma which is a version of the fundamental result on spread due to Shapirovskii (see [8;T.3]).

Lemma. 3.3 ([6;Prop. 3.3]). *Let X be a Urysohn space with $hs_\theta(X) \leq \tau$. Then there is a subset A of X such that $|A| \leq 2^\tau$ and $\cup\{Cl_\theta B : B \in [A]^{\leq\tau}\} = X$.*

Proposition 3.4. *For every Urysohn H-closed space X, we have*

$$sqL_\theta(X) \leq hs_\theta(X).$$

Proof. Let $hs_\theta(X) = \tau$. By Lemma 3.3 there exists a set $A \subset X$ with $|A| \leq 2^\tau$ such that $X = \cup\{Cl_\theta(B) : B \in [A]^{\leq\tau}\}$. Let us show that A witnesses $sqL_\theta(X) \leq \tau$. Take a collection $\mathcal{U}$ of open subsets of X. By Lemma 3.1 there exist $\mathcal{V} \in [\mathcal{U}]^{\leq\tau}$ and $M \in [\cup\mathcal{U}]^{\leq\tau}$ such that $\cup\mathcal{U} \subset Cl_\theta M \cup (\cup\overline{\mathcal{V}})$. For every $p \in M$ there exists some $S_p \in [A]^{\leq\tau}$ with $p \in Cl_\theta S_p$. Put $S = \cup\{S_p : p \in M\}$. Then $S \in [A]^{\leq\tau}$ and $M \subset \cup\{Cl_\theta S_p : p \in M\} \subset Cl_\theta(\cup\{S_p : p \in M\}) = Cl_\theta S$. As the θ-closure operator is idempotent in Urysohn H-closed spaces we have $Cl_\theta M \subset Cl_\theta(Cl_\theta S) = Cl_\theta S$. Hence, $\cup\mathcal{U} \subset Cl_\theta S \cup (\cup\overline{\mathcal{V}})$ and the proposition is proved. □

Example. Let X be the Niemytzki plane T equipped with the topology $\mathcal{T} = \{U \setminus C : U$ is open in T and $C \subset T$ is countable$\}$. Then $hs_\theta(X) = s(T) = 2^\omega$ and $sqL_\theta(X) = sqL(T) = \omega$. □

Theorem 3.5. *For every Urysohn H-closed space X, we have*

$$\psi_c(X) \leq 2^{sqL_\theta(X)}.$$

Proof. Let $sqL_\theta(X) = \tau$ and let $A \subset X$ be a set witnessing this fact. Fix a point $x \in X$. Since X is Urysohn, for every $y \in X \setminus \{x\}$ there are neighbourhoods U_y of x and V_y of y with $\overline{U}_y \cap \overline{V}_y = \emptyset$. Applying the definition of $sqL_\theta(X)$ to the family $\mathcal{V} = \{V_y : y \in X \setminus \{x\}\}$ (and A) one can find sets $Y = \{y_\alpha : \alpha \in \tau\} \in [X \setminus \{x\}]^{\leq\tau}$ and $B \in [A]^{\leq\tau}$ such that

$$X \setminus \{x\} \subset Cl_\theta B \cup (\cup\{\overline{V}_{y_\alpha} : \alpha \in \tau\}).$$

Put $\mathcal{U}_x = \{X \setminus Cl_\theta C : C \subset B, x \notin Cl_\theta C\} \cup \{U_{y_\alpha} : \alpha \in \tau\}$. Then $|\mathcal{U}_x| \leq 2^\tau$ so that we need to check $\{x\} = \cap\{\overline{U} : U \in \mathcal{U}_x\}$.

Let $p \in X \setminus \{x\} \subset Cl_\theta B \cup (\cup\{\overline{V}_{Y_\alpha} : \alpha \in \tau\})$. Consider two possibilities:

(i) $p \in Cl_\theta B$. Take neighbourhoods U_p of p and V_p of x such that $\overline{U}_p \cap \overline{V}_p = \emptyset$. It is easy to see that $p \in Cl_\theta(B \cap \overline{V}_p) \subset Cl_\theta \overline{V}_p = \overline{V}_p$ (in Urysohn H-closed spaces it holds $Cl_\theta \overline{G} = \overline{G}$ for each open set G). Therefore, $C = B \cap \overline{V}_p$ provides a subset of B with $\overline{U}_p \cap Cl_\theta C = \emptyset$, hence $\overline{U}_p \subset X \setminus Cl_\theta C = \emptyset$ and thus $\overline{U}_p \subset \overline{X \setminus Cl_\theta C}$ which gives $\{x\} = \cap\{\overline{U} : U \in \mathcal{U}_x\}$.

(ii) $p \in \cup\{\overline{V}_{y_\alpha} : \alpha \in \tau\}$. Then $x \in \cap\{\overline{U}_{y_\alpha} : \alpha \in \tau\}$, but $p \notin \cap\{\overline{U}_{y_\alpha} : \alpha \in \tau\}$. □

The following theorem is an improvement of Lemma 3.3.

Theorem 3.6. *Let X be a Urysohn H-closed space with $sqL_\theta(X) \leq \tau$. Then there is a subset A of X such that $|A| \leq 2^\tau$ and $\cup\{Cl_\theta B : B \in [A]^{\leq\tau}\} = X$.*

Proof. Let S be a set in X witnessing $sqL_\theta(X) \leq \tau$. According to Theorem 3.5, for every $x \in X$ one can choose a collection $\mathcal{U}_x$ of neighbourhoods of x such that $|\mathcal{U}_x| \leq 2^\tau$ and $\cap\{\overline{U} : U \in \mathcal{U}_x\} = \{x\}$. By transfinite induction we shall construct a sequence $\{M_\alpha : \alpha < \tau^+\}$ of subsets of X and a sequence $\{\mathcal{U}_\alpha : \alpha < \tau^+\}$ of families of open subsets of X satisfying the following conditions:

(a) $|M_\alpha| \leq 2^\tau$, $\alpha < \tau^+$;

(b) $\mathcal{U}_\alpha = \cup\{\mathcal{U}_x : x \in \cup\{M_\beta : \beta < \alpha\}\}$ (so $|\mathcal{U}_\alpha| \leq 2^\tau$), $\alpha < \tau^+$;

(c) If $T \in [S]^{\leq\tau}$, $\mathcal{V} \in [U_\alpha]^{\leq\tau}$ and $Cl\theta T \cup \cup\overline{\mathcal{V}} \neq X$, then $M_\alpha \setminus (Cl_\theta T \cup \cup\overline{\mathcal{V}}) \neq \emptyset$.

Suppose we have already defined all M_β and $\mathcal{U}_\beta$ for $\beta < \alpha$. Let us define M_α and $\mathcal{U}_\alpha$. For every $T \in [S]^{\leq\tau}$ and every $\mathcal{V} \in [\mathcal{U}_\beta]^{\leq\tau}$ choose a point $x(T, \mathcal{V}) \in X \setminus (Cl_\theta T \cup \cup\overline{\mathcal{V}})$ whenever the last set is not empty (otherwise the construction has been finished). Let

$$M_\alpha = \{x(T,\mathcal{V}) : T \in [S]^{\leq\tau} \text{ and } \mathcal{V} \in [U_\beta]^{\leq\tau}\}$$
$$\mathcal{U}_\alpha = \cup\{\mathcal{U}_x : x \in \cup\{M_\beta : \beta < \alpha\}\}.$$

It is easy to check that M_α and $\mathcal{U}_\alpha$ satisfy (a), (b) and (c). Put $M = \cup\{M_\alpha : \alpha < \tau^+\}$, $A = M \cup S$ and prove that A is the set we are looking for. First of all $|A| \leq 2^\tau$. Let $x \in X$. If $x \in A$ there is nothing to prove. Let $x \in X \setminus A$. Then $x \notin M$ so that for every $y \in M$ one can find a neighbourhood $V_y \in \mathcal{U}_y$ of y such that $x \notin \overline{V}_y$. So, $x \notin \cup\{\overline{V}_y : y \in M\}$. By the properties of S one can choose $B \in [S]^{\leq\tau}$ and $\{y_\gamma : \gamma \in \tau\} \in [M]^{\leq\tau}$ such that $M \subset \cup\{V_y : y \in M\} \subset Cl_\theta B \cup (\cup\{\overline{V}_{y_\gamma} : \gamma \in \tau\})$. Let us prove $x \in Cl_\theta B$. Suppose not. Then $Cl_\theta B \cup (\cup\{\overline{V}_{y_\gamma} : \gamma \in \tau\}) \neq X$. Since τ^+ is regular, there exists some $\delta < \tau^+$ such that $\{y_\gamma : \gamma \in \tau\} \subset M_\delta$. Then $\{V_{y_\gamma} : \gamma \in \tau\} \in [\mathcal{U}_\delta]^{\leq\tau}$. By (c), $M_{\delta+1} \setminus (Cl_\theta B \cup (\cup\{\overline{V}_{y_\gamma} : \gamma \in \tau\})) \neq \emptyset$. But this contradicts the fact $Cl_\theta B \cup (\cup\{\overline{V}_{y_\gamma} : \gamma \in \tau\}) \supset M \supset M_{\delta+1}$. The theorem is proved. □

The next two theorems improve Theorems 3.4 and 3.5, respectively, from [6]. The first of them is an immediate corollary of the previous theorem.

Theorem 3.7. *For every Urysohn H-closed space X we have*
$$d_\theta(X) \leq 2^{sqL_\theta(X)}. \quad \square$$

Theorem 3.8. *For every Urysohn H-closed space X we have*

$$|X| \le 2^{sqL_\theta(X) bt_\theta(X)}.$$

Proof. Theorem 2.3 in [6] states that for every Urysohn space X, $|X| \le [d_\theta(X)]^{bt_\theta(X)}$. Using now Theorem 3.7 we have $|X| \le (d_\theta(X))^{bt_\theta(X)} \le \left(2^{sqL_\theta(X)}\right)^{bt_\theta(X)} = 2^{sqL_\theta(X) bt_\theta(X)}$. $\square$

The famous theorem of Hajnal-Juh'asz says: if X is a Hausdorff space, then $|X| \le 2^{2^{s(X)}}$ [3], [4], [5]. In [9], it was shown that for a Urysohn space X this inequality can be improved to $|X| \le 2^{2^{s_\theta(X)}}$. Our next result is an improvement of the last estimation for Urysohn H-closed spaces.

Theorem 3.9. *For every Urysohn H-closed space X we have*

$$|X| \le 2^{2^{sqL_\theta(X)}}.$$

Proof. By Theorem 2.6 in [6], $|X| \le 2^{d_\theta(X)\psi_c(X)}$ so that, by Theorems 3.5 and 3.7, one obtains $|X| \le 2^{d_\theta(X)\psi_c(X)} \le 2^{2^{sqL_\theta(X)}} \cdot 2^{sqL_\theta(X)} = 2^{2^{sqL_\theta(X)}}$. $\square$

References

[1] F. Cammaroto, Lj. Kočinac, *On θ-tightness*, Facta Universitatis (Niš), Ser. Math. Inform. 8(1993), 77-85.

[2] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[3] A. Hajnal, I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math. 29(1967), 343-356.

[4] R. Hodel, *Cardinal functions I*, In: K. Kunen, J.E. Vaughan (eds.), Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, 1-61.

[5] I. Juhász, *Cardinal functions in topology - ten years later*, Mathematical Centre Tracts 123, Amsterdam, 1980.

[6] Lj. Kočinac, *On the cardinality of Urysohn spaces*, Q & A in General Topology, 13(1995), 211–216.

[7] X.S. Liu, *Two topological cardinal functions*, Acta Math. Sinica 29 (1987), 494-497 (In Chinese).

[8] B. Shapirovskii, *On discrete subspaces of topological spaces. Weight, tightness and the Souslin number*, Doklady AN SSSR 202(1972), 779-782 (In Russian).

[9] J. Schröder, *Urysohn cellularity and Urysohn spread*, Math. Japonica 38(1993), 1129-1133.

[10] S.H. Sun, K.G. Choo, *Some new cardinal inequalities involving a cardinal function less than the spread and the density*, Comment. Math. Univ. Carolinae 31(1990), 395-401.

29. novembra 132, 37230 Aleksandrovac, Yugoslavia