FILOMAT (Niš) 9:1 (1995), 39–55

ON APPROXIMATION BY ANGLE FOR 2π PERIODIC FUNCTIONS

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ABSTRACT. *Approximations by angle from singular integrals of functions belonging to the space L_p, $1 \leq p \leq \infty$ are estimated using best aaproximations by angle from the trigonometric polynomials. The applications to Riesz's singular integrals are given.*

1. Introduction

It is well known that integrable 2π periodic functions can be obtained by different means of summation of their Fourier series. Approximations by sums of Fourier series can be compared with the best approximations as in paper [3] and [4]. In the paper [3] for function of one variable several inequalities are established by which the approximations are compared depending on whether $p > 1$ or $p = 1$. Those inequalities allow to compare classes of functions which are defined by approximations. Those are classes of Nikolski and saturation classes.

In the paper [4] we proved inequality concerning the approximation by angle for $1 < p < \infty$. The aim in this paper is to prove the inequality concerning also the approximation by angle but which concerns the space L_1 (the case $p = 1$).

To realize this aim we use one theorem of Timan of [3] (Theorem 1, inequality (3.11)) and one equality of [4] which in this paper we give as Lemma 2.

The difference between the quoted result of Timan and the results of this paper is following:

1) We generalise the result of Timan so that we consider an n-dimensional case of approximation by angle.

2) We give a theorem in a form which is more suitable for application in order to compare Nikolski's classes with saturation classes.

Received 02.03.1995; Revised 28.06.1995

1991 *Mathematics Subject Classification.* 42A10.

2. Auxiliary results

We say that $f \in L_p([o,2\pi]^n)$ if $f = f(x_1,\dots,x_n)$ is measurable on Δ_n and is a 2π periodic function with respect to every variable $x_1,\dots,x_n$ for which $\|f\| < \infty$, where

$$\|f\| = \|f\|_p = \left(\int_{\Delta_n} |f(x_1,\dots,x_n)|^p\right)^{1/p}, \quad 1 \le p < \infty,$$

$$\|f\|_\infty = \sup \mathrm{vrai}\, |f(x)|,$$

$$\Delta_n = \{x = (x_1,\dots,x_n), 0 \le x_i \le 2\pi, i = 1,\dots,n\}.$$

We will use the set of all sets of indices $i_1,\dots,i_m$ such that $1 \le i_j \le n$, $1 \le j \le m \le n$.

Let $T_{l_{i_j}}(x_1,\dots,x_n) \in L_p$ be a trigonometrical polynomial of order l_{i_j} with respect to variable x_{i_j} but with respect to all other variables $T_{l_{i_j}}$ is a arbitrary function.

The best approximation by m-dimensional angle for the function f to variables $x_{i_1},\dots,x_{i_m}$ is the quantity (see [2]):

$$Y_{l_{i_1}\dots l_{i_m}}(f)_p = \inf_T \left\| f - \sum_{j=1}^{m} T_{l_{i_j}} \right\|_p, \; l_{i_j} = 0,1,2,\dots \tag{2.1}$$

Let $\mathcal{X}^j_{l_j}(t)$, $j = 1,\dots,n$, $l_j = 1,2,\dots$ be the kernels such that $\mathcal{X}(-t) = \mathcal{X}(t)$, and

$$\int_0^{2\pi} \mathcal{X}(t)dt = 2\pi, \quad \int_0^{2\pi} |\mathcal{X}(t)|dt \le M, \quad \lim_{l_j\to\infty} \int_{0<\delta\le|t|} |\mathcal{X}_{l_j}(t)| = 0, \tag{2.2}$$

where the constant M does not depend on l_j.

A Fourier series of the kernel $\mathcal{X}^j_{l_j}(t)$ can be stated in the form

$$\mathcal{X}^j_{l_j}(t) \equiv 1 + \sum_{k_j=1}^{\infty} \gamma^j_{l_j}(k_j)\cos k_j t, \quad (j = 1,\dots,n). \tag{2.3}$$

For the function $f \in L_p$ by these kernels we can define singular integrals

$$I^j_{l_j} f = \frac{1}{2\pi}\int_0^{2\pi} f(x_1,\dots,x_j - t_j,\dots,x_n)\mathcal{X}^j_{l_j}(t_j)dt_j, \tag{2.4}$$

$$I_{l_i l_j} f = I_{l_i}^i I_{l_j}^j f, \dots, I_{l_1,\dots,l_n} f = I_{l_1}^1 \dots I_{l_n}^n f.$$

By these singular integrals we can determine all m-dimensional angles ($1 \le i_j \le n$, $1 \le j \le m \le n$);

$$\text{(2.5)} \quad \begin{aligned} A_{l_{i_1},\dots,l_{i_m}} f = I_{l_{i_1}}^{i_1} f + \dots + I_{l_{i_m}}^{i_m} f - I_{l_{i_1} l_{i_2}} f - \dots - I_{l_{i_{m-1}} l_{i_m}} f \\ + \dots + (-1)^{m-1} I_{l_{i_1} \dots l_{i_m}} f. \end{aligned}$$

Without loss of generality, in order to simplify the exposition, we will give a proof for the case $n = 2$, i.e. for a function of two variables. In that case we have three angles

$$A_{l_1} f = I_{l_1}^1 f, \quad A_{l_2} f = I_{l_2}^2 f, \quad A_{l_1 l_2} f = I_{l_1}^1 f + I_{l_2}^2 f - I_{l_1 l_2} f.$$

two one-dimensional angles and one two-dimensional angle.

For a function $f(x_1, x_2) \in L_p$ we will use singular integrals

$$S_{l_1} f = S_{l_1 \infty} f = \frac{1}{\pi} \int_0^{2\pi} f(x_1 - t_1, x_2) D_{l_1}(t_1) dt_1$$

$$S_{l_2} f = S_{\infty l_2} f = \frac{1}{\pi} \int_0^{2\pi} f(x_1, x_2 - t_2) D_{t_2}(t_2) dt_2, \quad S_{l_1 l_2} f = S_{l_1}(S_{l_2} f).$$

where $D_l(t) = \dfrac{\sin(l + 1/2)t}{2 \sin t/2}$ is the Dirichlet's kernel.

In order to prove our main result, we need the singular integrals of de la Vallee-Poussin (see [2]) $V_{l_1} f = V_{l_1 \infty} f$, $V_{l_2} f = V_{\infty l_2} f$, $V_{l_1 l_2} f = V_{l_1}(V_{l_2} f)$, $W_{l_1 l_2} f = V_{l_1} f + V_{l_2} f - V_{l_1 l_2} f$, $l_j = 0, 1, 2, \dots$.

The functions $V_{l_j} f$, $j = 1, 2$, are trigonometrical polynomials of degree $2l_j - 1$ with respect to x_j and satisfies $\|V_{l_j} f\| \le B\|f\|$, $1 \le p \le \infty$, where B is an absolute constant.

Lemma 1 ([2], lemma 3). *Let $f(x_1, x_2) \in L_p$, $1 \le p \le \infty$. Then*

$$\text{(2.6)} \quad \left\|f - W_{l_1 l_2} f\right\|_p \le C Y_{l_1 l_2}(f)_p, \quad \left\|f - V_{l_j} f\right\|_p \le C Y_{l_j}(f)_p, \quad l_j = 0, 1, 2, \dots$$

where C is an absolute constant.

The most important tool in the proof will be the following lemma:

Lemma 2. *For a* $f \in L_p$, $1 \le p \le \infty$ *and* $l_j, s_j = 1, 2, \dots$ *the equalities*

$$f - A_{l_1 l_2} f = \sum_{i=1}^{9} B_j \tag{2.7}$$

hold, where

$$\begin{aligned} &B_1 = f - W_{2^{s_1}2^{s_2}} f, \quad B_2 = -I^1_{l_1} B_1, \quad B_3 = -I^2_{l_2} B_1, \quad B_4 = I_{l_1 l_2} B_1, \\ &B_5 = V_{2^{s_1}}\big(f - I^1_{l_1} f - V_{2^{s_2}} f + I^1_{l_1} V_{2^{s_2}} f\big), \quad B_6 = -I^2_{l_2} B_5, \\ &B_7 = V_{2^{s_2}}\big(f - I^2_{l_2} f - V_{2^{s_1}} f + I^2_{l_2} V_{2^{s_1}} f\big), \quad B_8 = -I^1_{l_1} B_7, \\ &B_9 = V_{2^{s_1}2^{s_2}}\big(f - A_{l_2 l_2} f\big). \end{aligned} \tag{2.8}$$

Proof.. The equality in the lemma is obtained by using the theorem of Fubini. □

We note that similar equalities were established in the paper [4] in which de la Vallee-Poussin sums are replaced by Dirichlet's sums.

Now we will use the function $F^j_l(m, \theta)$ which is defined in [3]:

$$\begin{aligned} F^j_l(m,\theta) &= \frac{1 - \gamma^j_l(m)}{2} + \sum_{k=1}^{m-1} [1 - \gamma^j_l(m-k)] \cos k\theta \\ &= \frac{1 - \gamma^j_l(m)}{2} + \sum_{k=1}^{m-1} [1 - \gamma^j_l(k)] \cos (m-k)\theta, \end{aligned} \tag{2.9}$$

for $m = 2, 3, \dots$ and

$$F^j_l(1,\theta) = \frac{1 - \gamma^j_l(1)}{2}, \quad F^j_l(0,\theta) = 1 - \gamma^j_l(1), \quad j = 1, 2, \dots, n.$$

Lemma 3. *If*

$$T_m(t) = \sum_{\nu=0}^{m} \alpha_\nu \cos \nu t + \beta_\nu \sin \nu t$$

is a trigonometrical polynomial of order m *in one variable* t, *and if the function* F_l *is defined by (2.9), then the following equalities hold*

$$\sum_{k=1}^{m} [1 - \gamma_l(k)](\alpha_k \cos kx + \beta_k \sin kx) = \frac{2}{\pi} \int_0^{2\pi} F_l(m,\theta) T_m(x+\theta) \cos m\theta \, d\theta \tag{2.10}$$

$$\int_0^{2\pi} F_l(\nu,\theta) T_m(t+\theta) \cos \nu\theta \, d\theta = \int_0^{2\pi} F_l(m,\theta) T_m(t+\theta) \cos m\theta \, d\theta, \quad \nu > m. \tag{2.11}$$

Proof. Equality (2.10) is proved in [3] (the equality (3.11)). Equality (2.11) can be proved in the same way. □

3. The main result

For every kernel $\mathcal{X}^j_{l_j}(t)$, $j = 1, \dots, n$ we can identify the quantities $\phi = \phi_j(l_j) > 0$, $\psi = \psi^j_{l_j}(k_j)$, $K = K_j(\psi^j, l_j, k_j)$, using equalities

$$(3.1) \qquad 1 - \gamma^j_{l_j}(k_j) = \phi_j(l_j)\psi^j_{l_j}(k_j), \quad k_j, l_j = 1, 2, \dots,$$

$$(3.2) \qquad K = K_j(\psi^j, l_j, k_j) = \frac{1}{2} + \sum_{\nu_j=1}^{2^{k_j-2}} \frac{\psi^j_{l_j}(2^{k_j} - 1 - \nu_j)}{\psi^j_{l_j}(2^{k_j} - 1)} \cos \nu_j\theta_j .$$

For a fixed number l_j we choose the number s_j such that $2^{s_j} \le l_j \le 2^{s_j+1}$. We will say that the quantities ϕ, ψ, K satisfy conditions $(\alpha), (\beta), (\gamma), (\delta)$ if

(α) $$\left|\psi^j_{l_j}(k'_j)\right| \le C_1\left|\psi^j_{l_j}(k''_j)\right|, \quad 0 \le k'_j \le k''_j \le 2^s$$

$$\left|\psi^j_{l_j}(1)\right| \le C_2, \quad (\psi^j_{l_j}(1) = \psi^j_{l_j}(0)),$$

(β) $$\left|\psi^j_{l_j}(2k_j)\right| \le C_3\left|\psi^j_{l_j}(k_j)\right|, \quad 2k_j \le 2^{s_j}$$

(γ) $$0 < C_4 \le \phi_j(l_j)\left|\psi^j_{l_j}(2^{s_j})\right|,$$

(δ) $\left\|K_j(\psi^j, l_j, k_j)\right\|_1 \le C_5,$

where the constants $C_1, \dots, C_5$ don't depend on k_j and l_j.

We will use symbol the [ ] such that $\left[2^{k-1}\right] = 2^{k-1}$ for $k \ge 1$ and $\left[2^{0-1}\right] = 0$.

By $a << b$, $a > 0$, $b > 0$, we will denote the inequality $a \le Cb$, where C is some positive constant.

The following theorem gives the estimation of the approximation $\|f - A_{l_{i_1} \dots l_{i_m}} f\|$ by the best approximation by angle.

Theorem 1. *Let the quantities ϕ, ψ, K satisfy the conditions (α), (β), (γ), (δ) and let $f \in L_p$, $1 \le p \le \infty$. Then for all natural numbers i_j and m such that $1 \le i_j \le n$, $1 \le j \le m \le n$ the following inequalities hold*

$$(3.3) \qquad \left\|f - A_{l_{i_1} \dots l_{i_m}} f\right\|_p \le C \prod_{j=1}^{m} \phi_{i_j}(l_{i_j}) \left[\sum_{k_{i_1}=0}^{l_{i_1}} \cdot \sum_{k_{i_m}=0}^{l_{i_m}} \prod_{j=1}^{m} \frac{\|\psi^{i_j}_{l_{i_j}}(k_{i_j})\|}{k_{i_j} + 1} Y_{k_{i_1} \dots k_{i_m}}(f)_p \right]$$

with constant C independent on f and $l_j = 1, 2, \dots$.

To prove this theorem we need

Theorem 2. *Let the singular integrals $X^j_{l_j}(t)$, $j = 1,2$ satisfy the condition (2.2) and let the function $f(x_1, x_2) \in L_p([0, 2\pi]^2)$, $1 \le p \le \infty$. Then for approximation by angle the following inequalities hold*

$$\|f - A_{l_j} f\|_p \le C_1 \Big[Y_{2^{s_j}}(f)_p + \sum_{k_j=0}^{s_j} Y_{[2^{k_j}-1]}(f)_p \int_0^{2\pi} \big|F^j_{l_j}(2^{k_j} - 1, \theta_j)\big| d\theta_j \Big], \tag{3.4}$$

$$\begin{aligned}\|f - A_{l_1 l_2} f\|_p \le C_2 \Big[& Y_{2^{s_1} 2^{s_2}}(f)_p + \\ & + \sum_{k_1=0}^{s_1} Y_{[2^{k_1}-1] 2^{s_2}}(f)_p \int_0^{2\pi} \big|F^1_{l_1}(2^{k_1} - 1, \theta_1)\big| d\theta_1 \Big] + \\ & + C_2 \Big[\sum_{k_2=0}^{s_2} Y_{2^{s_1} [2^{k_1}-1]}(f)_p \int_0^{2\pi} \big|F^2_{l_2}(2^{k_2} - 1, \theta_2)\big| d\theta_2 \Big] + \\ & + C_2 \Big[\sum_{k_1=0}^{s_1} \sum_{k_1=0}^{s_2} Y_{[2^{k_1}-1][2^{k_2}-1]}(f)_p \prod_{j=1}^{2} \int_0^{2\pi} \big|F^j_{l_j}(2^{k_j} - 1, \theta_j)\big| d\theta_j \Big],\end{aligned} \tag{3.5}$$

where the constants C_1, C_2 do not depend on f and $s_j, l_j = 1, 2, \dots$ $(j = 1, 2)$.

Proof of Theorem 2. We have

$$\|f - A_{l_j} f\| = \|f - I^j_{l_j} f\| = \|f - V_{2^{s_j}} f + V_{2^{s_j}} f - I^j_{l_j} V_{2^{s_j}} f + I^j_{l_j} V_{2^{s_j}} f - I^j_{l_j} f\|$$

and therefore

$$\|f - A_{l_j} f\|_p \le C_3 Y_{2^{s_j}}(f)_p + \|V_{2^{s_j}} f - I^j_{l_j} V_{2^{s_j}} f\|_p \tag{3.6}$$

where the constant C_3 does not depend on f, l_j, s_j and the numbers l_j and s_j are arbitrary.

We consider

$$\begin{aligned} V_{2^{s_j}} f - I^j_{l_j} V_{2^{s_j}} f = G_{l_j}(f, 2^{s_j}, x) = & \sum_{k=0}^{2^{s_j}-1} \delta_k^{(2^{s_j})} A_k \\ - \sum_{k=0}^{2^{s_j+1}-1} \gamma^j_{l_j}(k) \delta_k^{(2^{s_j})} A_k = & \sum_{k=1}^{2^{s_j+1}-1} [1 - \gamma^j_{l_j}(k)] \delta_k^{(2^{s_j})} A_k \end{aligned} \tag{3.7}$$

where A_k is the term of the Fourier series of f as a function with respect to the variable x_j and δ_j is a factor of a product which is determined by the sum of Vallee-Poussin.

For G we will use the expression

$$G_{l_j}(f,2^{s_j},x) = \sum_{k=1}^{s_j}\left[G_{l_j}(f,2^k,x) - G_{l_j}(f,2^{k-1},x)\right] + G_{l_j}(f,1,x). \tag{3.8}$$

In view of (3.7) the function $G_{l_j}(f,2^k,x)$ is a trigonometrical polynomial with respect to x_j. Therefore, using Lemma 3, (2.10), we have

$$\begin{aligned} &G_{l_j}(f,2^{k_1},x) = \\ &= \frac{2}{\pi}\int_0^{2\pi} F^1_{l_1}(2^{k_1+1}-1,\theta_1)V_{2^{k_1}}f(x_1+\theta_1,x_2)\cos(2^{k_1+1}-1)\theta_1 d\theta_1. \end{aligned} \tag{3.9}$$

$$\begin{aligned} &G_{l_j}(f,2^{k_1-1},x) = \\ &= \frac{2}{\pi}\int_0^{2\pi} F^1_{l_1}(2^{k_1}-1,\theta_1)V_{2^{k_1-1}}f(x_1+\theta_1,x_2)\cos(2^{k_1}-1)\theta_1 d\theta_1. \end{aligned} \tag{3.10}$$

and similar equalities with respect to the variable x_2.

It follows by Lemma 3, (2.11), that holds

$$\begin{aligned} &G_{l_j}(f,2^{k_1},x) \\ &= \frac{2}{\pi}\int_0^{2\pi} F^1_{l_1}(2^{k_1}-1,\theta_1)V_{2^{k_1}}f(x_1+\theta_1,x_2)\cos(2^{k_1}-1)\theta_1 d\theta_1. \end{aligned} \tag{3.11}$$

The equalities (3.10) and (3.11) give

$$\begin{aligned} &G_{l_j}(f,2^{k_1},x) - G_{l_1}(f,2^{k_1-1},x) = \frac{2}{\pi}\int_0^{2\pi} F^1_{l_1}(2^{k_1}-1,\theta_1) \\ &\left[V_{2^{k_1}}f(x_1+\theta_1,x_2) - V_{2^{k_1-1}}f(x_1+\theta_1,x_2)\right]\cos(2^{k_1}-1)\theta_1 d\theta_1. \end{aligned} \tag{3.12}$$

Since by Lemma 1

$$\left\|V_{2^{k_1}}f - V_{2^{k_1-1}}f\right\| \le \left\|V_{2^{k_1}}f - f\right\| + \left\|f - V_{2^{k_1-1}}f\right\| << 2Y_{[2^{k_1-1}]}(f) \tag{3.13}$$

we conclude from (3.12) that

$$\begin{aligned} &\left\|G_{l_j}(f,2^{k_1},x) - G_{l_1}(f,2^{k_1-1},x)\right\| \\ &<< Y_{[2^{k_1-1}]}(f)\int_0^{2\pi}\left|F^1_{l_1}(2^{k_1}-1,\theta_1)\right|d\theta_1 \end{aligned} \tag{3.14}$$

holds.

For the quantity $G_{l_j}(f,1,x)$ we have

$$\begin{aligned} G_{l_j}(f,1,x) &= V_1 f - I^1_{l_1} V_1 f = S_1 f - I^1_{l_1} S_1 f \\ &= A_1 f - \gamma^1_{l_1}(1) A_1 f = \left[1-\gamma^1_{l_1}(1)\right] A_1 f \\ &= \left[1-\gamma^1_{l_1}(1)\right]\left[S_1 f - S_0 f\right] = \left[1-\gamma^1_{l_1}(1)\right]\left[V_1 f - V_0 f\right] \end{aligned}$$

because $V_1 f = S_1 f$, $V_0 f = S_0 f$. Therefore

$$\|G_{l_1}(f,1,x)\| \ll Y_0(f)\left|1-\gamma^1_{l_1}(1)\right|$$

hence, using the definition for $F^1_{l_1}(0,\theta)$ we obtain

$$\|G_{l_1}(f,1,x)\|_p \ll Y_0(f)_p \int_0^{2\pi} \left|F^1_{l_1}(0,\theta_1)\right| d\theta_1. \tag{3.15}$$

Now, in view of (3.7), (3.8), (3.14) and (3.15) , we obtain

$$\left\|V_{2^{s_1}} f - I^1_{l_1} V_{2^{s_1}} f\right\| \ll \sum_{k_1=0}^{s_1} Y_{[2^{k_1}-1]}(f) \int_0^{2\pi} \left|F^1_{l_1}\left(2^{k_1}-1,\theta_1\right)\right| d\theta_1. \tag{3.16}$$

From (3.6) using (3.16) it follows the inequality (3.4) for $j=1$. In the same way we establish the inequality (3.16) for $j=2$. Thus, the inequality (3.4) is proved.

To establish the inequality (3.5) concerning the approximation by two-dimensional angle we use Lemma 2.

It is clear that

$$\|B_j\| \ll Y_{2^{s_1}2^{s_2}}(f)_p, \quad j=1,2,3,4, \tag{3.17}$$

holds.

To estimate the quantity B_5 we will write

$$B_5 = V_{2^{s_1}}\Phi - I^1_{l_1} V_{2^{s_1}}\Phi \tag{3.18}$$

where

$$\Phi = f - V_{2^{s_2}} f.$$

We consider the function B_5 as a function of the variable x_1 and apply the metod by which we estimated the expression $G_{l_1}(f,2^{s_1},x)$. So we derive

(3.8')
$$B_5 = B_5\left(2^{s_1}\right) = \sum_{k_1=1}^{s_1} \left[B_5\left(2^{k_1}\right) - B_5\left(2^{k_1-1}\right)\right] + B_5(1),$$

(3.12')
$$\begin{aligned} &B_5\left(2^{k_1}\right) - B_5\left(2^{k_1-1}\right) = \\ &= \frac{2}{\pi}\int_0^{2\pi} F_{l_1}^1\left(2^{k_1}-1,\theta_1\right)\left[V_{2^{k_1}}\Phi\left(x_1+\theta_1,x_2\right) - \right. \\ &\left. - V_{2^{k_1-1}}\Phi\left(x_1+\theta_1,x_2\right)\right]\cos\left(2^{k_1}-1\right)\theta_1 d\theta_1, \end{aligned}$$

$$\left\|V_{2^{k_1}}\Phi - V_{2^{k_1}}\Phi\right\| \leq \left\|V_{2^{k_1}}\Phi - \Phi\right\| + \left\|\Phi - V_{2^{k_1-1}}\Phi\right\|.$$

Since
$$V_{2^{k_1}}\Phi - \Phi = -\left[f - \left(V_{2^{k_1}}f + V_{2^{s_2}}f - V_{2^{k_1}2^{s_2}}f\right)\right]$$
we obtain
$$\begin{aligned} &\left\|V_{2^{k_1}}\Phi - \Phi\right\|_p \ll Y_{2^{k_1}2^{s_2}}(f)_p, \\ &\left\|\Phi - V_{2^{k_1-1}}\Phi\right\|_p \ll Y_{2^{k_1-1}2^{s_2}}(f)_p. \end{aligned}$$

Thus

(3.19)
$$\left\|V_{2^{k_1}}\Phi - V_{2^{k_1-1}}\Phi\right\|_p \ll Y_{2^{k_1-1}2^{s_2}}(f)_p.$$

For $B_5(1)$ we have
$$B_5(1) = V_1\Phi - I_{l_1}^1\Phi = \left[1-\gamma_{l_1}^1(1)\right]A_1\Phi = \left[1-\gamma_{l_1}^1(1)\right]\left[V_1\Phi - V_0\Phi\right].$$

Since
$$\begin{aligned} &\left\|V_1\Phi - V_0\Phi\right\| \leq \left\|V_1\Phi - \Phi\right\| + \left\|\Phi - V_0\Phi\right\|, \\ &\quad\left\|V_1\Phi - \Phi\right\| = \left\|f - \left(V_1 f + V_{2^{s_2}}f - V_{12^{s_2}}f\right)\right\| \ll Y_{12^{s_2}}(f), \\ &\quad\left\|\Phi - V_0\Phi\right\| = \left\|f - \left(V_0 f + V_{2^{s_2}}f - V_{02^{s_2}}f\right)\right\| \ll Y_{02^{s_2}}(f), \\ &\quad\left|1-\gamma_{l_1}^1(1)\right| = \left|F_{l_1}^1\left(0,\theta_1\right)\right| \end{aligned}$$

we derive

(3.20)
$$\left\|B_5(1)\right\| \ll Y_{02^{s_2}}(f)\int_0^{2\pi}\left|F_{l_1}^1\left(0,\theta_1\right)\right| d\theta_1.$$

In view of (3.18) , (3.8') , (3.12') , (3.19) , (3.20) it follows that

$$\|B_5\| \ll \sum_{k_1=0}^{s_1} Y_{[2^{k_1}-1]2^{s_2}}(f) \int_0^{2\pi} \left|F_{l_1}^1\left(2^{k_1}-1,\theta_1\right)\right| d\theta_1. \tag{3.21}$$

It is clear that

$$\|B_6\| \ll \|B_5\|. \tag{3.22}$$

In the same way we obtain

$$\|B_7\| \ll \sum_{k_2=0}^{s_2} Y_{2^{s_1}[2^{k_2}-1]}(f) \int_0^{2\pi} \left|F_{l_2}^2\left(2^{k_2}-1,\theta_2\right)\right| d\theta_2 \tag{3.23}$$

$$\|B_8\| \ll \|B_7\|. \tag{3.24}$$

To estimate B_9 we use the equality

$$B_9 = V_{2^{s_1}}P - V_{2^{s_1}}I_{l_1}^1 P, \quad P = V_{2^{s_2}}f - I_{l_2}^2 V_{2^{s_2}}f.$$

In the same way as we obtained the expression for B_5 we derive

$$\begin{aligned} B_9 &= \sum_{k_1=0}^{s_1}\left[B_9\left(P,2^{k_1}\right) - B_9\left(P,\left[2^{k_1-1}\right]\right)\right] \\ &= \sum_{k_1=0}^{s_1}\frac{2}{\pi}\int_0^{2\pi}\left\{V_{2^{k_1}}P\left(x_1+\theta_1,x_2\right) - V_{[2^{k_1-1}]}P\left(x_1+\theta_1,x_2\right)\right\}\cdot \\ &\cdot F_{l_1}^1\left(2^{k_1}-1,\theta_1\right)\cos\left(2^{k_1}-1\right)\theta_1 d\theta_1. \end{aligned} \tag{3.25}$$

We consider the function P as a function with respect to x_2 and obtain

$$\begin{aligned} &P(x_1+\theta_1,x_2) = V_{2^{s_2}}f(x_1+\theta_1,x_2) - V_{2^{s_2}}I_{l_2}^2 f(x_1+\theta_1,x_2) = \\ &\sum_{k_2=0}^{s_2}\left\{B_7\left(f,2^{k_2}\right) - B_7\left(f,\left[2^{k_2-1}\right]\right)\right\} = \\ &\sum_{k_2=0}^{s_2}\frac{2}{\pi}\int_0^{2\pi}\left\{V_{2^{k_2}}f\left(x_1+\theta_1,x_2+\theta_2\right) - \right. \\ &\left. V_{[2^{k_2-1}]}f\left(x_1+\theta_1,x_2+\theta_2\right)\right\}\cdot F_{l_2}^2\left(2^{k_2}-1,\theta_2\right)\cos\left(2^{k_2}-1\right)\theta_2 d\theta_2. \end{aligned} \tag{3.26}$$

Using (3.25) and (3.26) we get

$$B_9 = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2}\frac{4}{\pi}\int_0^{2\pi}\int_0^{2\pi}\{V_{2^{k_1}2^{k_2}}f - V_{2^{k_1}[2^{k_2}-1]}- \tag{3.27}$$
$$- V_{[2^{k_1}-1]2^{k_2}}f + V_{[2^{k_1}-1][2^{k_2}-1]}f\}\cdot$$
$$\cdot\prod_{j=1}^{2} F_{l_j}^{j}\left(2^{k_j}-1,\theta_j\right)\cos\left(2^{k_j}-1\right)\theta_j d\theta_j.$$

Since

$$\begin{aligned} Q &= V_{2^{k_1}2^{k_2}}f - V_{2^{k_1}[2^{k_2}-1]}f - V_{[2^{k_1}-1]2^{k_2}}f + V_{[2^{k_1}-1][2^{k_2}-1]}f \\ &= W_{2^{k_1}2^{k_2}}f - W_{2^{k_1}[2^{k_2}-1]}f - W_{[2^{k_1}-1]2^{k_2}}f + W_{[2^{k_1}-1][2^{k_2}-1]}f \end{aligned} \tag{3.28}$$

we obtain

$$\|Q\| \ll Y_{[2^{k_1}-1][2^{k_2}-1]}(f). \tag{3.29}$$

From (3.27) , in view of (3.28) and (3.29) it follows that

$$\|B_9\| \ll \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2} Y_{[2^{k_1}-1][2^{k_2}-1]}(f)\prod_{j=1}^{2}\int_0^{2\pi}\left|F_{l_j}^{j}\left(2^{k_j}-1,\theta_j\right)\right| d\theta_j. \tag{3.30}$$

Finally, using Lemma 2 and the inequalities (3.17), (3.2), (3.22), (3.23), (3.24), (3.30) we obtain the inequality (3.5). The proof of Theorem 2 is complete .

Proof of Theorem 1. First we establish the inequality (3.3) for $m = 1$, $n = 2$. We will use the inequality (3.4), (Theorem 2), and the conditions of Theorem 1.

In view of (3.1) and (3.2) we derive

$$F_{l_1}^{1}\left(2^{k_1}-1,\theta_1\right) = \phi_1(l_1)\psi_{l_1}^{1}\left(2^{k_l}-1\right)K_1\left(\psi^1,l_1,k_1\right), \tag{3.31}$$

hence, using the condition (δ), it follows that

$$\left\|F_{l_1}^{1}\left(2^{k_1}-1,\theta_1\right)\right\| \ll \phi_1(l_1)\left|\psi_{l_1}^{1}\left(2^{k_1}-1\right)\right|. \tag{3.32}$$

From (3.4) by (3.32) we obtain

$$(3.33)\qquad \|f - A_{l_1} f\| \ll Y_{2^{s_1}}(f) + \phi_1(l_1) \sum_{k_1=0}^{s_1} \left|\psi^1_{l_1}(2^{k_1} - 1)\right| Y_{[2^{k_1-1}]}(f).$$

Now, from (3.33) using the condition (γ) it follows that

$$(3.34)\qquad \begin{aligned} \|f - A_{l_1} f\| \ll \phi_1(l_1)\{ &\left|\psi^1_{l_1}(2^{s_1})\right| Y_{2^{s_1}}(f) \\ &+ \sum_{k_1=0}^{s_1} \left|\psi^1_{l_1}(2^{k_1-1})\right| Y_{[2^{k_1-1}]}(f)\}. \end{aligned}$$

hence by the conditions (α) and (β) we derive

$$(3.35)\qquad \|f - A_{l_1} f\| \ll \phi_1(l_1) \sum_{k_1=0}^{s_1+1} \left|\psi^1_{l_1}([2^{k_1-1}])\right| Y_{[2^{k_1-1}]}(f).$$

We conclude, using the conditions (β), (α) that (see [4]):

$$(3.36)\qquad \sum_{k=1}^{s} \left|\psi_l(2^k)\right| Y_{2^k} \ll \sum_{\nu=2}^{2^s} \frac{|\psi_l(\nu)|}{\nu} Y_\nu.$$

Finally, from (3.35) by (3.36) we obtain the inequality (3.3) for the case $n = 2$, $m = 1$ (with respect to the variable x_1).

In the same way using the inequality (3.5) of Theorem 2 and the conditions of Theorem 1 (see the proof of the corresponding theorem in [4]), we obtain the inequality (3.3) for $m = n = 2$.

The proof of Theorem 1 is complete .

4. Applications

The obtained result (Theorem 1) we apply to Riesz's singular integrals.

Riesz's singular integral is given by the kernel (see [1])

$$(4.1)\qquad \chi_l^{(\lambda)}(t) = 1 + \sum_{j=1}^{l} \left(1 - \frac{\lambda_j}{\lambda_{l+1}}\right) \cos jt$$

where the sequence λ_l, $l = 1, 2, \ldots$ satisfies following conditions: (i) $0 < \lambda_l < \lambda_{l+1}$, (ii) $\lambda_l \to \infty$, $l \to \infty$, (iii) $\Delta_2\lambda < 0$ or $\Delta_2\lambda \geq 0$, (iv) $\lambda_{2l} = O(\lambda_l)$ if $\Delta_2\lambda_l \geq 0$.

For this singular integral (this method of summation of Fourier series of functions) we have

$$\gamma_l^{(\lambda)}(0) = 1,\ \gamma_l^{(\lambda)}(j) = 1 - \frac{\lambda_j}{\lambda_{l+1}},\ j = 1,\ldots,l,$$
$$\gamma_l^{(\lambda)}(j) = 0,\ j = l+1, l+2,\ldots.$$

The quantities ϕ, ψ, K are

$$\psi_l^{(\lambda)}(j) = \lambda_j,\ j = 1,2,\ldots,l$$
$$\psi_l^{(\lambda)}(j) = \lambda_{l+1},\ j \geq l+1,\ \phi^{(\lambda)}(l) = \frac{1}{\lambda_{l+1}},\ l = 1,2,\ldots \tag{4.3}$$
$$K = K\left(\psi^{(\lambda)}, k, \theta\right) = \frac{1}{2} + \sum_{\nu=1}^{2^k-2} \frac{\lambda_{2^k-1-\nu}}{\lambda_{2^k-1}} \cos\nu\theta.$$

We will prove that the quantities ϕ, ψ, K satisfy the conditions (α), (β), (γ), (δ) of Theorem 1.

Since (i) and (4.3) the condition (α) holds for ψ.

To prove that the condition (β) is satisfied we use the inequality

$$\lambda_{k+1} - \lambda_k \leq C\frac{\lambda_k}{k},\quad (C = \text{constant}),$$

which is proved in the paper [1] of Aljancic (if $\Delta_2\lambda \geq 0$ the condition (β) is obviously satisfied, the condition (iv)). From this inequality it follows that

$$\frac{\lambda_{k+1}}{\lambda_k} \leq 1 + C\frac{1}{k}.$$

Putting $k, k+1, \ldots, 2k-1$ in this inequality, by multiplication we derive

$$\frac{\lambda_{2k}}{\lambda_k} \leq \left(1 + \frac{C}{k}\right)^k,$$

and then $\lambda_{2k} = O(\lambda_k)$.

The condition (γ) is equivalent to the condition

$$\frac{\lambda_l}{\lambda_{2^s}} \leq C,\ 2^s \leq l < 2^{s+1}.$$

We have

$$\frac{\lambda_l}{\lambda_{2^s}} \leq \frac{\lambda_{2^{s+1}}}{\lambda_{2^s}} \leq C,$$

that means that condition (γ) is satisfied.

Now we will prove that the function K satisfies the condition (δ).

By applying Abel's identity we have

$$K\left(\psi^{(\lambda)},k,\theta\right) - \frac{1}{\lambda_{2^k-1}}\sum_{\nu=0}^{2^k-3}\Delta\lambda_{2^k-1-\nu}D_\nu(\theta) + \frac{\lambda_1}{\lambda_{2^k-1}}D_{2^k-2}(\theta) \tag{4.4}$$

where D is Dirichlet's kernel.

To estimate the free term in (4.4) we introduce the new condition

$$\frac{l}{\lambda_l} \le C \quad (C = \text{constant}), \tag{4.5}$$

independent on l.

In view of (4.5) we have

$$K\left(\psi^{(\lambda)},k,\theta\right) = \frac{1}{\lambda_{2^k-1}}\sum_{\nu=0}^{2^k-3}\Delta\lambda_{2^k-1-\nu}D_\nu(\theta) + O(1). \tag{4.6}$$

If we apply Abel's identity again, from (4.6) we obtain

$$\begin{aligned} K\left(\psi^{(\lambda)},k,\theta\right) &= \frac{1}{\lambda_{2^k-1}}\sum_{\nu=0}^{2^k-4}\Delta_2\lambda_{2^k-1-\nu}\sum_{j=0}^{\nu}D_j(\theta) \\ &+ \frac{1}{\lambda_{2^k-1}}\Delta\lambda_2\sum_{j=0}^{2^k-3}D_j(\theta) + O(1). \end{aligned} \tag{4.7}$$

Since

$$\sum_{j=0}^{\nu}D_j(\theta) = (\nu+1)F_\nu(\theta)$$

where F_ν is Fejer's kernel , it follows that

$$\begin{aligned} K\left(\psi^{(\lambda)},k,\theta\right) &= \frac{1}{\lambda_{2^k-1}}\sum_{\nu=0}^{2^k-4}(\nu+1)F_\nu(\theta)\Delta_2\lambda_{2^k-1-\nu} + \\ &+ \frac{1}{\lambda_{2^k-1}}\Delta\lambda_2\left(2^k-2\right)F_k(\theta) + O(1). \end{aligned} \tag{4.8}$$

The equality (4.8) implies

$$\left\|K\left(\psi^{(\lambda)},k,\theta\right)\right\|_1 \ll \frac{1}{\lambda_{2^k-1}}\sum_{\nu=0}^{2^k-4}(\nu+1)\left|\Delta_2\lambda_{2^k-1-\nu}\right| \cdot \frac{|\Delta\lambda_2|}{\lambda_{2^k-1}}\left(2^k-2\right)+O(1). \tag{4.9}$$

Using (4.5) from (4.9) we obtain

$$\left\|K\left(\psi^{(\lambda)},k,\theta\right)\right\|_1 \ll \frac{1}{\lambda_{2^k-1}}\sum_{\nu=0}^{2^k-4}(\nu+1)\left|\Delta_2\lambda_{2^k-1-\nu}\right|+O(1) \tag{4.10}$$

where $\Delta\lambda_\nu=\lambda_\nu-\lambda_{\nu+1}$, $\Delta_2\lambda=\Delta(\Delta\lambda)$.
Since

$$\sum_{\nu=0}^{j}(\nu+1)\Delta_2\mu_\nu = 2\mu_0-\mu_1+(\mu_{j+2}-\mu_{j+1})(j+1)-\mu_{j+1} \tag{4.11}$$

then, putting $\mu_\nu=\lambda_{2^k-1-\nu}$, $j=2^k-4$ we derive

$$\sum_{\nu=0}^{2^k-4}(\nu+1)\left|\Delta_2\lambda_{2^k-1-\nu}\right| = 2\lambda_{2^k-1}-\lambda_{2^k-2}+(\lambda_1-\lambda_2)\left(2^k-3\right)-\lambda_2. \tag{4.12}$$

Finally, from (4.10) in view of (4.12) we obtain that $\|K\|_1\le C$. This means that the function K of Riesz's singular integrals satisfies the condition (δ).
In this way we prove the following

Theorem 3. *Let the sequences $\lambda_l^{(k)}$, $k=1,\dots,n$, $l=1,2,\dots,$ satisfy the conditions (i) - (iv) and (v) $l=O(\lambda_l)$, $l\to\infty$. Let $A^{(\lambda)}_{l_{i_1},\dots,l_{i_m}}f$ be m-dimensional angles which are obtained from singular integrals which are associated with the given sequences .*

Then, for $f\in L_p$, $1\le p\le\infty$, and all natural numbers i_j and m such that $1\le i_j\le n$, $1\le j\le m\le n$, the following inequalities hold

$$\left\|f-A^{(\lambda)}_{l_{i_1},\dots,l_{i_m}}f\right\| \le C\left[\prod_{j=1}^{m}\lambda^{(i_j)}_{l_{i_j}}\right]^{-1}\sum_{k_{i_1}=0}^{l_{i_1}}\cdots \sum_{k_{i_m}=0}^{l_{i_m}}\prod_{j=1}^{m}\frac{\lambda^{(i_j)}_{k_{i_j}}}{k_{i_j}+1}Y_{k_{i_1}\dots k_{i_m}}(f) \tag{4.13}$$

where the constant C does not depend on f and $l_j = 1, 2, \ldots$.

Particularly we consider the sequences

$$\lambda_j = \lambda_j^{(r,s)} = \lambda_j(r,s) = j^r \log^s(j+2), \; j = 1, 2, \ldots \tag{4.14}$$

where the real numbers r and s satisfy $r \geq 0$, $s \geq 0$.

Conditions (i) - (iv) of Theorem 3 are satisfied. Condition (iii) is satisfied because the function $\lambda(x) = x^r \log^s x$ has derivative $\lambda''(x) \geq 0$ for $r \geq 1$, $s \geq 0$, $x \geq b$ where b is the base of the logarithm.

Thus, we can apply Theorem 3 and obtain

Theorem 4. *Let $A^{(r,s)}_{l_{i_1}\ldots l_{i_m}} f$ be an m-dimensional angle from singular integrals which are determined by the sequences $\lambda_j(r,s) = j^r \log^s(j+2)$, $j = 1, 2, \ldots$ for $r \geq 1$, $s \geq 0$. Then for $f \in L_p([0,2\pi]^n)$, $1 \leq p \leq \infty$, the following inequalities hold*

$$\begin{aligned} \left\| f - A^{(r,s)}_{l_{i_1}\ldots l_{i_m}} f \right\| &\leq C \prod_{j=1}^{m} l_{i_j}^{-r_{i_j}} \log^{-s_{i_j}} (l_{i_j} + 2) \cdot \\ &\cdot \sum_{k_{i_1}=0}^{l_{i_1}} \cdots \sum_{k_{i_m}=0}^{l_{i_m}} \prod_{j=1}^{m} (k_{i_j} + 1)^{r_{i_j} - 1} \log^{s_{i_j}} (k_{i_j} + 2)\, Y_{k_{i_1}\ldots k_{i_m}}(f) \end{aligned} \tag{4.15}$$

where the constant C does not depend on f and $l_j = 1, 2, \ldots$, $1 \leq i_j \leq n$, $1 \leq j \leq m \leq n$.

Puting $s_j = 0$, $j = 1, \ldots n$ we obtain from (4.15) the following inequalities

$$\begin{aligned} \left\| f - A^{(r)}_{l_{i_1}\ldots l_{i_m}} f \right\|_p &\leq C \prod_{j=1}^{m} l_{i_j}^{-r_{i_j}} \cdot \sum_{k_{i_1}=0}^{l_{i_1}} \cdots \\ &\sum_{k_{i_m}=0}^{l_{i_m}} \prod_{i=1}^{m} (k_{i_j} + 1)^{r_{i_j} - 1} Y_{k_{i_1}\ldots k_{i_m}}(f) \end{aligned} \tag{4.16}$$

where $f \in L_p([0,2\pi]^n)$, $1 \leq p \leq \infty$, $r_j \geq 1$, $1 \leq i_j \leq n$, $1 \leq j \leq m \leq n$.

For $n = 1$ we have the case of a function of one variable. Then $Y = E$ and from (4.16) we obtain

$$\left\| f - A_l^{(r)} f \right\|_p \leq C \frac{1}{l^r} \sum_{k=0}^{l} (k+1)^{r-1} E_k(f)_p \tag{4.17}$$

where $1 \le p \le \infty$, $r \ge 1$, $l = 1, 2, \ldots$ and

$$A_l^{(r)} f = \frac{a_0}{2} + \sum_{k=0}^{l} \left[1 - \frac{k^r}{(l+1)^r}\right] (a_k \cos kx + b_k \sin kx),$$

a_k, b_k are the Fourier coeficients of the function f.

The theorem proved above make it possible to compare the classes of functions which are defined by the approximations. We will show that comparing the following classes.

Let the numbers $r_1, \ldots r_n$, $r_j \ge 1$, $j = 1, \ldots, n$, be given. We identify the classes

$$S_p^r H = \Bigg\{ f \in L_p : Y_{l_{i_1} \ldots l_{i_m}}(f)_p = O\Big(\prod_{j=1}^{m} l_{i_j}^{-r_{i_j}}\Big),$$

$$l_j = 1, 2, \ldots,\ 1 \le i_j \le n,\ 1 \le j \le m \le n \Bigg\}$$

$$V_p^r R = \Bigg\{ f \in L_p : \Big\| f - A^{(r)}_{l_{i_1} \ldots l_{i_m}} f \Big\|_p = O\Big(\prod_{j=1}^{m} l_{i_j}^{-r_{i_j}}\Big),$$

$$l_j = 1, 2, \ldots,\ 1 \le i_j \le n,\ 1 \le j \le m \le n \Bigg\}$$

where $A^{(r)}$ are angles which are determined by the sequances $\lambda_j(r_k) = j^{r_k}$, $k = 1, \ldots, n$, $j = 1, 2, \ldots$.

Then in view of the inequalities (4.16) we conclude that

$$S_p^{r+\varepsilon} H \subset V_p^r R \subset S_p^r H, \quad 1 \le p \le \infty,$$

where $r + \varepsilon$ is determined by numbers $r_j + \varepsilon_j$, $r_j \ge 1$, $\varepsilon_j > 0$, $j = 1, \ldots, n$.

The classes $S_p^r H$ are the classes of Nikolski which are defined by the mixed dominated modulus of smoothness (see [2]).

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