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A SUFFICIENT UNIVALENCE CONDITION

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ABSTRACT. *This paper is concerned with a sufficient univalence condition for analytic functions in the unit disc. This condition generalizes some well-known univalence criteria.*

1. Introduction and preliminaries

Let U_r denote the disc $\{z \in \mathbb{C} : |z| < r\}$, $r \in (0,1]$ and let A denote the class of functions f which are analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

Definition 1. *Let $f, g : U \mapsto \mathbb{C}$ be analytic function in U. The function f is subordinate to the function g ($f \prec g$) if there is an analytic function φ in U, which satisfies the conditions $\varphi(0) = 0$, $|\varphi(z)| < 1$, $z \in U$, and $f = g \circ \varphi$.*

Definition 2. *The function $L : U \times I \mapsto \mathbb{C}$, $I = [0, \infty)$ is a Loewner chain if the function $L(z,t)$ is analytic and univalent in U for all $t \in I$ and $L(z,s) \prec L(z,t)$ for all $0 \le s < t$.*

Definition 3 [5]. *The function $F : U_r \times \mathbb{C} \mapsto \mathbb{C}$, $F = F(u,v)$ satisfies the Pommerenke's conditions in U_r if:*

i) the function $L(z,t) = F(e^{-t}z, e^t z)$ is analytic in U_r, for all $t \in I$, locally absolutely continuous in I, locally uniform with respect to U_r.

ii) the function $G(e^{-t}z, e^t z)$, where $G(u,v) = \dfrac{u}{v}\dfrac{\partial F}{\partial u} / \dfrac{\partial F}{\partial v}$ is analytic in U_r for all $t \in I$ and has an analytic extension in $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ for all $t > 0$ and in U for $t = 0$. The analytic extension of the function G is denoted by $H = H(e^{-t}z, e^t z)$ and is called the associate function of F.

iii) $\dfrac{\partial F}{\partial v}(0,0) \neq 0$ and $\dfrac{\partial F}{\partial u}(0,0) \Big/ \dfrac{\partial F}{\partial v}(0,0) \notin (-\infty, -1]$.

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iv) the family of functions $\left\{ F(e^{-t}z, e^t z) \Big/ \left[e^{-t}\frac{\partial F}{\partial u}(0,0) + e^t \frac{\partial F}{\partial v}(0,0)\right]\right\}_{t\in I}$ *forms a normal family in* U_r.

We shall need the following theorem to prove our results:

Theorem 1 [5]. *Let* $F : U_r \times \mathbb{C} \mapsto \mathbb{C}$, $F = F(u,v)$ *be a function which satisfies Pommerenke's conditions in* U_r *and let* H *be the associate function of* F. *If*

$$\begin{aligned} &|H(z,z)| < 1, \quad z \in U \text{ and} \\ &|H(z,1/\bar{z})| \le 1, \quad z \in U \setminus \{0\}, \end{aligned} \tag{1}$$

then the function $F(e^{-t}z, e^t z)$ *for all* $t \in I$, *has an analytic and univalent extension in* U.

2. Main results

Theorem 2. *Let* $f, h \in A$ *and* $\alpha \in \mathbb{C}$. *If* $\operatorname{Re}\alpha > 1/2$ *and*

$$\begin{aligned} &\left| \frac{1-\alpha}{\alpha}\left[|z|^2 + \frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right]\left[|z|^2 + \frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)} \right.\right. \\ &\left. + (1-|z|^2)\frac{zf'(z)}{f(z)}\right] + |z|^2(1-|z|^2)\left[\frac{zh''(z)}{h'(z)} + \frac{zf''(z)}{f'(z)}\right] \\ &\left. + \frac{(1-|z|^2)^2}{2}\frac{zh''(z)}{h'(z)}\frac{zf''(z)}{f'(z)} - \frac{(1-|z|^2)^2}{2} z^2 S_h(z)\right| \le |z|^2 \end{aligned} \tag{2}$$

for all $z \in U$, *then* f *is an univalent function in* U.

Remark: We denote by $S_h(z)$ the Schwarz's derivative of the function h, i.e.

$$S_h(z) = \left[\frac{h''(z)}{h'(z)}\right]' - \frac{1}{2}\left[\frac{h''(z)}{h'(z)}\right]^2.$$

Proof: Let $F : U \times \mathbb{C} \mapsto \mathbb{C}$ be the function

$$F(u,v) = [f(u)]^{1-\alpha}\left[f(u) + (v-u)f'(u)\left(1 + \frac{v-u}{2}\frac{h''(u)}{h'(u)}\right)^{-1}\right]^{\alpha}, \tag{3}$$

$(u,v) \in U \times \mathbb{C}$, and let $L : U \times I \mapsto \mathbb{C}$ be the function

$$\begin{aligned} L(z,t) = &F(e^{-t}z, e^t z) = f(e^{-t}z)\left[1 + (e^{2t}-1)\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)}\right. \\ &\left.\left(1 + \frac{e^{2t}-1}{2}\frac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)}\right)^{-1}\right]^{\alpha}, \ (z,t) \in U \times I. \end{aligned} \tag{4}$$

Since, there is $r' \in (0,1)$ such that $f(z) \neq 0$ for all $z \in U_{r'} \setminus \{0\}$, the function $f_1(z,t) = \dfrac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} = 1 + \cdots$ is analytic in $U_{r'}$.

The function $f_2(z,t) = \dfrac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)} = b_1 e^{-t}z + \cdots$ is analytic in U.

Then, there exists $r \in (0,r')$ such that the function $f_3(z,t) = 1 + \dfrac{(e^{2t}-1)f_1(z,t)}{1+2^{-1}(e^{2t}-1)f_2(z,t)} = e^{2t} + \cdots$ is analytic in U_r and $f_3(z,t) \neq 0$ for all $z \in U_r$ and $t \in I$. Hence, for the function $f_4(z,t) = \left[f_3(z,t)\right]^\alpha = e^{2\alpha t} + \cdots$, we can choose an analytic branch in U_r and the function $L(z,t) = f(e^{-t}z)f_4(z,t) = e^{(2\alpha-1)t}z + \cdots$ is analytic in U_r.

Using (4) we obtain

$$\frac{\partial L(z,t)}{\partial t} = -e^{-t}z\frac{\partial F}{\partial u}(e^{-t}z, e^t z) + e^t z\frac{\partial F}{\partial v}(e^{-t}z, e^t z)$$

and we observe that $\left|\dfrac{\partial L(z,t)}{\partial t}\right|$ is bounded on $[0,T]$, for any $T > 0$ fixed and for all $z \in U_r$. Therefore, the function $L(z,t) = F(e^{-t}z, e^t z)$ is locally absolutely continuous in I, locally uniform with respect to U_r.

Since

$$a_1(t) = e^{-t}\frac{\partial F}{\partial u}(0,0) + e^t\frac{\partial F}{\partial v}(0,0) = e^{(2\alpha-1)t}$$

we obtain $a_1(t) \neq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = \lim\limits_{t\to\infty} e^{t\,\mathrm{Re}(2\alpha-1)} = \infty$.

It is easy to prove that there exists $k > 0$ such that $|F(e^{-t}z, e^t z)/a_1(t)| \leq k$ for all $z \in U_r$ and $t \in I$. Hence $\{F(e^{-t}z, e^t z)/a_1(t)\}_{t\in I}$ is a normal family in U_r.

Using (3) we obtain

$$\begin{aligned} G(u,v) = \frac{u}{v}\frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v} = \frac{1-\alpha}{\alpha}\frac{u}{v}&\left[1 + \frac{v-u}{2}\frac{h''(u)}{h'(u)}\right]\left[1 + \frac{v-u}{2}\frac{h''(u)}{h'(u)}\right. \\ &\left. + (v-u)\frac{f'(u)}{f(u)}\right] + \frac{u}{v}(v-u)\left[\frac{h''(u)}{h'(u)} + \frac{f''(u)}{f'(u)}\right] \\ &+ \frac{u}{v}\frac{(v-u)^2}{2}\left[\frac{f''(u)}{f'(u)}\frac{h''(u)}{h'(u)} - S_h(u)\right]. \end{aligned}$$

It results that the function $G(e^{-t}z, e^t z)$ has an analytic extension

$H(e^{-t}z, e^{t}z)$, where

$$H(e^{-t}z,e^{t}z)=\frac{1-\alpha}{2}e^{-2t}\left[1+\frac{e^{2t}-1}{2}\frac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)}\right]\cdot$$
$$\left[1+\frac{e^{2t}-1}{2}\frac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)}+(e^{2t}-1)\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)}\right]$$
$$+e^{-2t}(e^{2t}-1)\left[\frac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)}+\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t})}\right]$$
$$+e^{-2t}\frac{(e^{t}-e^{-t})^2}{2}z^2\left[\frac{h''(e^{-t}z)}{h'(e^{-t}z)}\frac{f''(e^{-t}z)}{f'(e^{-t}z)}-S_h(e^{-t}z)\right].$$

We have:
$|H(z,z)|=|\frac{1-\alpha}{\alpha}|<1$ for all $z\in U$ and $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>1/2$ and

$$|H(z,1/\bar{z})|=\left|\frac{1-\alpha}{\alpha}\frac{1}{|z|^2}\left[|z|^2+\frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right]\left[|z|^2\frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right.\right.$$
$$\left.+(1-|z|^2)\frac{zf'(z)}{f(z)}\right]+(1-|z|^2)\left[\frac{zh''(z)}{h'(z)}+\frac{zf''(z)}{f'(z)}\right]$$
$$\left.+\frac{z}{\bar{z}}\frac{(1-|z|^2)^2}{2}\left[\frac{h''(z)}{h'(z)}\frac{f''(z)}{f'(z)}-S_h(z)\right]\right|\leq 1,$$

for all $z\in U\setminus\{0\}$.

Therefore we can conclude, using Theorem 1, that the function $F(e^{-t}z, e^{t}z)$, $t\in I$ has an analytic and univalent extension $F_1(e^{-t}z,e^{t}z)$ in U for all $t\in I$. In particular, the function $f(z)=F_1(z,z)$, $z\in U$, is an univalent function in U.

3. Remarks

1. For $\alpha=1$ we obtain the following sufficient univalence condition:

Corollary 1. *If $f,h\in A$ and*

$$\left||z|^2(1-|z|^2)\left[\frac{zh''(z)}{h'(z)}+\frac{zf''(z)}{f'(z)}\right]+\frac{(1-|z|^2)^2}{2}\frac{zh''(z)}{h'(z)}\frac{zf''(z)}{f'(z)}\right.$$
$$\left.-\frac{(1-|z|^2)^2}{2}z^2S_h(z)\right|\leq|z|^2, \tag{5}$$

for all $z\in U$, then the function f is univalent in U.

2. For $\alpha\to\infty$ we obtain an another sufficient univalence condition:

Corollary 2. *If $f, h \in A$ and*

$$
\begin{aligned}
&\Big|\frac{(1-|z|^2)^2}{2}\frac{zh''(z)}{h'(z)}\frac{zf''(z)}{f'(z)} - \frac{(1-|z|^2)^2}{2}z^2 S_h(z)\\
(6)\quad &- \left[|z|^2 + \frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right]\left[|z|^2 + \frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)} + (1-|z|^2)\frac{zf'(z)}{f(z)}\right]\\
&+ |z|^2(1-|z|^2)\left[\frac{zh''(z)}{h'(z)} + \frac{zf''(z)}{f'(z)}\right]\Big| \le |z|^2,
\end{aligned}
$$

for all $z \in U$, then the function f is univalent in U.

3. If $h' = 1/f'$ we obtain the following univalence criterion which generalize the generalize the criterion due to Nehari:

Theorem 3 [7]. *Let $f \in A$ and $\alpha \in \mathbb{C}$. If $\operatorname{Re}\alpha > 1/2$ and*

(7)

$$
\begin{aligned}
&\Big|\frac{1-\alpha}{\alpha}\left[|z|^2 - \frac{1-|z|^2}{2}\frac{zf''(z)}{f'(z)}\right]\left[|z|^2 + (1-|z|^2)\left(\frac{zf'(z)}{f(z)} - \frac{1}{2}\frac{zf''(z)}{f'(z)}\right)\right]\\
&+ z^2\frac{(1-|z|^2)^2}{2}S_f(z)\Big| \le |z|^2, \quad z \in U,
\end{aligned}
$$

then f is a univalent function in U.

$\alpha = 1$ in Theorem 3 gives us the Nehari's sufficient univalence condition:

Theorem 4 [4]. *If $f \in A$ and*

$$|S_f(z)| \le 2(1-|z|^2)^{-2}, \quad z \in U,$$

then f is univalent in U.

4. If $h' = g' \cdot (f')^{-1}$ we obtain an univalence condition which generalize the criterion due to Epstein:

Theorem 5 [8]. *Let $f, g \in A$ and $\alpha \in \mathbb{C}$. If $\operatorname{Re}\alpha > 1/2$ and*

$$
\begin{aligned}
&\Big|\frac{1-\alpha}{\alpha}\left[|z|^2 - \frac{1-|z|^2}{2}\left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right)\right.\\
(8)\quad &\left. + (1-|z|^2)\frac{zf'(z)}{f(z)}\right]\left[|z|^2 - \frac{1-|z|^2}{2}\left(\frac{zf''(z)}{f'(z)} + \frac{zg''(z)}{g'(z)}\right)\right]\\
&+ |z|^2(1-|z|^2)\frac{zg''(z)}{g'(z)} + z^2\frac{(1-|z|^2)^2}{2}[S_f(z) - S_g(z)]\Big| \le |z|^2, \, z \in U,
\end{aligned}
$$

then f is a univalent function in U.

$\alpha = 1$ in Theorem 4 gives us the Epstein's sufficient univalence condition:

Theorem 6 [2]. *If $f, g \in A$ and*

$$(9)\qquad \left|\frac{(1-|z|^2)^2}{2}\left[S_f(z)-S_g(z)\right]+(1-|z|^2)\bar{z}\frac{g''(z)}{g'(z)}\right|\le 1,\ z\in U,$$

then f is a univalent function in U.

5. If $h(z)=z$ we obtain the following univalence criterion which generalize the criterion due to Becker:

Theorem 7 [3]. *Let $f \in A$ and $\alpha \in \mathbb{C}$. If $\operatorname{Re}\alpha > 1/2$ and*

$$(10)\qquad \left|\frac{1-\alpha}{\alpha}\left[|z|^2+(1-|z|^2)\frac{zf'(z)}{f(z)}\right]+(1-|z|^2)\frac{zf''(z)}{f'(z)}\right|\le 1,\ z\in U,$$

then the function f is univalent in U.

$\alpha = 1$ in Theorem 7 gives us the Becker's sufficient univalence condition:

Theorem 8 [1]. *If $f \in A$ and*

$$(11)\qquad (1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right|\le 1,\quad z\in U,$$

then the function f is univalent in U.

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