Filomat 31:10 (2017), 3015–3021
https://doi.org/10.2298/FIL1710015A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Properties of Convex Functions

Miloljub Albijanić[a]

[a]*Faculty of Economics, Finance and Administration, Belgrade, Serbia*

Abstract. We treat two problems on convex functions of one real variable. The first one is concerned with properties of tangent lines to the graph of a convex function and essentially is related to the questions on the first derivative (if it exists). The second problem is related to Schwarz's derivative, in fact its upper limit modification. It gives an interesting characterization of convex functions.

Let us recall the definition of a convex functions.

Definition. A function $f : (a, b) \to \mathbb{R}$, defined on an interval (a, b), is convex if for all distinct points $x_1, x_2 \in (a, b)$ and every real number $\lambda \in (0, 1)$ the following inequality holds:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

If the reverse inequality holds for a function $f : (a, b) \to \mathbb{R}$

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2).$$

then the function f is said to be concave.

If we write $<$ instead of $\leq$, then we say that the function is strictly convex (or instead $\geq$ we write $>$, the function is strictly concave).

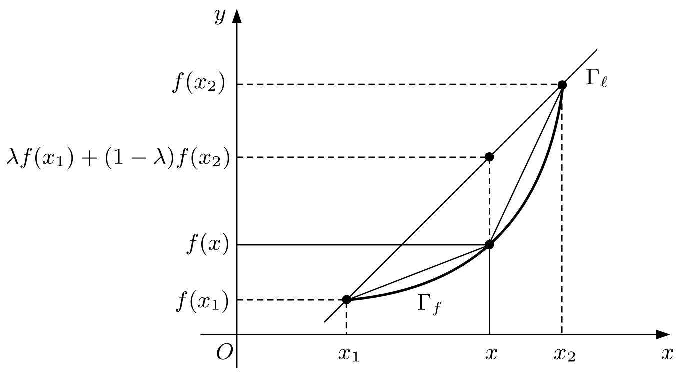

Figure 1: Convexity

2010 *Mathematics Subject Classification*. Primary 26A51; Secondary 26A24
Keywords. Convexity, asymptote, Schwarz derivative
Received: 21 Septempber 2016; Revised: 24 December 2016; Accepted: 03 January 2017
Communicated by Eberhard Malkowsky
Email address: malbijanic@fefa.edu.rs (Miloljub Albijanić)

An equivalent definition of convexity on an interval (a,b) is given by the following condition:

$$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x)}{x_2-x} \qquad \big(x_1 < x < x_2,\ x_1, x_2 \in (a,b)\big).$$

Convexity and tangent line. *A differentiable function $f:(a,b)\to\mathbb{R}$ is (strictly) convex if and only if for each point $P_0=(x_0,f(x_0))$ on the graph of f, the graph of f lies above the tangent L to the graph at P_0 (the graph of f is strictly above the tangent L, except at the point P_0). An analogous statement holds for (strictly) concave differentiable functions.*

Proof. (*Necessity*). Let $x_0 \in (a,b)$. The equation of L is given by $y = L(x) = f(x_0) + f'(x_0)(x-x_0)$.

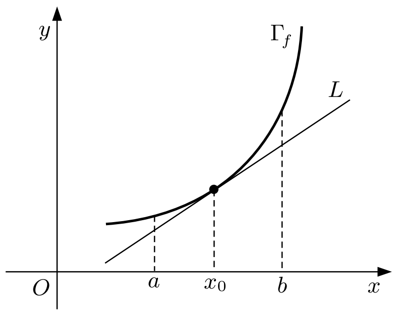

Figure 2: Convex function

Using Lagrange's theorem we obtain

$$f(x)-L(x) = f(x)-f(x_0)-f'(x_0)(x-x_0) = \big(f'(\xi)-f'(x_0)\big)(x-x_0), \quad \text{for some} \quad \xi \in (x_0,x).$$

Since f is convex, f' is increasing on (a,b) and the difference $f'(\xi)-f'(x_0)$ has the same sign as $x-x_0$. Hence $f(x)-L(x)\ge 0$, for $x\in(a,b)$, as desired. If the function f is strictly convex, then $f(x)-L(x)>0$ for $x\in(a,b)$ and $x\ne x_0$.

(*The sufficiency.*) By the assumption we have, for all $x, x_0 \in (a,b)$:

$$f(x)-L(x) = f(x)-f(x_0)-f'(x_0)(x-x_0) \ge 0,$$

then

$$\frac{f(x)-f(x_0)}{x-x_0} \le f'(x_0) \quad \text{for} \quad x < x_0,$$
$$\frac{f(x)-f(x_0)}{x-x_0} \ge f'(x_0) \quad \text{for} \quad x > x_0.$$

Therefore, for all $x_1, x_0, x_2 \in (a,b)$, such that $x_1 < x_0 < x_2$ we have

$$\frac{f(x_0)-f(x_1)}{x_0-x_1} \le \frac{f(x_2)-f(x_0)}{x_2-x_0},$$

which is the second definition of convexity. □

The inspiration for this paper comes from the Bourbaki treatise [2].

Solutions to some of the problems from [2] are presented in [1]. We direct the reader to the classical monograph on inequalities [4] for a thorough exposition of classical inequalities. Convexity plays a crucial role in such considerations, there is extensive literature on these topics. It is present in classical texts on Analysis, like Hardy [3] and in modern ones, like Zorich [9]. An example of this line of research can be found in [6].

Let us mention a more specialized monograph D. Mitrinović [8] on inequalities. Serbian mathematicians made their contributions to that field, for example M. Petrovic's inequality and Karamata's inequality, which will be stated below.

M. Petrovic's inequality. *If $f : [0, +\infty) \to \mathbb{R}$ is a convex function and $x_1, x_2, \ldots, x_n$ is a sequence of positive numbers. Then*

$$f(x_1) + \cdots + f(x_n) \le f(x_1 + \cdots + x_n) + (n-1)f(0).$$

Majorization. *Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two finite sequences of real numbers. We say that the sequence a majorizes the sequence b and write $a \succ b$, if*

$$a_1 \ge a_2 \ge \cdots \ge a_n, \quad b_1 \ge b_2 \ge \cdots \ge b_n,$$
$$a_1 + a_2 + \cdots + a_k \ge b_1 + b_2 + \cdots + b_k, \quad k = 1, 2, \ldots, n-1,$$
$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n.$$

Karamata's inequality. *Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two sequences of real numbers in an interval (α, β). If $a \succ b$ and if $f : (\alpha, \beta) \to \mathbb{R}$ is convex, then*

$$f(a_1) + \cdots + f(a_n) \ge f(b_1) + \cdots + f(b_n).$$

Proposition A. *Let f be differentiable and convex on (a, b), $a \ge 0$.*

(1) *Then $f(x) - xf'(x)$ decreases (strictly decreases if f is strictly convex) on (a, b).*

(2) *If f admits a finite right limit at a, then $\lim_{x \to a+0}(x-a)f'(x) = 0$.*

(3) *Function $\dfrac{f(x)}{x}$ on (a, b) either increases or decreases or there exists $c \in (a, b)$ so that $\dfrac{f(x)}{x}$ decreases on (a, c) and increases on (c, b).*

(4) *Let us assume that $b = +\infty$. If*

$$\beta = \lim_{x \to +\infty} \big(f(x) - xf'(x)\big)$$

is finite, then the limit $\alpha = \lim\limits_{x \to +\infty} \dfrac{f(x)}{x}$ also exists and is finite. The straight line $y = \alpha x + \beta$ is an asymptote of the function f and it lies below the graph of f.

Proof. (1) If f has a second derivative then it is easy to prove that $h(x) = f(x) - xf'(x)$ decreases, because

$$h'(x) = f'(x) - f'(x) - xf''(x) = -x \cdot f''(x) \le 0.$$

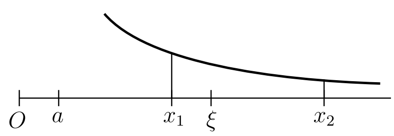

Figure 3:

Let us now present a proof assuming only existence of the first derivative. We choose $x_1 < x_2$.

$$h(x_2) - h(x_1) = f(x_2) - x_2 f'(x_2) - f(x_1) + x_1 f'(x_1) =$$
$$= f(x_2) - f(x_1) - \big(x_2 f'(x_2) - x_1 f'(x_1)\big) =$$
$$= f(x_2) - f(x_1) - x_2 f'(x_2) + x_1 f'(x_2) - x_1 f'(x_2) + x_1 f'(x_1)$$
$$= f(x_2) - f(x_1) - f'(x_2)(x_2 - x_1) - x_1\big(f'(x_2) - f'(x_1)\big)$$
$$= f'(\xi)(x_2 - x_1) - f'(x_2)(x_2 - x_1) - x_1\big(f'(x_2) - f'(x_1)\big) \quad \text{(Lagrange's theorem)}$$
$$= (x_2 - x_1)\big(f'(\xi) - f'(x_2)\big) - x_1\big(f'(x_2) - f'(x_1)\big),$$

where ξ is between x_1 and x_2. Since f is convex, the first derivative f' is increasing and therefore we obtain

$$f'(\xi) - f'(x_2) \le 0, \quad \xi < x_2,$$
$$f'(x_2) - f'(x_1) \le 0, \quad x_1 < x_2$$

and the result follows immediately.

(2) Let us extend the function f at point a by setting $f(a) = f(a+0) = c$. Let us assume that, for some $x_0 > a$ we have $f(x_0) = f(a)$. Then, by convexity, $f(x) \le f(a)$ for all $a < x < x_0$. In fact, there are only two possibilities: either $f(x) = c$ for all $a < x < x_0$ or $f(x) < c$ for all $a < x < x_0$. Since the first possibility is trivial we can assume that $f(x) \neq c$ for x near a. Then, by the Lagrange's theorem, there is $\xi = \xi(x) \in (a, x)$ such that

$$(x-a)f'(x) = \frac{f'(x)}{\dfrac{f(x)-f(a)}{x-a}} \cdot (f(x) - f(a)) = \frac{f'(x)}{f'(\xi)} \cdot \big(f(x) - f(a)\big).$$

Case a. If for some sufficiently small $x - a$, $f'(x) < 0$, then $f'(\xi) < f'(x) < 0$ so

$$\left|\frac{f'(x)}{f'(\xi)}\right| \le 1,$$

while $f(x) - f(a) \to c - c = 0$ when $x \to a+0$. We conclude $(x-a)f'(x) \to 0$.

Case b. $f'(x) \ge 0$, for every $x \in (a, b)$. Then f is increasing on (a, b). Let us fix $d \in (a, b)$. For $x < d$ we have $0 \le f'(x) \le f'(d)$. Because x tends to a we can assume that $x < d$. We can conclude at once

$$|(x-a)f'(x)| \le |x-a| f'(x) \to 0,$$

which suffices.

(3) Let $h(x) = \dfrac{f(x)}{x}$. Then,

$$h'(x) = \frac{xf'(x) - f(x)}{x^2} = -\frac{f(x) - xf'(x)}{x^2} = -\frac{g(x)}{x^2}.$$

We already proved, in (1), that $g(x)$ is decreasing function on (a, b). Let us consider three cases for $g(x)$:

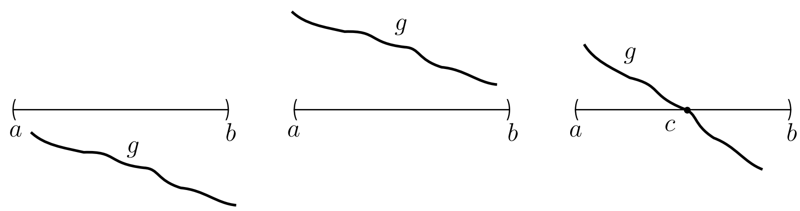

Figure 4:

In the first case $g(x) < 0$ for all $x \in (a, b)$, so $h'(x) > 0$, which means that $h(x)$ is increasing.

The second case is that g takes both positive and negative values on (a, b). This easily implies that h at first decreases and then increases; similarly we deal with the case when g is strictly positive on (a, b).

(4) $f'(x)$ is monotonically increasing because of the convexity of f (for $x > a$). Let us fix $c > a$ and let, for $x > a$, $\varphi(x)$ be the unique real number such that the points $M(c, f(c))$, $L(x, f(x))$ and $N(0, \varphi(x))$ are collinear. We distinguish two cases: (a) $f'(x)$ is not bounded from above on $(a, +\infty)$; (b) $f'(x)$ is bounded from above on $(a, +\infty)$.

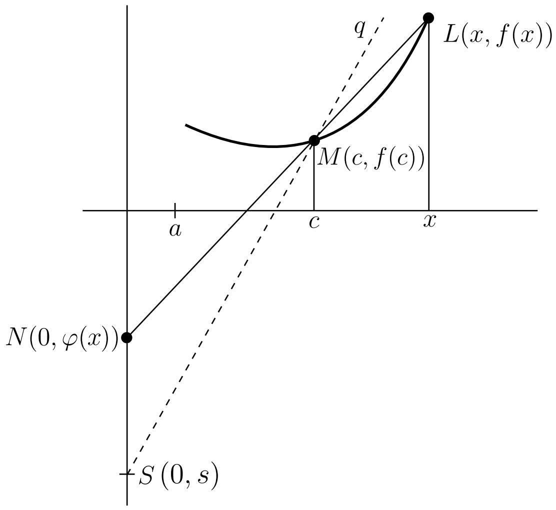

Figure 5:

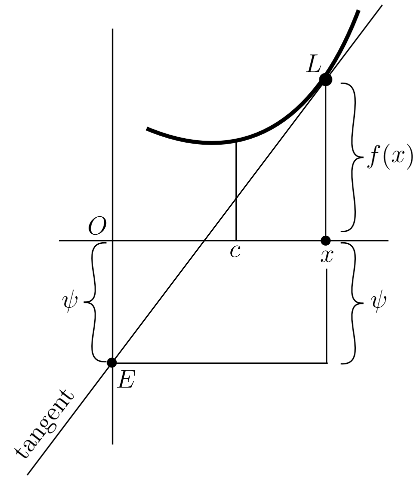

Figure 6:

The first case (a).

When $x \to +\infty$, then $\varphi(x) \to -\infty$ (see Figure 5).

Indeed, assuming the contrary, we would have $\varphi(x) \geq s$ for all sufficiently large x. That gives: the graph of f lies below the line q determined by points M and $S(0, s)$, for large x. Let $y = \gamma x + \delta$ be the equation of the line q. Then, for large x:

$$f(x) \leq \gamma x + \delta \qquad \text{and therefore}$$

$$\limsup_{x\to+\infty} \frac{f(x)}{x} \leq \gamma.$$

Next, there is an x_0 such that $f'(x) \geq \gamma + 1$ for all $x \geq x_0$. Using Lagrange's theorem we have

$$f(x) - f(x_0) = (x - x_0)f'(\xi) \geq (\gamma + 1)(x - x_0) \qquad x \geq x_0,$$

but the last inequality easily implies $\limsup_{x\to+\infty} \frac{f(x)}{x} \geq \gamma + 1$ which gives a contradiction and we proved $\lim_{x\to+\infty} \varphi(x) = -\infty$.

(b) The equation of the tangent line at point $L(x, f(x))$ is

$$y = f(x) + f'(x)(t - x), \quad t > a,$$

where t is the independent variable. Let us denote the value of this function at $t = 0$ by $\psi = \psi(x)$, i.e.

$$\psi(x) = f(x) - xf'(x),$$

see Figure 6. Let us note that, under our assumptions, $f'(x)$ is bounded from above. Hence, since f' is increasing and bounded from above, it has a finite limit α as $x \to +\infty$.

Next, by assumption, $f(x) = xf'(x) + \beta + o(1)$ which gives, passing to the limit:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} f'(x) + \frac{\beta + o(1)}{x} = \alpha.$$

Note that we proved:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} f'(x).$$

The straight line $y = \alpha x + \beta$ is an asymptote of the function f. The asymptote is under the graph of f because f is convex. $\square$

Proposition B. Let $f : I \to \mathbb{R}$ be upper semi-continuous on an open interval $I \subset \mathbb{R}$. Then: f is convex on I if

$$\limsup_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} \geq 0, \tag{1}$$

for every $x \in I$.

Proof. The following should be proved: if f is not convex, then the condition (1) fails.

Let us assume that f is not convex on I. That means that there are $a < c < b$ in I such that $f(c) > g(c)$ where $g(x)$ is represents a straight line which contains $(a, f(a))$ and $(b, f(b))$. Let $\Phi(x) = f(x) + \varepsilon x^2$. For sufficiently small $\varepsilon > 0$ we also have $\Phi(c) > G(c)$ where $G(x)$ represents a straight line which contains points $(a, \Phi(a))$ and $(b, \Phi(b))$ that is, the function Φ is not convex on $[a, b]$.

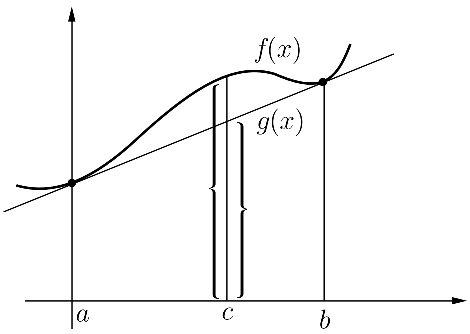

Figure 7:

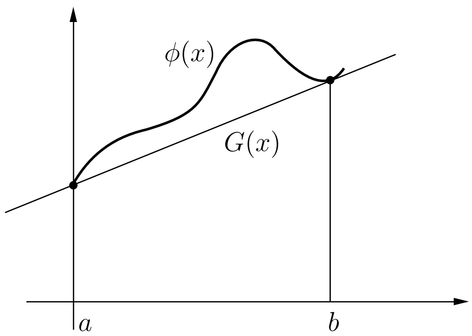

Figure 8:

Let

$$\Psi(x) = \Phi(x) - G(x), \quad x \in [a, b].$$

The function Ψ attains its maximum on $[a, b]$ at some point $\xi \in (a, b)$ because Φ is upper semi-continuous on the compact interval $[a, b]$. Therefore, there is $\delta > 0$ such that

$$\frac{\Psi(\xi - h) + \Phi(\xi + h)}{2} \leq \Psi(\xi), \qquad \text{for all} \quad |h| < \delta.$$

That means,

$$\Psi(\xi - h) + \Psi(\xi + h) \leq 2\Psi(\xi),$$

so

$$\Big(\Phi(\xi - h) - G(\xi - h)\Big) + \Big(\Phi(\xi + h) - G(\xi + h)\Big) \leq 2\Big(\Phi(\xi) - G(\xi)\Big),$$

and since $G(x)$ is linear we obtain

$$G(\xi - h) + G(\xi + h) = 2G(\xi).$$

Now we complete the proof by establishing the following inequalities:

$$\begin{aligned}
&\Phi(\xi-h)+\Phi(\xi+h)\le 2\Phi(\xi),\\
&f(\xi-h)+\varepsilon(\xi-h)^2+f(\xi+h)+\epsilon(\xi+h)^2\le 2f(\xi)+2\varepsilon\xi^2,\\
&f(\xi-h)+f(\xi+h)-2f(\xi)\le -2\varepsilon h^2,\\
&\frac{f(\xi-h)+f(\xi+h)-2f(\xi)}{h^2}\le -2\varepsilon \quad \text{and finally}\\
&\limsup_{h\to 0}\frac{f(\xi-h)+f(\xi+h)-2f(\xi)}{h^2}\le -2\varepsilon<0.
\end{aligned}$$

□

Conclusion

The proofs presented have a strong geometrical flavor and we stress the role played by Lagrange's theorem. Although these results are quite classical and appeared in the literature, we believe the novelty of presentation will be of interest. These and related results have generalizations in many directions.

References

[1] M. Albijanić, Abstraction and Application in Mathematical Analysis (in Serbian), Zavod za udžbenike, Beograd, 2016.
[2] N. Bourbaki, Fonctions d'une variable réelle: Théorie élémentaire, Springer, 2007.
[3] G. H. Hardy, A Course of Pure Mathematics, Cambridge University Press, 1945.
[4] G. H. Hardy, J. E. Littlewood and G. Plya Inequalities, Cambridge Mathematical Library, 2001.
[5] P. Loh, Convexity, http://www.math.cmu.edu/ ploh/docs/math/mop2013/convexity-soln.pdf, 2013.
[6] M. Mateljević, M. Svetlik, M. Albijanić, N. Savić, Generalizations of the Lagrange mean value theorem and applications, FILOMAT 27:4 (2013), 515-528.
[7] N. Merentes, K. Nikodem, Remarks on strongly convex functions, Aequationes Mathematicae 80(1) (2010), 193-199.
[8] D. S. Mitrinović, Analytic Inequalities, Springer, 1970.
[9] V. A. Zorich Mathematical Analysis I, Springer, 2015.