

ON PROPERTIES OF SOME NONCLASSICAL
ORTHOGONAL POLYNOMIALS

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ABSTRACT. *In this paper we consider some sequences of nonclassical orthogonal polynomials which were studied in the papers [3], [4] and [6]. We find some new relations which they satisfy and discuss their zeros.*

1. Polynomials of the Laguerre type

We consider the generalized Laguerre functional

$$I^{(s)}(P) = \int_0^{+\infty} P(x)x^s e^{-x} dx, \quad s \in \mathbb{N}_0, \quad I^{(0)} = I,$$

and the monic generalized Laguerre polynomials $\{\widehat{L}_n^{(s)}(x)\}$, which satisfy the following three-term recurrence relation

$$(1.1) \quad \begin{aligned} \widehat{L}_{n+1}^{(s)}(x) &= (x - 2n - s - 1)\widehat{L}_n^{(s)}(x) - n(n + s)\widehat{L}_{n-1}^{(s)}(x), \\ \widehat{L}_{-1}^{(s)}(x) &= 0, \quad \widehat{L}_0^{(s)}(x) = 1. \end{aligned}$$

These polynomials can be expressed in the form

$$(1.2) \quad \widehat{L}_n^{(s)}(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (k + s + 1)_{n-k} x^k,$$

where

$$(m)_s = m(m + 1) \cdots (m + s - 1), \quad (m)_0 = 1.$$

We introduce the functional Δ_a by

$$\Delta_a(f) = f(a),$$

Received 22.07.1995

1991 *Mathematics Subject Classification*: 33C45

Supported by Grant 0401A of RFNS through Math. Inst. SANU

and define the functional I_1 by

$$I_1 = I + c\Delta_0, \quad c \in \mathbb{R}.$$

By $\{L_n(x; c)\}$ we denote the corresponding sequence of orthogonal polynomials of the *Laguerre type*. Such polynomials were expressed as a linear combination of $L_n^{(0)}(x)$ and $xL_n^{(2)}(x)$ (see [6]). In the same paper, it was proved that the n -th polynomial of this sequence can be expressed in the form

$$(1.3) \quad L_n(x; c) = (-1)^n n! \sum_{k=0}^n (-1)^k \frac{1 + k(1 + c(n+1))}{(k+1)!} \binom{n}{k} x^k.$$

In this section we prove some properties of these polynomials.

Denoting the monic polynomials of the Laguerre type by $\widehat{L}_n(x; c)$, we yield

$$(1.3') \quad L_n(x; c) = (1 + nc)\widehat{L}_n(x; c), \quad c \neq -\frac{1}{n}, \quad n \in \mathbb{N}.$$

From (1.3) we see that

$$\deg L_n(x; -1/n) = n - 1 \quad \text{and} \quad L_n(x; -1/n) = -nL_{n-1}(x; -1/n).$$

Hence, for each $n \in \mathbb{N}$, the sequences $\{\widehat{L}_\nu(x; -1/n)\}_{\nu=0}^\infty$ are *quasi-orthogonal* of the order one with respect to I_1 , with $c = -1/n$, and the next recurrence relations are not valid for $L_n(x; -1/n)$ and its neighbours.

Theorem 1.1. *The polynomial $\widehat{L}_n(x; c)$ can be expressed in the form*

$$(1.4) \quad \widehat{L}_n(x; c) = \widehat{L}_n^{(1)}(x) + \lambda_n \widehat{L}_{n-1}^{(1)}(x),$$

where

$$\lambda_n = n \frac{1 + (n+1)c}{1 + nc},$$

i.e., $\{\widehat{L}_n(x; c)\}$ is *quasi-orthogonal* of the order one with respect to the functional $I^{(1)}$.

Proof. Suppose that λ_n exists such that (1.4) is true. Using (1.2) and (1.3)–(1.3'), for the coefficient of the term x^k we yield

$$\frac{1 + k(1 + c(n+1))}{1 + nc} \cdot \frac{n}{n-k} = \frac{n(n+1)}{n-k} - \lambda_n.$$

After some computation we find λ_n which does not depend on k . \square

Using the previous relations we can prove the three-term recurrence relation for polynomials $\widehat{L}_n(x; c)$:

Theorem 1.2. *The sequence $\{\widehat{L}_n(x; c)\}$ satisfies the three-term recurrence relation*

$$\widehat{L}_{n+1}(x; c) = (x - \alpha_n)\widehat{L}_n(x; c) - \beta_n\widehat{L}_{n-1}(x; c),$$

where

$$\alpha_n = \frac{2n + 1 + 4n(n + 1)c + n(n + 1)(2n + 1)c^2}{(1 + nc)(1 + (n + 1)c)},$$

$$\beta_n = n^2 \frac{(1 + (n - 1)c)(1 + (n + 1)c)}{(1 + nc)^2}.$$

The norm of $\widehat{L}_n(x; c)$ is given by

$$\|\widehat{L}_n(x; c)\|^2 = \beta_0\beta_1 \cdots \beta_n = (n!)^2 \frac{1 + (n + 1)c}{1 + nc}.$$

Theorem 1.3. *The zeros of $\widehat{L}_n(x; c)$ are real, simple and positive, except for $c < -1/n$ when one of them is negative. Denoting these zeros by*

$$\zeta_{n1} < \zeta_{n2} < \cdots < \zeta_{nn},$$

we have, for $c < -2/(n + 1)$, that the lowest zero ζ_{n1} satisfies

$$(1.5) \quad -\frac{\zeta_{nn}}{n - 1} < \zeta_{n1} < 0 \quad \wedge \quad |\zeta_{n1}| < |\zeta_{n2}|.$$

Proof. Let ζ_{nj} , $j = k, \dots, n$, be positive zeros of odd multiplicity. Defining $q(x) = \prod_{j=k}^n (x - \zeta_{nj})$, we see that the polynomial $xq(x)\widehat{L}_n(x; c)$ does not change sign for $x > 0$. Hence

$$I_1(xq(x)\widehat{L}_n(x; c)) = \int_0^{+\infty} xq(x)\widehat{L}_n(x; c)e^{-x} dx \neq 0.$$

Since $\{\widehat{L}_n(x; c)\}$ is quasi-orthogonal of the order one, with respect to the functional $I^{(1)}$, it follows $\deg q(x) \geq n - 1$.

From (1.3) we have that $\widehat{L}_n(0; c) = (-1)^n n! / (1 + nc)$. Hence

$$\text{sign } \widehat{L}_n(0; c) = \begin{cases} (-1)^n, & \text{for } c > -1/n, \\ (-1)^{n+1}, & \text{for } c < -1/n. \end{cases}$$

Since $\widehat{L}_n(x; c) = \prod_{i=1}^n (x - \zeta_{ni})$ and $\widehat{L}_n(0; c) = (-1)^n \prod_{i=1}^n \zeta_{ni}$, we conclude that all zeros are positive for $c > -1/n$, and only one of them is negative for $c < -1/n$. Also, differentiating (1.3) with respect to x , we find

$$\widehat{L}'_n(0; c) = (-1)^{n+1} \frac{n!n}{2} \cdot \frac{2 + c(n+1)}{1 + nc}.$$

Thus, for $c < -2/(n+1)$, it is $\widehat{L}'_n(0; c)/\widehat{L}_n(0; c) > 0$. Because of

$$\frac{\widehat{L}'_n(x; c)}{\widehat{L}_n(x; c)} = \sum_{i=1}^n \frac{1}{x - \zeta_{ni}},$$

we have $\sum_{i=1}^n (-1/\zeta_{ni}) > 0$. Then, from $-1/\zeta_{n1} > \sum_{i=2}^n (1/\zeta_{ni})$, we yield

$$\frac{1}{-\zeta_{n1}} > \frac{n-1}{\zeta_{nn}} \quad \text{and} \quad \frac{1}{-\zeta_{n1}} > \frac{1}{\zeta_{n2}},$$

from which follows (1.5). \square

EXAMPLE 1.1. The polynomial $\widehat{L}_3(x; 1) = x^3 - \frac{11}{4}x^2 + \frac{27}{2}x - \frac{3}{2}$, has positive zeros: $x_1 \approx 0.119747$, $x_2 \approx 2.06541$, $x_3 \approx 6.06483$, but the polynomial $\widehat{L}_3(x; -2) = x^3 - \frac{39}{5}x^2 + \frac{54}{5}x + \frac{6}{5}$, has one negative zero: $x_1 \approx -0.103301$, $x_2 \approx 1.95187$, $x_3 \approx 5.95143$.

Theorem 1.4. *The zeros of $\widehat{L}_n^{(1)}(x)$ and $\widehat{L}_{n-1}^{(1)}(x)$ interlace the zeros of the polynomial $\widehat{L}_n(x; c)$.*

Proof. Let $x_{m,i}^{(1)}$, $i = 1, \dots, m$, be the zeros of $\widehat{L}_m^{(1)}(x)$. Then from (1.4), we have $\widehat{L}_n(x_{n,i}^{(1)}; c) = \lambda_n \widehat{L}_{n-1}^{(1)}(x_{n,i}^{(1)})$ and $\widehat{L}_n(x_{n,i+1}^{(1)}; c) = \lambda_n \widehat{L}_{n-1}^{(1)}(x_{n,i+1}^{(1)})$. Since between $x_{n,i}^{(1)}$ and $x_{n,i+1}^{(1)}$ there exists a unique zero of $\widehat{L}_{n-1}^{(1)}(x)$, we conclude that $\widehat{L}_{n-1}^{(1)}(x_{n,i}^{(1)})$ and $\widehat{L}_{n-1}^{(1)}(x_{n,i+1}^{(1)})$ must have the opposite signs. Hence, a zero of $\widehat{L}_n(x; c)$ exists in the interval $(x_{n,i}^{(1)}, x_{n,i+1}^{(1)})$ for $i = 1, \dots, n-1$.

In the same way, we can conclude that $\widehat{L}_n(x; c)$ has a zero in the interval $(x_{n-1,i}^{(1)}, x_{n-1,i+1}^{(1)})$ ($i = 1, \dots, n-2$). \square

Remark. In [8] it was proved that $\widehat{L}_n(x)$ and $\widehat{L}_{n-1}(x)$ interlace the zeros of $\widehat{L}_n(x; c)$.

2. Polynomials of the Jacobi type

In this section we consider the functional

$$I^{(\beta, \alpha)}(P) = \int_0^1 x^\beta (1-x)^\alpha P(x) dx, \quad \beta, \alpha > -1,$$

and the monic Jacobi polynomials $\{\widehat{Q}_n^{(\beta, \alpha)}(x)\}$ orthogonal with respect to the functional $I^{(\beta, \alpha)}$. Such polynomials can be expressed by the sum

$$\widehat{Q}_n^{(\beta, \alpha)}(x) = \frac{(-1)^n n!}{(n + \alpha + \beta + 1)_n} \sum_{k=0}^n \frac{(-1)^k}{k!(n-k)!} (n + \alpha + \beta + 1)_k \frac{(\beta + 1)_n}{(\beta + 1)_k} x^k,$$

and they satisfy the three-term recurrence relation

$$\widehat{Q}_{n+1}^{(\beta, \alpha)}(x) = (x - a_n) \widehat{Q}_n^{(\beta, \alpha)}(x) - b_n \widehat{Q}_{n-1}^{(\beta, \alpha)}(x), \quad \widehat{Q}_{-1}^{(\beta, \alpha)}(x) = 0, \quad \widehat{Q}_0^{(\beta, \alpha)}(x) = 1,$$

where

$$a_n = \gamma_{n+1} - \gamma_n, \quad \gamma_n = \frac{n(n + \beta)}{2n + \alpha + \beta},$$

$$b_n = \left\{ \frac{(n-1)(\beta + n - 1)}{2(2n + \alpha + \beta - 1)} - a_n \right\} \gamma_n - \frac{n(\beta + n)}{2(2n + \alpha + \beta + 1)} \gamma_{n+1}.$$

Let $I_2 = I^{(0, \alpha)} + c\Delta_0$, $c \in \mathbb{R}$, be a new functional and let $\{P_n^{(0, \alpha)}(x; c|0)\}$ be the corresponding monic orthogonal polynomials of the *Jacobi type*. In [6] it was proved that the n -th polynomial can be expressed by

$$P_n^{(0, \alpha)}(x; c|0) = (-1)^n n! \sum_{k=0}^n (-1)^k \frac{1 + k(1 + c(n+1)(n+\alpha))}{(k+1)!(n+\alpha+k+1)_{n-k}} \binom{n}{k} x^k.$$

Denoting the monic polynomials of the Jacobi type by $\{\widehat{P}_n^{(0, \alpha)}(x; c|0)\}$, we have

$$P_n^{(0, \alpha)}(x; c|0) = (1 + n(n + \alpha)c) \widehat{P}_n^{(0, \alpha)}(x; c|0), \quad c \neq -\frac{1}{n(n + \alpha)}, \quad n \in \mathbb{N}.$$

Hence it is $\deg P_n^{(0, \alpha)}(x; -\frac{1}{n(n+\alpha)}|0) = n - 1$ and the next recurrence relations are not valid for that polynomial and its neighbours.

Like in Section 1 we can prove the following results:

Theorem 2.1. The polynomial $\widehat{P}_n^{(0,\alpha)}(x; c|0)$ can be expressed in the form

$$\widehat{P}_n^{(0,\alpha)}(x; c|0) = \widehat{Q}_n^{(1,\alpha)}(x) + \lambda_n \widehat{Q}_{n-1}^{(1,\alpha)}(x),$$

where

$$\lambda_n = \frac{n(n+\alpha)}{(2n+\alpha)(2n+\alpha+1)} \cdot \frac{1+(n+1)(n+\alpha+1)c}{1+n(n+\alpha)c}.$$

Theorem 2.2. The sequence $\{\widehat{P}_n^{(0,\alpha)}(x; c|0)\}$ satisfies the three-term recurrence relation

$$\widehat{P}_{n+1}^{(0,\alpha)}(x; c|0) = (x - \alpha_n) \widehat{P}_n^{(0,\alpha)}(x; c|0) - \beta_n \widehat{P}_{n-1}^{(0,\alpha)}(x; c|0),$$

where

$$\alpha_n = a_n + \lambda_{n+1} - \lambda_n, \quad \beta_n = b_{n-1} \frac{\lambda_n}{\lambda_{n-1}}.$$

Theorem 2.3. All zeros of $\widehat{P}_n^{(0,\alpha)}(x; c|0)$ are real, simple and positive, with exception one of them for $c < -1/(n(n+\alpha))$. Furthermore, for $c < -2/((n+1)(n+\alpha))$ the inequalities (1.5) hold.

EXAMPLE 2.1. The polynomial $\widehat{P}_3^{(0,0)}(x; 1|0) = x^3 - \frac{27}{20}x^2 + \frac{21}{50}x - \frac{1}{200}$, has all zeros in $(0, 1)$: $x_1 \approx 0.0123940$, $x_2 \approx 0.459337$, $x_3 \approx 0.878268$, but the polynomial $\widehat{P}_3^{(0,0)}(x; -2|0) = x^3 - \frac{45}{34}x^2 + \frac{33}{85}x + \frac{1}{340}$, has one negative zero: $x_1 \approx -0.103301$, $x_2 \approx 1.95187$, $x_3 \approx 5.95143$.

Theorem 2.4. The zeros of $\widehat{Q}_n^{(1,\alpha)}(x)$ and $\widehat{Q}_{n-1}^{(1,\alpha)}(x)$ interlace the zeros of the polynomial $\widehat{P}_n^{(0,\alpha)}(x; c|0)$.

Remark. It was proved the interlacing property for $\widehat{Q}_n^{(0,\alpha)}(x)$ and $\widehat{Q}_{n-1}^{(0,\alpha)}(x)$ with respect to $\widehat{P}_n^{(0,\alpha)}(x; c|0)$ (see [8]).

3. Polynomials of the Legendre type

The sequence of Legendre polynomials $\{P_n(x)\}$ is orthogonal with respect to the functional

$$L(P) = \int_{-1}^1 P(x) dx.$$

This sequence satisfies the three-term recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

with initial values $P_{-1}(x) = 0$ and $P_0(x) = 1$. The polynomial $P_n(x)$ can be expressed in the form

$$P_n(x) = \frac{1}{2^n} \sum_{i=0}^{[n/2]} (-1)^i \binom{n}{i} \binom{2n-2i}{n} x^{n-2i}.$$

Let

$$I_3 = L + c(\Delta_{-1} + \Delta_1), \quad c \in \mathbb{R},$$

be a new functional, and $\{P_n(x; c | -1, 1)\}$ the corresponding orthogonal polynomials of the *Legendre type*. Then the moments $\tilde{\mu}_n = I_3(x^n)$ are given by

$$\tilde{\mu}_n = \begin{cases} 0 & \text{for odd } n, \\ \frac{2}{n+1} + 2c & \text{for even } n. \end{cases}$$

The polynomial $P_n(x; c | -1, 1)$ can be expressed in a determinant form,

$$P_n(x; c | -1, 1) = \det \begin{vmatrix} \tilde{\mu}_0 & \tilde{\mu}_1 & \dots & \tilde{\mu}_{n-1} \\ \tilde{\mu}_1 & \tilde{\mu}_2 & & \tilde{\mu}_n \\ \vdots & & & \\ \tilde{\mu}_{n-1} & \tilde{\mu}_n & & \tilde{\mu}_{2n-1} \\ 1 & x & & x^n \end{vmatrix}.$$

Using a method as in Gautschi and Milovanović [5], we determine

$$\tilde{\Delta}_n = \det[\tilde{\mu}_{i+j}]_{i,j=0,\dots,n-1}.$$

It is known that for determinants

$$H_n = \det \left[\frac{1}{2i+2j-3} \right]_{i,j=1}^n, \quad H_0 = 1,$$

holds

$$H_n = \frac{(2n-2)!!^2}{(2n-1)^2 \dots (4n-5)^2 (4n-3)} H_{n-1}, \quad n = 3, 4, \dots$$

Introducing

$$H_n(c) = \left[\frac{1}{2i+2j-3} + c \right]_{i,j=1}^n,$$

we obtain

$$H_n(c) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ -c & 1 & \frac{1}{3} & & \frac{1}{2n-1} \\ -c & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1} \\ \vdots & & & & \\ -c & \frac{1}{2n-1} & \frac{1}{2n+1} & & \frac{1}{4n-3} \end{vmatrix}.$$

Lemma 3.1. *We have*

$$H_n(c) = (1 + c \binom{2n}{2}) H_n.$$

Proof. For $H_n(c)$ we have

$$H_n(c) = H_n + cD_n,$$

where

$$D_n = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 1 & \frac{1}{3} & & \frac{1}{2n-1} \\ -1 & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1} \\ \vdots & & & & \\ -1 & \frac{1}{2n-1} & \frac{1}{2n+1} & & \frac{1}{4n-3} \end{bmatrix}.$$

We can prove that $D_n = \binom{2n}{2} H_n$, $n \in \mathbb{N}$. Subtracting the last row in D_n from all others, except the first one, we obtain

$$D_n = \frac{(2n-2)!!}{(2n-1)\cdots(4n-3)} \begin{bmatrix} 0 & 2n-1 & 2n+1 & \cdots & 4n-3 \\ 0 & 1 & \frac{1}{3} & & \frac{1}{2n-1} \\ 0 & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1} \\ \vdots & & & & \\ -1 & 1 & 1 & & 1 \end{bmatrix}.$$

Also, subtracting the last column from all others, except the first one, we obtain

$$D_n = \frac{(2n-2)!!^2}{(2n-1)^2 \cdots (4n-5)^2 (4n-3)} \{(4n-3)H_{n-1} + D_{n-1}\},$$

from which, by induction, we finish the proof. \square

Lemma 3.2. *If*

$$H'_n(c) = \det \left[\frac{1}{2i+2j-1} + c \right]_{i,j=1}^n, \quad H'_n = H'_n(0),$$

then

$$H'_n(c) = [1 + c \binom{2n+1}{2}] H'_n, \quad n = 1, 2, \dots$$

Thus,

$$\tilde{\Delta}_n = \Delta_n [1 + \binom{n}{2} c] [1 + \binom{n+1}{2} c],$$

where

$$\Delta_n = \det [L(x^{i+j})]_{i,j=0,\dots,n-1}.$$

Theorem 3.3. *The polynomials $P_n(x; c | -1, 1)$ satisfy the following three-term recurrence relation*

$$(n+1) \left(1 + \binom{n}{2} c\right) P_{n+1}(x; c | -1, 1) = (2n+1) \left(1 + \binom{n+1}{2} c\right) x P_n(x; c | -1, 1) - n \left(1 + \binom{n+2}{2} c\right) P_{n-1}(x; c | -1, 1).$$

Proof. By $\{\widehat{P}_n(x; c | -1, 1)\}$ we denote the sequence of monic polynomials of the Legendre type. According to the property $(xf, g) = (f, xg)$ of the inner product defined by the functional I_3 : $(f, g) = I_3(fg)$, we conclude (cf. [2] and [9]) that this sequence satisfies a three-term recurrence relation of the form

$$\widehat{P}_{n+1}(x; c | -1, 1) = x \widehat{P}_n(x; c | -1, 1) - \beta_n \widehat{P}_{n-1}(x; c | -1, 1),$$

where (see [3])

$$\beta_n = \frac{\widetilde{\Delta}_{n-1} \widetilde{\Delta}_{n+1}}{\widetilde{\Delta}_n^2}.$$

Knowing a relation for the monic Legendre polynomials and the determinant Δ_n , we yield

$$\begin{aligned} \widehat{P}_{n+1}(x; c | -1, 1) &= x \widehat{P}_n(x; c | -1, 1) \\ &\quad - \frac{n^2}{4n^2 - 1} \frac{(1 + \binom{n-1}{2} c)(1 + \binom{n+2}{2} c)}{(1 + \binom{n}{2} c)(1 + \binom{n+1}{2} c)} \widehat{P}_{n-1}(x; c | -1, 1). \end{aligned}$$

Putting

$$P_n(x; c | -1, 1) = \frac{(2n)!}{2^n n!^2} \widehat{P}_n(x; c | -1, 1),$$

we finish the proof. \square

By induction and using the three-term recurrence relation, we obtain:

Theorem 3.4. *The polynomial $P_n(x; c | -1, 1)$ can be expressed in the form*

$$P_n(x; c | -1, 1) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \left\{ 1 + \left(\binom{n}{2} + 2k \right) c \right\} x^{n-2k},$$

where $n = 2, 3, \dots$

Remark. This formula was derived in [6] with a mistake.

The last formula shows that the sequence $\{P_n(x; c | -1, 1)\}$ has a degenerated property in the sense that is

$$\widehat{P}_n(x; -\binom{n}{2}^{-1} | -1, 1) = \widehat{P}_{n-2}(x; -\binom{n}{2}^{-1} | -1, 1).$$

Theorem 3.5. *The polynomials $P_n(x; c | -1, 1)$ are quasi-orthogonal of the second order with respect to the functional*

$$J^{(1,1)}(P) = \int_{-1}^1 P(x)(1-x^2) dx.$$

The polynomial $P_n(x; c | -1, 1)$ has at least $n - 2$ different zeros with odd multiplicity in $(-1, 1)$.

Proof. Because of the orthogonality, we have for any polynomial $p(x)$ of degree k ($k \leq n - 3$) that

$$I_3(P_n(x; c | -1, 1)p(x)(1-x^2)) = 0,$$

i.e.,

$$\int_{-1}^1 P_n(x; c | -1, 1)p(x)(1-x^2) dx = 0.$$

So, we yield quasi-orthogonality of the order 2. Finally, let $x_{n1}, x_{n2}, \dots, x_{nk}$ be all distinct zeros of $P_n(x; c | -1, 1)$ with odd multiplicity and which are in $(-1, 1)$. If we introduce the node polynomial $p(x) = (x-x_{n1})(x-x_{n2}) \cdots (x-x_{nk})$, then the polynomial $P_n(x; c | -1, 1)p(x)(1-x^2)$ does not change sign in $(-1, 1)$. Therefore,

$$\int_{-1}^1 P_n(x; c | -1, 1)p(x)(1-x^2) dx \neq 0.$$

Because of that, we conclude that $\deg p(x) \geq n - 2$. \square

EXAMPLE 3.2. The polynomial

$$P_3(x; c | -1, 1) = 5(1+3c)x^3 - 3(1+5c)x,$$

has the zeros

$$x_1 = 0, \quad x_{2,3} = \pm \sqrt{\frac{3(1+5c)}{5(1+3c)}}.$$

All zeros of $P_3(x; c | -1, 1)$ are in $(-1, 1)$ if $c > -1/3$. For $c < -1/3$, two of them are out of $(-1, 1)$.

Remark. According to Favard's theorem (see [2]) there are exist the weight distributions $w_i(x)$, $i = 1, 2, 3$, corresponding to the previous functionals J_i :

$$J_i(P) = \int_{-\infty}^{+\infty} P(x) dw_i(x), \quad i = 1, 2, 3.$$

These distributions are (see [3])

$$w_1(x) = e^{-x} + c\delta(x), \quad w_2(x) = (1-x)^\alpha + c\delta(x), \quad w_3(x) = 1 + c(\delta(x-1) + \delta(x+1)),$$

where $\delta(x)$ is the Dirac's delta function.

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