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THE GENERAL CONCEPT OF CLEAVABILITY OF MAPPINGS

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ABSTRACT. *The aim of the paper is to give some answers to the following general question: "If X and Y are topological spaces and $f : X \to Y$ is a continuous mapping cleavable over the class $\mathcal{P}$ of topological spaces, is it true that f is a $\mathcal{P}$-mapping? ". Answers are given for some classes of topological spaces.*

Introduction and preliminary. In 1985 Arhangel'skii ([1], [2]), introduced the notion of cleavability for topological spaces. Following a general idea ([22]) to investigate mappings instead of spaces, in this paper we want to introduce the notion of cleavability for mappings. So, the concept of $\mathcal{P}$-mapping ([14]) is a basic notion. Let $\mathcal{P}$ be a topological property; a continuous mapping is called a *$\mathcal{P}$-mapping* if it satisfies a property $G_{\mathcal{P}}$ depending on $\mathcal{P}$ and every continuous mapping on a $\mathcal{P}$-space has the property $G_{\mathcal{P}}$. We want to study the $\mathcal{P}$-mappings when the property $\mathcal{P}$ is the *cleavability over a class of topological spaces*; in this way we want to obtain a more general notion of cleavability of mappings over a class of spaces as a generalization of the notion of cleavability of a space over the same class of spaces.

In particular we are interested in answering the following question: "If $f : X \to Y$ is a continuous mapping *cleavable* over a class $\mathcal{P}$ of topological spaces, is it true that f is a $\mathcal{P}$-mapping?"In this paper we shall use the following notations: (X, τ) or simply X means a topological space; $\overline{A}$, A^o are the closure and the interior of A respectively, where A is a subset of X; if $\overline{A}^o = A$ ($\overline{A^o} = A$) we say that A is a regular open (regular closed) subset of X; $C(X,Y)$ is the set of all continuous mappings from X to Y, where Y is

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a topological space. For notations not explicitly mentioned here, the reader is referred to [6], [15] and [19].

Let $\mathcal{P}$ be a class of topological spaces and $\mathcal{M}$ a class of continuous mappings. We recall the following

Definition 1. **[1].** *A space X is $\mathcal{M}$-cleavable over $\mathcal{P}$ if for every $A \subset X$ there exist $Y \in \mathcal{P}$ and $f \in \mathcal{M}$, $f : X \to Y$, such that $A = f^{-1}f(A)$ (or equivalently $f(A) \cap f(X - A) = \varnothing$).*

If $\mathcal{M}$ is the class of all continuous mappings, we shall just say that X is *cleavable over $\mathcal{P}$*. If $\mathcal{M}$ is the class of all open, closed, perfect, quotient mappings, we shall say that X is respectively *open, closed, perfect, quotient cleavable over $\mathcal{P}$*.

Remark 1 Let f be a one-to-one continous mapping of a space X into a space $Y \in \mathcal{P}$. Then obviously X is cleavable over $\mathcal{P}$. Note, that in the definition of cleavability the mapping f depends on the subset A of X. Thus we might say that a space X is said to be *absolutely cleavable* over $\mathcal{P}$ if there exists a one-to-one continuous mapping of X into some space $Y \in \mathcal{P}$ ([5]). Then cleavability over $\mathcal{P}$ may be regarded as a generalization of continuous bijections (onto some $Y \in \mathcal{P}$).

Definition 2. **[6].** *A space X is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ if for every point $x \in X$, there exist $Y \in \mathcal{P}$ and $f \in \mathcal{M}$, $f : X \to Y$, where such that $\{x\} = f^{-1}f(x)$.*

Definition 3. **[6].** *A space X is $\mathcal{M}$-double cleavable over $\mathcal{P}$ if for any subsets A and B of X, there exist $Y \in \mathcal{P}$ and $f \in \mathcal{M}$, $f : X \to Y$, such that $A = f^{-1}f(A)$ and $B = f^{-1}f(B)$.*

Remark 2 If X is absolutely cleavable over $\mathcal{P}$, then X is double cleavable over $\mathcal{P}$; if X is double cleavable over $\mathcal{P}$, then X is cleavable over $\mathcal{P}$; moreover, if a space X is cleavable over $\mathcal{P}$, then X is pointwise cleavable over $\mathcal{P}$.

Then we can give the following definitions for the cleavability of a mapping.

Definition 4. *A continuous mapping $f : X \to Y$ is $\mathcal{M}$-cleavable over $\mathcal{P}$ if for every $y \in Y$ and $A \subset f^{-1}(y)$ there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$, such that $A = g^{-1}g(A)$.*

Remark 3 The previous definition is not trivial if f is onto.

If $\mathcal{M}$ is the class of all continuous mappings, we shall just say that f is *cleavable over $\mathcal{P}$*. If $\mathcal{M}$ is the class of all open, closed, perfect, quotient mappings, we shall say that f is respectively *open, closed, perfect, quotient cleavable over $\mathcal{P}$*.

Further f is said to be *absolutely cleavable* over $\mathcal{P}$ if the mapping g is one-to-one.

Definition 5. *A continuous mapping $f : X \to Y$ is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ if for every $y \in Y$ and $\{x\} \subset f^{-1}(y)$, there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$ such that $\{x\} = g^{-1}g(x)$.*

Remark 4 The previous definition is equivalent to the definition of pointwise cleavability of X over $\mathcal{P}$.

Definition 6. *A continuous mapping $f : X \to Y$ is $\mathcal{M}$-double cleavable over $\mathcal{P}$ if for every $y \in Y$ and for every subset A and B of $f^{-1}(y)$, there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$ such that $A = g^{-1}g(A)$ and $B = g^{-1}g(B)$.*

Remark 5 If $f : X \to Y$ is absolutely cleavable over $\mathcal{P}$, then f is double cleavable over $\mathcal{P}$; if f is double cleavable over $\mathcal{P}$, then f is cleavable over $\mathcal{P}$; moreover, if f is cleavable over $\mathcal{P}$, then f is pointwise cleavable over $\mathcal{P}$.

We have

Proposition 1. *A space X is $\mathcal{M}$-cleavable ($\mathcal{M}$-pointwise cleavable, ...) over $\mathcal{P}$ iff every continuous mapping $f : X \to Y$ is $\mathcal{M}$-cleavable ($\mathcal{M}$-pontwise cleavable, ...) over $\mathcal{P}$.*

Proof. ($\Rightarrow$) Let $f : X \to Y$ be a continuous mapping, $y \in Y$ and $A \subset f^{-1}(y)$. As X is $\mathcal{M}$-cleavable over $\mathcal{P}$, then there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$ such that $g^{-1}g(A) = A$; this proves that f is $\mathcal{M}$-cleavable over $\mathcal{P}$. ($\Leftarrow$) Now suppose that every continuous mapping with domain X is $\mathcal{M}$-cleavable over $\mathcal{P}$. Let $A \subset X$ and let $Y = (Y, \tau)$ be Sierpinski's 2-point space (i.e., $Y = \{0,1\}$ and $\tau = \{\varnothing, Y, \{1\}\}$. Define $f : X \to Y$ by $f(\overline{A}) = \{0\}$, $f(X - \overline{A}) = \{1\}$; then f is continuous. Since $A \subset f^{-1}(0)$ and f is a $\mathcal{M}$-cleavable mapping, there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$ such that $g^{-1}g(A) = A$. Thus X is $\mathcal{M}$-cleavable over $\mathcal{P}$. □

So we have the following natural question

Question - A. *Does there exist a continuous mapping f that is $\mathcal{M}$-cleavable over $\mathcal{P}$ such that its domain X is not $\mathcal{M}$-cleavable over $\mathcal{P}$?*

We have the following

Proposition 2. *A space X is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ iff every continuous one-to-one mapping $f : X \to Y$ is $\mathcal{M}$-cleavable over $\mathcal{P}$.*

Proof. ($\Rightarrow$) Let $f : X \to Y$ be a continuous one-to-one mapping. Then, for every $y \in Y$ the fiber $f^{-1}(y)$ is a single point of X. So, if X is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ we have that f is $\mathcal{M}$-cleavable over $\mathcal{P}$. ($\Leftarrow$) Now suppose that every continuous one-to-one continuous mapping with domain X is $\mathcal{M}$-cleavable over $\mathcal{P}$. Let $x \in X$. By hypothesis, the identity mapping on X, id_X, is $\mathcal{M}$-cleavable over $\mathcal{P}$; since $\{x\} = id_X^{-1}(x)$, X is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$. □

Note that if a space X is $\mathcal{M}$-pointwise cleavable but not $\mathcal{M}$-cleavable over $\mathcal{P}$, then the identity mapping on X, id_X, is $\mathcal{M}$-cleavable over $\mathcal{P}$; this shows that the notion of cleavability of a mapping is more general than the notion of cleavability of a space, in fact there exist mappings $f : X \to Y$ $\mathcal{M}$-cleavable over $\mathcal{P}$ such that X is not $\mathcal{M}$-cleavable over $\mathcal{P}$. Then we have an affirmative answer to the question A as the following example show

Example 1. *If $\mathcal{P} = \{\mathbb{R}\}$, the circumference S^1 is not cleavable over $\mathcal{P}$ ([4]) while the mapping $id : S^1 \to S^1$ is cleavable over $\mathcal{P}$.* □

Now we have the following natural question

Question - B. *Does there exist a continuous mapping f that is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ such that its domain is not $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$?*

By the definitions, the answer to the previous question is the following: "A continuous mapping $f : X \to Y$ is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ iff X is pointwise cleavable over $\mathcal{P}$".

Some particular forms of cleavability of mappings imply particular forms of cleavability of spaces, as show the following four results

Proposition 3. *A constant mapping $f : X \to Y$ is $\mathcal{M}$-cleavable ($\mathcal{M}$-pointwise cleavable, ...) over $\mathcal{P}$ iff X is $\mathcal{M}$-cleavable ($\mathcal{M}$-pointwise cleavable, ...) over $\mathcal{P}$.*

Proposition 4. *If $f : X \to Y$ is cleavable over $\mathcal{P}$, where $\mathcal{P}$ is a $card(Y)$-productive class of spaces, then X is cleavable over $\mathcal{P}$.*

Proof. Let $A \subset X$ and $y \in f(A)$. By hypothesis, there exist a space $Z_y \in \mathcal{P}$ and a continuous mapping $g_y : X \to Z_y$ such that $g_y^{-1} g_y(A \cap f^{-1}(y)) = A \cap f^{-1}(y)$. Let $Z = \prod_{y \in f(A)} Z_y$; then, by hypothesis, $Z \in \mathcal{P}$. Define a mapping $g : X \to Z$, by $g(x) = \{g_y(x)\}_{y \in f(A)}$, for all $x \in X$. We will show that $g^{-1}g(A \cap f^{-1}(y)) = A \cap f^{-1}(y)$. Only need to show that $g^{-1}g(A \cap f^{-1}(y)) \subseteq A \cap f^{-1}(y)$. Let $x \in g^{-1}g(A \cap f^{-1}(y)$; so, $g(x) \in g(A \cap f^{-1}(y))$. Then, there exists $a \in A \cap f^{-1}(y)$ such that $g(x) = g(a)$; in particular, $f(a) = y$. Then, for every $z \in f(A)$, we have that $g_z(x) = g_z(a)$. So $x = g_z^{-1} g_z(a)$, for all $z \in f(A)$, and then, by hypothesis, $x \in A \cap f^{-1}(y)$. Thus $g^{-1}g(A) = A$. □

Remark 6 In the case in which $\mathcal{P}$ is a $card(Y)$-productive class of spaces, the previous property gives a negative answer to the question A.

Definition 7. *If f is a mapping from the space X to a space Y, the cardinality of f is defined as the number*

$$card(f) = card(f(X)) \times Sup\{card(f^{-1}(y)) : y \in Y\}.$$

Proposition 5. *If $f : X \to Y$ is pointwise cleavable over $\mathcal{P}$, where $\mathcal{P}$ is a $card(f)$-productive class of spaces, then X is absolutely cleavable over $\mathcal{P}$.*

Proof. Let $y \in Y$ and $x \in f^{-1}(y)$; then there exist a space $Z_x \in \mathcal{P}$ and a continuous mapping $g_x : X \to Z_x$ such that $\{x\} = g_x^{-1} g_x(x)$. Let $Z_y = \prod_{x \in f^{-1}(y)} Z_x$; by hypothesis, $Z_y \in \mathcal{P}$. Define the mapping $g_y : X \to Z_y$, by $g_y(z) = \{g_x(z)\}_{x \in f^{-1}(y)}$ for all $z \in X$. The mapping g_y is continuous: recall that g_y is continuous iff $p_s g_y$ is continuous, for $s \in f^{-1}(y)$, where $p_s : \prod_{x \in f^{-1}(y)} Z_x \to Z_s$ is the s^{th} projection mapping; since $p_s g_s(t) = g_s(t)$ for all $t \in X$, we have that $p_s g_y$ is a continuous mapping. Further $g_y | f^{-1}(y) : f^{-1}(y) \to Z_y$ is one-to-one: let $s, t \in f^{-1}(y)$ such that $s \neq t$. By hypothesis, $g_s(t) \neq g_s(s)$; then $\{g_x(s)\}_{x \in f^{-1}(y)} \neq \{g_x(t)\}_{x \in f^{-1}(y)}$, or equivalentely, $g_y | f^{-1}(s) \neq g_y | f^{-1}(t)$. Let $Z = \prod_{y \in f(X)} Z_y$; by hypothesis, $Z \in \mathcal{P}$. Define the mapping $g : X \to Z$, by $g(z) = \{\{g_x(z)\}_{x \in f^{-1}(y)}\}_{y \in f(X)}$. The mapping g is continuous: let $p_t : \prod_{y \in f(X)} Z_y \to Z_t$ the t^{th} projection mapping (recall that $Z_t = \prod_{x \in f^{-1}(t)} Z_x$); since $p_t g(s) = g_t(s)$, for all $s \in X$ and we have proved that g_t is continuous for all $t \in Y$, we have that $p_t g$ is continuous for all $t \in f(X)$ and then g is continuous. Since $g_y | f^{-1}(y) : f^{-1}(y) \to Z_y$ is one-to-one, for all $y \in Y$, we have that g is one-to-one. Then X is absolutely cleavable over $\mathcal{P}$. □

Remark 7 In the case in which $\mathcal{P}$ is a $card(f)$-productive class of spaces, the previous property gives a negative answer to the question A.

Proposition 6. *If $f : X \to Y$ is closed pointwise cleavable over $\mathcal{P}$, where $\mathcal{P}$ is a $card(f)$-productive class of spaces, then X can be embedded as subspace into some space of $\mathcal{P}$.*

Proof. The proof is similar to the proof of Proposition 5 noting that, by hypothesis, every continuous mapping $g_x : X \to Z_y$ is closed and then $g : X \to g(X)$ is a closed mapping. Now we prove this fact. Let $A \subset X$ be closed. We want to prove that $g(A) = \prod_{y \in f(X)} \prod_{x \in f^{-1}(y)} g_x(A) \cap g(X)$, where $\prod_{y \in f(X)} \prod_{x \in f^{-1}(y)} g_x(A)$ is a closed subset of Z. The inclusion $g(A) \subseteq$

$\prod_{y\in f(X)} \prod_{x\in f^{-1}(y)} g_x(A) \cap g(X)$ is obvious. Let $t \in \prod_{y\in f(X)} \prod_{x\in f^{-1}(y)} g_x(A) \cap g(X)$ and $s \in X$ such that $g(s) = t$. Then $g_y(s) = \{g_x(s)\}_{x\in f^{-1}(y)} \in \prod_{x\in f^{-1}(y)} g_x(A)$, for all $y \in f(X)$. Let $\overline{y} = f(s)$. Then $g_x(s) \in g_x(A)$, for all $x \in f^{-1}(\overline{y})$. Since $s \in f^{-1}(\overline{y})$, we have that $g_s(s) \in g_s(A)$; so, there exists $a \in A$ such that $g_s(s) = g_s(a)$. Then, by hypothesis, $s \in A$ and the proof is complete. □

Remark 9 If $\mathcal{P}$ is a $card(f)$-productive and hereditary class of spaces, the previous property is equivalent to say that if X is pointwise cleavable over $\mathcal{P}$, then X is closed absolutely cleavable over $\mathcal{P}$.

Remark 10 In the following we will use the terms *e-cleavable mapping* or *e-cleavable space* over $\mathcal{P}$ to indicate that cleavability, pointwise cleavability, double cleavability and absolute cleavability of a mapping or of a space over $\mathcal{P}$ are equivalent.

By Propositions 5 and 6 we have the following:

Theorem 1. *Let $f : X \to Y$ be a continuous mapping and let $\mathcal{P}$ be a $card(f)$-productive class of spaces. The following conditions are equivalent:*
(i) f is e-cleavable over $\mathcal{P}$;
(ii) X is e-cleavable over $\mathcal{P}$;

Theorem 2. *Let $f : X \to Y$ be a continuous mapping and let $\mathcal{P}$ be a $card(f)$-productive and hereditary class of spaces. The following conditions are equivalent:*
(i) f is closed e-cleavable over $\mathcal{P}$;
(ii) X is closed e-cleavable over $\mathcal{P}$.

1. Cleavability over T_0, T_1, T_2, functionally Hausdorff and Urysohn spaces.

Note that, by the previous results, in the case in which $\mathcal{P}$ is a productive class of spaces, we have that the classic problem on cleavability: "If X is (closed) e-cleavable over the class $\mathcal{P}$, is it true that X belongs to $\mathcal{P}$?", can be reformulated in the following way:"If $f : X \to Y$ is (closed) e-cleavable over $\mathcal{P}$, is it true that $X \in \mathcal{P}$?". Further, in the case in which the answer is affirmative, the mapping f is a $\mathcal{P}$-mapping.

Following [15], we give

Definition 1.1. *A class $\mathcal{P}$ of topological spaces is said to be expansive if the existence of a continuous bijection $f : Y \to X$ from a space Y onto a space $X \in \mathcal{P}$ implies $Y \in \mathcal{P}$.*

By Corollary 1.1 in [10] in the case in which $\mathcal{P}$ is a productive, hereditary and expansive class of spaces, we have that if $f : X \to Y$ is e-cleavable over $\mathcal{P}$, then $X \in \mathcal{P}$, and f is a $\mathcal{P}$-mappings. In particular, the previous result is true for the classes $\mathcal{P}$ of T_0, T_1, T_2, *functionally Hausdorff* or *Urysohn* spaces. Recall the definitions of $\mathcal{P}$-mapping in these cases.

Definition 1.2 [14].

- *$f \in C(X,Y)$ is T_0 if for every pair of distinct points $x, y \in X$ such that $f(x) = f(y)$, there exists some neighbourhood U of x which not contains y or some neighborhood V of y which not contains x;*
- *$f \in C(X,Y)$ is T_1 if for every pair of distinct points $x, y \in X$ such that $f(x) = f(y)$, there exist two neighbourhoods U and V of x and y respectively, such that U does not contains y and V does not contains x;*
- *$f \in C(X,Y)$ is T_2 if for every pair of distinct points $x, y \in X$ such that $f(x) = f(y)$, there exist two disjoint open neighbourhoods U and V of x and y respectively;*
- *$f \in C(X,Y)$ is Urysohn if for every pair of distinct points x, y such that $f(x) = f(y)$, there exist a neighbourhood W of $f(x)$ and two open subsets U, V of $f^{-1}(W)$ such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \varnothing$, where the closures are in $f^{-1}(W)$.*

Further we give the following:

Definition 1.3.

- *$f \in C(X,Y)$ is functionally Hausdorff if for every pair of distinct points $x, y \in X$ such that $f(x) = f(y)$, there exists a continuous mapping $g : X \to [0,1]$ such that $g(x) = 0$ and $g(y) = 1$;*

2. Cleavability over regular, completely regular, semiregular and almost regular spaces.

Now we consider the classes of *regular* and *completely regular spaces.*

Definition 2.1 [14].

- *$f \in C(X,Y)$ is regular if for every point $x \in X$ and every closed $C \subset X$ such that $x \notin C$ there exist an open neighbourhood W of $f(x)$ and two open subsets U, V of $f^{-1}(W)$ such that $x \in U$, $C \cap f^{-1}(W) \subset V$ and $U \cap V = \varnothing$.*

Further we give the following

Definition 2.2.

- *$f \in C(X,Y)$ is completely regular if for every point $x \in X$ and every closed $C \subset X$ such that $x \notin C$ and $f(x) \in f(C)$, there exists a continuous mapping $g : X \to [0,1]$ such that $g(x) = 0$ and $g(C) = \{1\}$.*

Note that we can not consider the previous remarks for the classes of regular and completely regular spaces because they are not expansive. However, e-cleavability of a mapping $f : X \to Y$ over the class $\mathcal{P}$ of regular or completely regular spaces does not imply that X belongs to $\mathcal{P}$, and, in particular, that f is a $\mathcal{P}$-mapping; in fact there exists the following

Example 2. *Let τ^* be a topology on $\mathbb{R}$ generated by adding to the natural topology τ on the real line the set of rational numbers. $(\mathbb{R}, \tau)$ is regular (completely regular) while $(\mathbb{R}, \tau^*)$ is not regular (completely regular). Since $id : (\mathbb{R}, \tau^*) \to (\mathbb{R}, \tau)$ is a continuous bijection, $(\mathbb{R}, \tau^*)$ is absolutely cleavable over the class $\mathcal{P}$ of regular (completely regular) spaces; so id is absolutely cleavable over $\mathcal{P}$. However id is not regular (completely regular); in fact if id would be regular (completely regular), then $(\mathbb{R}, \tau^*)$ would be regular (completely regular), a contadiction.* □

By Corollary 1.3 in [10] in the case in which $\mathcal{P}$ is a productive and hereditary class of spaces, we have that if $f : X \to Y$ is closed e-cleavable over $\mathcal{P}$, then $X \in \mathcal{P}$, and f is a $\mathcal{P}$-mapping. In particular, the previous result is true for the classes $\mathcal{P}$ of regular or completely regular spaces.

Now we consider the classes of *semiregular* ([23]) and *almost regular* ([24]) *spaces*.

Definition 2.3 [14].

- *$f \in C(X, Y)$ is semiregular if for every open $A \subset X$ and every point $x \in A$ there exist an open neighbourhood W of $f(x)$ and a regular open subset R of $f^{-1}(W)$ such that $x \in R \subset (A \cap f^{-1}(W))$.*
- *$f \in C(X, Y)$ is almost regular if for every point $x \in X$ and every regular closed $C \subset X$ such that $x \notin C$ and $f(x) \in f(C)$, there exist an open neighbourhood W of $f(x)$ and two disjoint open subsets U, V of $f^{-1}(W)$ such that $x \in U$, $C \subset V$.*

Since every space can be embedded as a closed subspace into a semiregular space ([16]), every space is e-cleavable over the class of semiregular spaces and then *every continuous mapping is e-cleavable* over that class of spaces. Note that the classes of semiregular and almost regular space are productive but not hereditary, so we can not consider the previous remarks for these classes of spaces. However, the closed e-cleavability of a mapping $f : X \to Y$ over the classes $\mathcal{P}$ of semiregular or almost regular spaces does not imply that $X \in \mathcal{P}$ and, in particular, that f is a $\mathcal{P}$-mapping. In fact for the class of semiregular spaces we can consider Example 2 noting that the mapping *id* is closed, while for the class of almost regular we have the following

Example 3. *Let τ^{**} be a topology on $\mathbb{R}$ generated by adding to the natural topology τ on the real line the sets $\mathbb{Q}_1$ and $\mathbb{Q}_2$ such that $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ is a par-*

tition of $\mathbb{Q}$. *By Example 4 in [10], we have that* $(\mathbb{R}, \tau^{**})$ *is absolutely closed cleavable over the class* $\mathcal{P}$ *of almost regular spaces, but it does not belongs to* $\mathcal{P}$. *Then every constant mapping* f *on* $(\mathbb{R}, \tau^{**})$ *is absolutely closed cleavable over the class* $\mathcal{P}$ *but it is not almost regular; in fact if* f *would be almost regular then* $(\mathbb{R}, \tau^{**})$ *would be almost regular, a contradiction.* □

3. Cleavability over *H-closed* spaces.

Now we consider the class of *H-closed spaces* (see [23],[15]).

Definition 3.1 [12].

- *Let* X, Y, Z, W *be spaces and* $f : X \to Y$ *and* $g : Z \to W$ *be continuous mappings.* f *is said to be embedded in* g *if* $Y = W$, X *is a subspace of* Z *and the restriction* $g|X$ *is equal to* f.
- *A mapping* $f : X \to Y$ *is called H-closed if it is a Hausdorff mapping and for every embedding of* f *into a Hausdorff mapping* $g : Z \to Y$, X *is closed in* Z.

We will need the following known result

Proposition 3.1. *Every Hausdorff space can be embedded as a closed subspace into a H-closed space.*

Theorem 3.1. *Let* $\mathcal{H}$ *be the class of H-closed spaces and let* $f : X \to Y$ *be a contonuous mapping. The following conditions are equivalent*

(1) X *is e-cleavable over* $\mathcal{H}$;
(2) f *is e-cleavable over* $\mathcal{H}$;
(3) X *is Hausdorff;*
(4) X *is closed absolutely cleavable over* $\mathcal{H}$;
(5) X *is closed double cleavable over* $\mathcal{H}$;
(6) X *is closed cleavable over* $\mathcal{H}$;
(7) X *is closed pointwise cleavable over* $\mathcal{H}$;
(8) f *is closed absolutely cleavable over* $\mathcal{H}$;
(9) f *is closed double cleavable over* $\mathcal{H}$;
(10) f *is closed cleavable over* $\mathcal{H}$;
(11) f *is closed pointwise cleavable over* $\mathcal{H}$.

Proof. The equivalence (1)⇔(2) follows by Theorem 1 . Now we prove that (1)⇔(3). Let $\mathcal{P}$ the class of Hausdorff spaces and suppose that X is e-cleavable over $\mathcal{H}$. Since $\mathcal{H} \subset \mathcal{P}$, X is e-cleavable over $\mathcal{P}$; then, by Corollary 1.2 in [10], $X \in \mathcal{P}$. Now suppose that X is Hausdorff; then , by Proposition 3.1, X can be embedded as a closed subspace into a H-closed space, that is X is closed absolutely cleavable over $\mathcal{H}$ and then X is absolutely cleavable over $\mathcal{H}$. Now we prove the equicalences (3)-(6). By Proposition 3.1, (3)⇒(4);

the implications (4)⇒(5)⇒(6)⇒(7) are obvious. Further, (7) implies that X is pointwise cleavable over $\mathcal{H}$ and then, by the equivalence (1)⇔(3), X is Hausdorff. Now we prove the equivalences (7)-(11). We know that (7)⇔(4) and the implications (4)⇒(9)⇒(10)⇒(11) are obvious. But (11) implies that X is closed pointwise cleavable over $\mathcal{H}$, so the proof is complete. □

Note that the class $\mathcal{H}$ is productive but not hereditary, so we can not consider the previous remarks for that class of spaces. However, the closed e-cleavability of a mapping $f : X \to Y$ over the class $\mathcal{H}$ does not imply that $X \in \mathcal{H}$ and, in particular, that f is a $\mathcal{H}$-mapping. In fact there exists the following

Example 4. *Let X be an Hausdorff but not an H-closed space. Then X can be embedded as a closed subspace into a H-closed space, that is X is closed absolutely cleavable over $\mathcal{H}$. So by Proposition 3, every continuous and constant mapping f on X is closed absolutely cleavable over $\mathcal{H}$. However, f is not an H-closed mapping, because otherwise we would have that $X \in \mathcal{H}$, a contradiction.* □

4. Open questions.

Note that all the classes of spaces we have considered are productive.

Question - 1. *Do there exist not-productive classes $\mathcal{P}$ of spaces such that the cleavability of a mapping $f : X \to Y$ over $\mathcal{P}$ is equivalent to the cleavability of the space X over $\mathcal{P}$?.*

Note that a metrizable separable space need not be cleavable over $\mathcal{P} = \{\mathbb{R}\}$; so we have the following natural question:"Does there exist a space Y and a continuous mapping $f : X \to Y$ such that X is a metrizable separable space and f is cleavable over $\mathcal{P}$?". However it is known that every metrizable space X is pointwise cleavable over $\{\mathbb{R}\}$ or, equivalently, if X is a metrizable space, then the mapping id_X is cleavable over $\{\mathbb{R}\}$.

Question - 2. *What classic results about cleavability of spaces can be generalized to cleavability of mapping?*

References

[1] A.V.Arhangel'skii, *A general conception of splittability of topological spaces*, Abstracts Tirasp, Symp. (1985), Štiinca Kišinev, (1985), 8-10 (in Russian).

[2] A.V.Arhangel'skii, *Some new trends in the theory of continuous mappings*, In: Continuous functions or topological spaces, LGU, Riga (1986), 5-35 (in Russian).

[3] A.V.Arhangel'skii, *The general concept of cleavability of a topologiacal space*, Topology Appl. 44(1992), 27-36.

[4] A.V.Arhangel'skii, *Cleavability over reals*, Topology Appl. 48(1992), 163-178.

[5] A.V.Arhangel'skii, *A survey on cleavability*, to appear.

[6] A.V.Arhangel'skii, F.Cammaroto, *On different types of cleavability of topological spaces: pointwise, closed, open and pseudoopen*, J. Austral. Math. Soc (Ser.A) 57(1995), 1-17.

[7] A.V.Arhangel'skii, D.B.Shakhmatov, *On pointwise approximation of arbitrary functions by countable collections of continuous functions*, Journ. of Soviet Math, 50(2) (1990), 1497-1511. (Translated from: Trudy Sem. I.G.Petrovskogo 13(1988), 206-227).

[8] A.Bella, F.Cammaroto, Lj.Kočinac, *Remarks on splittability of topological spaces*, Q and A in General Topology 9(1991), 89-99.

[9] M.Bonanzinga, *On cleavability over relative spaces*, Q and A in General Topology 12(1994), 185-192.

[10] M.Bonanzinga, F.Cammaroto, *On cleavability over $T_{i,\rho}$ spaces*, submitted.

[11] F.Cammaroto, *On splittability of topological spaces*, Proc. VI Brasilian Topological Meeting (1990) (to apprear).

[12] F.Cammaroto, V.V.Fedorchuk, J.R.Porter, *On H-closed extension of Hausdorff mappings*, submitted.

[13] F.Cammaroto, Lj.Kočinac, *Developable spaces and cleavability*, Rendiconti di Matem., Serie VII, 1(1995), 1-17.

[14] F.Cammaroto, G.Nordo, *On Urysohn, almost regular and semiregular functions*, Filomat 8(1994).

[15] R.Engelking, *General Topology*, PWN, Warszawa, 1977.

[16] M.Katětov, *On H-closed exstensions of topological spaces*, Časopis Pěst. Mat. Fys. 72(1947), 17-32.

[17] Lj.Kočinac, *Metrizability and cardinal invariants using splittability* , Comt. Rend. Acad. Bulg. Sci. 43(1990), 9-12.

[18] Lj.Kočinac, *Cleavability and divisibility of topological spaces*, Atti Accademia Pelor. "dei Pericolanti" 70(1992), 1-16.

[19] Lj.Kočinac, F.Cammaroto, A.Bella, *Some results on splittability of topological spaces*, Atti Accademia Pelor. "dei Pericolanti" 68(1990), 41-60.

[20] G.Marchisa, *Su una generalizzazione degli assiomi di separazione T_2, T_3, T_4, T_5*, Rend. Sem. Mat. Univ. e Politec. Torino (1974-75), 33(1976), 103-106.

[21] B.A.Pasynkov, *The dimension and geometry of mappings*, Doul. Akad. Nauk SSSR 221(1975), 543-546 = Soviet Math. Dokl., 16(1975), 384-388.

[22] B.A.Pasynkov, *On extension onto mappings of some notions and statements about space*, (in Russian) in Mappings and functors, IZD. Moskow Univ., Moskow W (1984), 72-102.

[23] J.R.Porter, R.G.Woods, *Extensions and absolutes of Hausdorff spaces*, Springer-Verlag, 1988.

[24] M.K.Singal, S.P.Arya, *On almost regular spaces*, Glasnik Mat. 4 (24) (1969), 89-99.

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