Filomat 31:9 (2017), 2611–2623
https://doi.org/10.2298/FIL1709611N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Genuine Family of Bernstein-Durrmeyer Type Operators Based on Polya Basis Functions

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Abstract. In this paper, we construct a genuine family of Bernstein-Durrmeyer type operators based on Polya basis functions. We establish some moment estimates and the direct results which include global approximation theorem in terms of classical modulus of continuity, local approximation theorem in terms of the second order Ditizian-Totik modulus of smoothness, Voronovskaya-type asymptotic theorem and a quantitative estimate of the same type. Lastly, we study the approximation of functions having a derivative of bounded variation.

1. Introduction

The approximation of functions by positive linear operators is an important research area in the classical approximation theory. It provides us key tools for exploring the computer-aided geometric design, numerical analysis and the solutions of ordinary and partial differential equations that arise in the mathematical modeling of real world phenomena. After the well known theorem of Weierstrass and the important theorem of Korovkin, many new sequences and classes of operators were constructed and studied for their approximation behavior by researchers. Some of the recently introduced sequences and classes of operators which have been extensively studied by researchers, are Srivastava-Gupta operators [36], Bernstein-Durrmeyer type operators ([12],[13] [17] etc.), Bernstein-Kantorovich type operators ([14], [32], [24] [30] etc.), Hybrid type operators ([3],[18], [22] etc.) Gamma type operators ([23], [26], [27] etc.), Chlodowsky and Stancu variant of operators ([4],[28],[33],[39] etc.), linear positive operators constructed by means of the Chan-Chayan-Srivastava multivariable polynomials [10] etc. Many researchers have studied the approximation properties of Srivastava-Gupta operators and its various generalizations over the past decade ([2], [7], [25], [40] [41] etc.). Erkus et al.[10] showed that the approximation method constructed by them by means of the Chan-Chayan-Srivastava polynomials is stronger than the corresponding classical aspects in approximation theory.

The study of the rate of convergence for functions of bounded variation by linear positive operators is another interesting area of research. Cheng [6] investigated the rate of convergence of Berntein polynomials for functions of bounded variation. Using probablistic approach, Srivastava and Gupta [37] studied the rate of convergence for the Bezier variant of the Bleimann-Butzer-Hahn operators for the functions of

2010 *Mathematics Subject Classification*. 41A25, 26A15, 40A35

Keywords. Genuine-linear positive operator; beta-function; Peetre's K-functional; Ditzian-Totik modulus of smoothness; least concave majorant; functions of bounded variation

Received: 21 April 2015; Accepted: 16 November 2015

Communicated by Hari M. Srivastava

Research supported by Ministry of Human Resource Development

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bounded variation. Bojanic and Cheng [5] studied the rate of convergence of Bernstein polynomials for functions with derivative of bounded variation. Srivastava et al. [35] discussed local and global results for a certain family of summation-integral type operators and estimate the rate of convergence for functions having derivative of bounded variation. Researchers studied these problems for several other sequences of linear positive operators we refer the readers to Book [16].

In 1968, Stancu [38] introduced a sequence of positive linear operators $P_n^{(\alpha)} : C[0,1] \longrightarrow C[0,1]$, depending on a non negative parameter α as

$$P_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) p_{n,k}^{(\alpha)}(x), \tag{1}$$

where $p_{n,k}^{\alpha}(x)$ is the Polya distribution with density function given by

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k} \frac{\prod_{v=0}^{k-1}(x+v\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}, \ \ x \in [0,1].$$

In case $\alpha = 0$, the operators (1) reduce to the classical Bernstein polynomials. For these operators , Lupas and Lupas [29] considered a special case for $\alpha = \frac{1}{n}$ which reduces to

$$P_n^{(\frac{1}{n})}(f;x) = \frac{2(n!)}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} f\Big(\frac{k}{n}\Big)(nx)_k (n-nx)_{n-k}, \tag{2}$$

where the rising factorial $(x)_n$ is given by $(x)_n = x(x+1)(x+2).....(x+n-1)$ with $(x)_0 = 1$.

Gupta and Rassias [19] introduced the Durrmeyer-type integral modification for the operators (2) and obtained local and global direct estimates and a Voronvskaya-type asymptotic formula. Later the same authors [20] considered a Durrmeyer type modification of the Jain operators and studied the asymptotic formula, error estimation in terms of the modulus of continuity and weighted approximation. Gupta et. al. [21] proposed certain Lupas-beta operators which preserve constant as well as linear functions and established some direct results and the approximation of functions having a derivative of bounded variation. Gonska and Păltănea [12] established a very interesting link between the Bernstein polynomials and their Bernstein Durrmeyer variants with several particular cases which preserve linear functions and gave recursion formula for moments and estimates for simultaneous approximation of derivatives. After that the same authors [13] established quantitative Voronovskaya-type assertions in terms of the first-order and second-order moduli of smoothness. Very recently, Gupta [15] defined a genuine Durrmeyer type modification of the operators given by (2) and obtained a Voronovskaya-type asymptotic theorem and a local approximation theorem. Motivated by these studies, for $f \in L_B[0,1]$, the space of bounded and Lebesgue integrable functions on $[0,1]$ and a parameter $\rho > 0$, we now propose a genuine Durrmeyer type modification of the operators given by (2), which preserve linear functions, as

$$U_n^{\rho}(f;x) = \sum_{k=0}^{n} F_{n,k}^{\rho} p_{n,k}^{(\frac{1}{n})}(x),$$

where

$$F_{n,k}^{\rho} = \begin{cases} \int_0^1 f(t)\mu_{n,k}^{\rho} dt, & 1 \le k \le n-1 \\ f(0), & k=0 \\ f(1), & k=n, \end{cases}$$

and

$$\mu_{n,k}^{\rho}(t) = \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)},$$

$B(m,n)$ being the beta-function defined as

$$B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}, \quad m,n>0.$$

For $\rho=1$, the operators U_n^{ρ} reduce to the operators defined by Gupta [15] and when $\rho\to\infty$, these operators reduce to the operators considered by Lupas and Lupas [29], in view of the fact that $F_{n,k}^{\rho}\to f\left(\frac{k}{n}\right)$, as shown by Gonska and Păltănea [3,Thm 2.3, p.786].

The purpose of this paper is to establish a Voronovskaya type asymptotic theorem, global approximation theorems in terms of the classical second order modulus of continuity and local-approximation theorem in terms of the second order Ditzian-Totik modulus of smoothness, Voronovskaya-type asymptotic theoreom and also quantitative estimate. In the last section of the paper, the approximation of functions having a derivative of bounded variation is also discussed.

2. Auxiliary Results

Lemma 2.1. *[31] For the operators defined by (2) , one has*

(i) $P_n^{(\frac{1}{n})}(1;x)=1,$

(ii) $P_n^{(\frac{1}{n})}(t;x)=x,$

(iii) $P_n^{(\frac{1}{n})}(t^2;x)=x^2+\frac{2x(1-x)}{n+1},$

(iv) $P_n^{(\frac{1}{n})}(t^3;x)=x^3+\frac{6nx^2(1-x)}{(n+1)(n+2)}+\frac{6x(1-x)}{(n+1)(n+2)},$

(v) $P_n^{(\frac{1}{n})}(t^4;x)=x^4+\frac{12(n^2+1)x^3(1-x)}{(n+1)(n+2)(n+3)}+\frac{12(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)}+\frac{2(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)}.$

Consequently, by simple computations we have the following:

Lemma 2.2. *For $U_n^{\rho}(t^m;x)$, $m=0,1,2,3,4,$ we obtain,*

(i) $U_n^{\rho}(1;x)=1,$

(ii) $U_n^{\rho}(t;x)=x,$

(iii) $U_n^{\rho}(t^2;x)=\frac{n\rho}{n\rho+1}\left(x^2+\frac{2x(1-x)}{n+1}\right)+\frac{x}{n\rho+1},$

(iv) $U_n^{\rho}(t^3;x)=\frac{1}{(n\rho+1)(n\rho+2)}\left\{n^2\rho^2\left(x^3+\frac{6nx^2(1-x)}{(n+1)(n+2)}+\frac{6x(1-x)}{(n+1)(n+2)}\right)+3n\rho\left(x^2+\frac{2x(1-x)}{n+1}\right)+2x\right\},$

(v) $$U_n^{\rho}(t^4;x)=\frac{1}{(n\rho+1)(n\rho+2)(n\rho+3)}\left\{n^3\rho^3\left(x^4+\frac{12(n^2+1)x^3(1-x)}{(n+1)(n+2)(n+3)}+\frac{12(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)}+\frac{2(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)}\right)\right.$$
$$\left.+6n^2\rho^2\left(x^3+\frac{6nx^2(1-x)}{(n+1)(n+2)}+\frac{6x(1-x)}{(n+1)(n+2)}\right)+11n\rho\left(x^2+\frac{2x(1-x)}{n+1}\right)+6x\right\}.$$

In our next Lemma, we find the central moment estimates required for the main results of the paper.

Lemma 2.3. *For $U_n^{\rho}((t-x)^m;x)$, $m\in\mathbb{N}\bigcup\{0\}$, we have,*

(i) $U_n^{\rho}((t-x);x)=0,$

(ii) $U_n^{\rho}((t-x)^2;x)=\frac{(2n\rho+n+1)x(1-x)}{(n+1)(n\rho+1)},$

(iii) $U_n^{\rho}((t-x)^3;x)=\frac{1}{(n\rho+1)(n\rho+2)}\left\{n^2\rho^2\left(x^3+\frac{6nx^2(1-x)}{(n+1)(n+2)}+\frac{6x(1-x)}{(n+1)(n+2)}\right)+3n\rho\left(x^2+\frac{2x(1-x)}{n+1}\right)+2x\right\}-\frac{3x^2(1-x)(2n\rho+n+1)}{(n+1)(n\rho+1)}-x^3$

(iv) $U_n^{\rho}((t-x)^4;x) = \frac{1}{(n\rho+1)(n\rho+2)(n\rho+3)}\Big\{n^3\rho^3\Big(x^4 + \frac{12(n^2+1)x^3(1-x)}{(n+1)(n+2)(n+3)} + \frac{12(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)} + \frac{2(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)}\Big)$
$+ 6n^2\rho^2\Big(x^3 + \frac{6nx^2(1-x)}{(n+1)(n+2)} + \frac{6x(1-x)}{(n+1)(n+2)}\Big) + 11n\rho\Big(x^2 + \frac{2x(1-x)}{n+1}\Big) + 6x\Big\} + \frac{6(2n\rho+n+1)x^3(1-x)}{(n+1)(n\rho+1)}$
$+ 3x^4 - \frac{4x}{(n\rho+1)(n\rho+2)}\Big\{n^2\rho^2\Big(x^3 + \frac{6nx^2(1-x)}{(n+1)(n+2)} + \frac{6x(1-x)}{(n+1)(n+2)}\Big) + 3n\rho\Big(x^2 + \frac{2x(1-x)}{n+1}\Big) + 2x\Big\}.$

Consequently, for every $x \in [0,1]$,

$$U_n^{\rho}((t-x)^2;x) \le \frac{2\rho+1}{n\rho+1}\phi^2(x)$$

where $\phi^2(x) = x(1-x)$ and

$$\lim_{n\to\infty} U_n^{\rho}((t-x)^4;x) = 12x^3(1+x) - 2x(7x+5) + \frac{12x^2}{\rho}(1-x+x^2) + \frac{3x^2}{\rho^2}(1-x)^2, \quad \text{uniformly in } x \in [0,1].$$

Remark 2.4. *From Lemma 2.3, we have*

$$\begin{aligned} U_n^{\rho}((t-x)^2;x) &\le \frac{(2\rho+1)}{(n\rho+1)}\phi^2(x) \le \frac{1}{4}\frac{(2\rho+1)}{(n\rho+1)}, \ \forall x \in [0,1] \\ &= \delta^2_{n,\rho}, \text{ (say)}. \end{aligned}$$

In what follows, $\|.\|$ will denote the uniform norm on $[0,1]$.

Lemma 2.5. *For every $f \in C[0,1]$, we have*

$$|U_n^{\rho}(f;.)\| \le \|f\|.$$

Proof. Using Lemma 2.1, the proof of this Lemma easily follows. Hence the details are omitted. □

In order to discuss the approximation of functions with derivatives of bounded variation, we express the operators U_n^{ρ} in an integral form as follows:

$$U_n^{\rho}(f;x) = \int_0^1 K_n^{\rho}(x,t)f(t)dt, \tag{3}$$

where,

$$K_n^{\rho}(x,t) = \sum_{k=1}^{n-1} p_{n,k}^{(1/n)}(x)\mu_{n,k}^{\rho}(t) + p_{n,0}^{(\frac{1}{n})}\delta(t) + p_{n,n}^{(\frac{1}{n})}\delta(1-t),$$

$\delta(u)$ being the Dirac-delta function.

Lemma 2.6. *For a fixed $x \in (0,1)$ and sufficiently large n, we have*

(i) $\xi_n^{\rho}(x,y) = \int_0^y K_n^{\rho}(x,t)dt \le \frac{(2\rho+1)}{(n\rho+1)}\frac{\phi^2(x)}{(x-y)^2}$, $0 \le y < x$,

(ii) $1 - \xi_n^{\rho}(x,z) = \int_z^1 K_n^{\rho}(x,t)dt \le \frac{(2\rho+1)}{(n\rho+1)}\frac{\phi^2(x)}{(z-x)^2}$, $x < z < 1$.

Proof. (i) Using Lemma 2.3, we get

$$\begin{aligned} \xi_n^{\rho}(x,y) &= \int_0^y K_n^{\rho}(x,t)dt \le \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_n^{\rho}(x,t)dt \\ &\le U_n^{\rho}((t-x)^2;x)(x-y)^{-2} \\ &\le \frac{(2\rho+1)}{(n\rho+1)}\frac{\phi^2(x)}{(x-y)^2}. \end{aligned}$$

The proof of (ii) is similar hence the details are omitted. □

Now we present a theorem which will be needed to obtain a quantitative Voronovskaya theorem using the least concave majorant of the first order modulus of continuity

Theorem 2.7. *[11] Let $q \in \mathbb{N} \bigcup \{0\}$ and $f \in C^q[0,1]$ and let $L : C[0,1] \to C[0,1]$ be a positive linear operator. Then*

$$\left| L(f;x) - \sum_{r=0}^{q} L\Big((t-x)^r; \frac{f^r(x)}{r!}\Big) \right| \leq \frac{L(|e_1 - x|^q; x)}{q!} \widetilde{\omega}\Big(f^{(q)}; \frac{1}{(q+1)} \frac{L(|t-x|^{q+1}; x)}{L(|t-x|^q; x)}\Big),$$

where $\widetilde{\omega}$ is the least concave majorant of the first-order modulus of continuity.

3. Main Results

First we will establish a global approximation theorem for the operators $U_n^\rho(f;x)$, using the classical modulus of continuity.
Let

$$W^2 = \Big\{ g \in C[0,1] : g'' \in C[0,1] \Big\},$$

endowed with the norm

$$\|f\|_{W^2} = \|f\| + \|f'\| + \|f''\|.$$

For any $\delta > 0$, the appropriate Peetre's K-functional [34] is defined by

$$K_2(f;\delta) = \inf_{g \in W^2} \Big\{ \|f - g\| + \delta \|g''\| \Big\}. \tag{4}$$

From [8], there exists an absolute constant $C > 0$, such that

$$K_2(f;\delta) \leq C\omega_2(f; \sqrt{\delta}), \tag{5}$$

where $\omega_2(f; \sqrt{\delta})$ is the second order modulus of continuity of $f \in C[0,1]$, defined as

$$\omega_2(f; \sqrt{\delta}) = \sup_{0<|h|\leq \sqrt{\delta}} \ \sup_{x,x+h,x+2h \in [0,1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

The usual modulus of continuity of $f \in C[0,1]$ is given by

$$\omega(f; \sqrt{\delta}) = \sup_{0<|h|\leq \sqrt{\delta}} \ \sup_{x,x+h \in [0,1]} |f(x+h) - f(x)|.$$

Theorem 3.1. *Let $f \in C[0,1]$ and $x \in [0,1]$. Then there exists a constant $C > 0$, such that*

$$\|U_n^\rho(f;.) - f(.)\| \leq C\omega_2(f; \delta_{n,\rho}),$$

where $\delta_{n,\rho}$ is as defined in Remark 2.4 and $C > 0$, is an absolute constant.

Proof. Let $g \in W^2$ and $t \in [0,1]$. Then by Taylor's expansion, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Now applying $U_n^\rho(.;x)$ to both sides of the above equation, we get

$$U_n^\rho(g;x) - g(x) = g'(x)U_n^\rho(t-x;x) + U_n^\rho\Big(\int_x^t (t-u)g''(u)du; x\Big).$$

Using Remark 2.4, we get

$$\begin{aligned} |U_n^{\rho}(g;x)-g(x)| &\leq U_n^{\rho}\Big(\Big|\int_x^t |(t-u)||g''(u)|du\Big|;x\Big)\\ &\leq \frac{\|g''\|}{2}U_n^{\rho}((t-x)^2;x)\\ &\leq \frac{\|g''\|}{2}\delta_{n,\rho}^2. \end{aligned} \tag{6}$$

Now, for $f\in C[0,1]$ and $g\in W^2$, using Lemma 2.5 and (6), we obtain

$$\begin{aligned} |U_n^{\rho}(f;x)-f(x)| &\leq |U_n^{\rho}(f-g;x)|+|U_n^{\rho}(g;x)-g(x)|+|f(x)-g(x)|\\ &\leq 2\|f-g\|+\frac{\|g''\|}{2}\delta_{n,\rho}^2. \end{aligned}$$

Taking infimum on the right side of the above inequality over all $g\in W^2$, we get

$$|U_n^{\rho}(f;x)-f(x)| \leq 2K_2(f;\delta_{n,\rho}^2),\ \ \forall x\in[0,1].$$

Consequently,

$$\|U_n^{\rho}(.;x)-f(.)\| \leq 2K_2(f;\delta_{n,\rho}^2),\ \ \forall x\in[0,1].$$

Using the relation (5) between K-functional and the second order modulus of continuity, we get the required result. This completes the proof. □

Next we shall prove a local approximation theorem by using the Ditzian-Totik modulus of smoothness. Let us define the space

$$W^2(\phi)=\Big\{g\in C[0,1]: g'\in AC[0,1]\ \textit{and}\ \phi^2g''\in C[0,1]\Big\},$$

where $g'\in AC[0,1]$ means that g' is absolutely continuous in $[0,1]$. The weighted K-functional of the second order for $f\in C[0,1]$, is defined as

$$K_{2,\phi}(f,\delta^2)=\inf\Big\{\|f-g\|+\delta^2\|\phi^2g''\|: g\in W^2(\phi),\delta>0\Big\}.$$

The Ditizian-Totik modulus of smoothness of the first order is given by

$$\overrightarrow{\omega}_{\phi}(f;\delta)=\sup_{0\leq|h|\leq\delta}\ \sup_{x+h\phi(x)\in[0,1]}|f(x+h\phi(x))-f(x)|.$$

The Ditizian-Totik modulus of smoothness of the second order is given by

$$\omega_2^{\phi}(f;\delta)=\sup_{0\leq|h|\leq\delta}\ \sup_{x-h\phi(x),x+h\phi(x)\in[0,1]}|f(x+h\phi(x))-2f(x)+f(x-h\phi(x))|,$$

where ϕ is an admissible step-weight function on $[0,1]$.
We know that weighted K-functional and Ditzian-Totik modulus of smoothness of the second order are equivalent [9] i.e. there exists a constant $C>0$, such that

$$C^{-1}\omega_2^{\phi}(f;\sqrt{\delta})\leq K_{2,\phi}(f;\delta)\leq C\omega_2^{\phi}(f;\sqrt{\delta}).$$

Now we will prove a local approximation theorem for the operators $U_n^{\rho}(f;x)$.

Theorem 3.2. *Let $f \in C[0,1]$. Then for every $x \in [0,1]$, we have*

$$|U_n^{\rho}(f;x) - f(x)| \leq C\omega_2^{\phi}\Big(f; \sqrt{\frac{2\rho+1}{n\rho+1}}\Big),$$

where $C > 0$, is an absolute constant and $\phi(x) = \sqrt{x(1-x)}$.

Proof. Let $g \in W^2(\phi)$ and $t \in [0,1]$. Then by Taylor's expansion, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Now applying $U_n^{\rho}(f;x)$ to both sides of the above equation, we get

$$\begin{aligned} U_n^{\rho}(g;x) - g(x) &= g'(x)U_n^{\rho}(t-x;x) + U_n^{\rho}\Big(\int_x^t (t-u)g''(u)du;x\Big) \\ |U_n^{\rho}(g;x) - g(x)| &= \Big|U_n^{\rho}\Big(\int_x^t (t-u)g''(u)du;x\Big)\Big| \\ &\leq U_n^{\rho}\Big(\Big|\int_x^t |t-u||g''(u)|du\Big|;x\Big). \qquad (7) \end{aligned}$$

Since $\phi^2(x)$ is concave function on $[0,1]$, so for $u = \lambda x + (1-\lambda)t$, $\lambda, t, x \in (0,1)$, we get,

$$\frac{|t-u|}{\phi^2(u)} = \frac{|t-\lambda x - (1-\lambda)t|}{\phi^2(\lambda x + (1-\lambda)t)} \leq \frac{\lambda|t-x|}{\lambda\phi^2(x) + (1-\lambda)\phi^2(t)} \leq \frac{|t-x|}{\phi^2(x)}.$$

Combining this inequality and equation (7) , we obtain

$$\begin{aligned} |U_n^{\rho}(g;x) - g(x)| &\leq U_n^{\rho}\Big(\Big|\int_x^t \frac{|t-u|}{\phi^2(u)}||\phi^2 g''||du\Big|;x\Big) \\ &\leq \frac{1}{\phi^2(x)}||\phi^2 g''||U_n^{\rho}((t-x)^2;x) \\ &\leq \frac{2\rho+1}{n\rho+1}||\phi^2 g''||, \quad \text{in view of Remark 2.4.} \end{aligned}$$

Now,

$$\begin{aligned} |U_n^{\rho}(f;x) - f(x)| &\leq |U_n^{\rho}(f-g;x)| + |U_n^{\rho}(g;x) - g(x)| + |g(x) - f(x)| \\ &\leq 2||f-g|| + \frac{2\rho+1}{n\rho+1}||\phi^2 g''|| \text{ using Lemma 2.5} \\ &\leq 2K_{2,\phi}\Big(f; \frac{2\rho+1}{n\rho+1}\Big) \\ &\leq C\omega_2^{\phi}(f; \sqrt{\frac{2\rho+1}{n\rho+1}}\Big). \end{aligned}$$

This completes the proof. □

Now we will establish a Voronovskaya-type asymptotic formula for the operators $U_n^{\rho}(f;x)$.

Theorem 3.3. *Let $f \in L_B[0,1]$. If f'' exists at a point $x \in [0,1]$, then*

$$\lim_{n\to\infty} n[U_n^{\rho}(f;x) - f(x)] = \frac{2\rho+1}{2\rho}\phi^2(x)f''(x). \qquad (8)$$

The convergence in (8) holds uniformly if $f'' \in C[0,1]$.

Proof. By Taylor's expansion for the function f, we may write

$$f(t)-f(x)=(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+\eta(t,x)(t-x)^2,$$

where $\eta(t,x)\to 0$ as $t\to x$ and is a bounded function, $\forall t\in[0,1]$. Now, applying U_n^{ρ} on the above Taylor's expansion and using Lemma 2.3, we get

$$\begin{aligned} U_n^{\rho}(f;x)-f(x) &= U_n^{\rho}((t-x)f'(x);x)+U_n^{\rho}\Big(\frac{(t-x)^2}{2}f''(x);x\Big)+U_n^{\rho}\Big(\eta(t,x)(t-x)^2;x\Big)\\ &= \frac{f''(x)}{2}\frac{(2n\rho+n+1)}{(n+1)(n\rho+1)}x(1-x)+U_n^{\rho}\Big(\eta(t,x)(t-x)^2;x\Big). \end{aligned}$$

Hence,

$$\lim_{n\to\infty} n[U_n^{\rho}(f;x)-f(x)] = \frac{(2\rho+1)}{2\rho}\phi^2(x)f''(x)+\lim_{n\to\infty} nU_n^{\rho}\Big(\eta(t,x)(t-x)^2;x\Big).$$

Let $F=\lim_{n\to\infty} nU_n^{\rho}\Big(\eta(t,x)(t-x)^2;x\Big)$. We shall show that $F=0$. Since $\epsilon(t,x)\to 0$ as $t\to x$, so for a given $\epsilon>0$, there exists a $\delta>0$, such that $|\eta(t,x)|<\epsilon$ whenever $|t-x|<\delta$. For $|t-x|\geq\delta$, the boundedness of $\eta(t,x)$ on $[0,1]$ implies that $|\eta(t,x)|\leq M\frac{(t-x)^2}{\delta^2}$, for some $M>0$. Let $\chi_\delta(t)$ be the characteristic function of the interval $(x-\delta,x+\delta)$. Then, from Lemma 2.3, for every $x\in[0,1]$, we have

$$\begin{aligned} \Big|U_n^{\rho}\Big(\eta(t,x)(t-x)^2;x\Big)\Big| &\leq U_n^{\rho}\Big(|\eta(t,x)|(t-x)^2\chi_\delta(t);x\Big)+U_n^{\rho}\Big(|\eta(t,x)|(t-x)^2(1-\chi_\delta(t));x\Big)\\ &\leq \epsilon U_n^{\rho}\Big((t-x)^2;x\Big)+\frac{M}{\delta^2}U_n^{\rho}\Big((t-x)^4;x\Big)\\ &= \epsilon O\Big(\frac{1}{n}\Big)+\frac{M}{\delta^2}O\Big(\frac{1}{n^2}\Big), \end{aligned}$$

Thus, for every $x\in[0,1]$, we get

$$n\Big|U_n^{\rho}\Big(\eta(t,x)(t-x)^2;x\Big)\Big| = \epsilon O(1)+\frac{M}{\delta^2}O\Big(\frac{1}{n}\Big).$$

Taking limit as $n\to\infty$, due to the arbitrariness of $\epsilon>0$, we get $F=0$. This completes the proof of the first assertion of the theorem.
To prove the uniformity assertion, it is sufficient to remark that $\delta(\epsilon)$ in the above proof can be chosen to be independent of $x\in[0,1]$ and all the other estimates hold uniformly on $[0,1]$. This completes the proof. □

In the next result we establish a quantitative-Voronovskaya type estimate for the operators U_n^{ρ}.

Theorem 3.4. *For $f\in C^2[0,1]$ and $x\in[0,1]$, we have*

$$\Big|U_n^{\rho}(f;x)-f(x)-\frac{f''(x)}{2!}(t-x)^2\Big|\leq\frac{1}{2!}\frac{(2\rho+1)}{(n\rho+1)}\phi^2(x)\widetilde{\omega}\Big(f'';\frac{M}{3\sqrt{n}}\Big),$$

where, $M>0$ and for any $f\in C[0,1]$, $\widetilde{\omega}(f;.)$ is the least concave majorant of first order of the function $\omega(f;.)$, defined as (see[11], Thm 2.1)

$$\widetilde{\omega}(f;\epsilon)=\begin{cases}\sup_{0\leq x\leq\epsilon\leq y\leq 1}\frac{(\epsilon-x)\omega(f;y)+(y-x)\omega(f;x)}{y-x}, & 0\leq\epsilon\leq 1,\\ \omega(f;1), & \epsilon>1.\end{cases}$$

Proof. Using the Cauchy–Schwarz inequality, we note that

$$\frac{U_n^\rho((t-x)^3;x)}{U_n^\rho((t-x)^2;x)} \leq \sqrt{\frac{U_n^\rho((t-x)^4;x)}{U_n^\rho((t-x)^2;x)}}. \tag{9}$$

For $q = 2$, using Theorem 2.7 and equation (9), we get

$$\begin{aligned}\left|U_n^\rho(f;x) - f(x) - \frac{f''(x)}{2!}U_n^\rho((t-x)^2;x)\right| &\leq \frac{U_n^\rho((t-x)^2;x)}{2!}\widetilde{\omega}\Big(f''; \frac{1}{3}\frac{U_n^\rho(|t-x|^3;x)}{U_n^\rho((t-x)^2;x)}\Big)\\ &\leq \frac{U_n^\rho((t-x)^2;x)}{2!}\widetilde{\omega}\Big(f''; \frac{1}{3}\sqrt{\frac{U_n^\rho((t-x)^4;x)}{U_n^\rho((t-x)^2;x)}}\Big)\\ &\leq \frac{1}{2!}\frac{(2\rho+1)}{(n\rho+1)}\phi^2(x)\widetilde{\omega}\Big(f''; \frac{M}{3\sqrt{n}}\Big).\end{aligned}$$

This completes the proof. □

Let $DBV[0,1]$ denote the class of all absolutely continuous functions f defined on $[0,1]$, having a derivative f' equivalent with a function of bounded variation on $[0,1]$. We observe that the functions $f \in DBV[0,1]$ possess a representation

$$f(x) = \int_0^x g(t)dt + f(0),$$

where $g \in BV[0,1]$, i.e., g is a function of bounded variation on $[0,1]$.

Theorem 3.5. *Let $f \in DBV([0,1])$. Then, for every $x \in (0,1)$ and sufficiently large n, we have*

$$\begin{aligned}|U_n^\rho(f;x) - f(x)| \leq\ & \sqrt{\frac{2\rho+1}{(n\rho+1)}}\phi(x)\left|\frac{f'(x+)-f'(x-)}{2}\right| + \frac{2\rho+1}{(n\rho+1)}\phi^2(x)x^{-1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}((f')_x)\\ &+\frac{2\rho+1}{(n\rho+1)}\frac{\phi^2(x)}{(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+(1-x)/k}((f')_x) + \frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+(1-x)/\sqrt{n}}((f')_x),\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of $f(x)$ on $[a,b]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+) & x < t < 1. \end{cases} \tag{10}$$

Proof. Since $U_n^\rho(1;x) = 1$, using (3), for every $x \in (0,1)$ we get

$$\begin{aligned}U_n^\rho(f;x) - f(x) &= \int_0^1 K_n^\rho(x,t)(f(t)-f(x))dt\\ &= \int_0^1 K_n^\rho(x,t)\int_x^t f'(u)dudt. \end{aligned}\tag{11}$$

For any $f \in DBV[0,1]$, from (10) we may write

$$\begin{aligned}f'(u) &= (f')_x(u) + \frac{1}{2}(f'(x+)+f'(x-)) + \frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)\\ &\quad+\delta_x(u)[f'(u) - \frac{1}{2}(f'(x+)+f'(x-))], \end{aligned}\tag{12}$$

where

$$\delta_x(u) = \begin{cases} 1 \;, \; u = x \\ 0 \;, \; u \neq x \end{cases} .$$

Obviously,

$$\int_0^1 \Big(\int_x^t \Big(f'(u) - \frac{1}{2}(f'(x+) + f'(x-))\Big)\delta_x(u)du\Big)K_n^{\rho}(x,t)dt = 0. \tag{13}$$

Using Lemma 2.3, we get

$$\begin{aligned} \int_0^1 \Big(\int_x^t \frac{1}{2}(f'(x+) + f'(x-))du\Big)K_n^{\rho}(x,t)dt &= \frac{1}{2}(f'(x+) + f'(x-)) \int_0^1 (t-x)K_n^{\rho}(x,t)dt \\ &= \frac{1}{2}(f'(x+) + f'(x-))U_n^{\rho}((t-x);x) \\ &= 0. \end{aligned} \tag{14}$$

And applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_0^1 K_n^{\rho}(x,t)\Big(\int_x^t \frac{1}{2}(f'(x+) - f'(x-))sgn(u-x)du\Big)dt &\leq \frac{1}{2} \mid f'(x+) - f'(x-) \mid \int_0^1 |t-x|K_n^{\rho}(x,t)dt \\ &= \frac{1}{2} \mid f'(x+) - f'(x-) \mid U_n^{\rho}(|t-x|;x) \\ &\leq \frac{1}{2} \mid f'(x+) - f'(x-) \mid \Big(U_n^{\rho}((t-x)^2;x)\Big)^{1/2}. \end{aligned} \tag{15}$$

Using Lemma 2.3 and equations (11 -15) , we obtain

$$\begin{aligned} |U_n^{\rho}(f;x) - f(x)| \leq\;& \frac{1}{2}|f'(x+) - f'(x-)| \sqrt{\frac{2\rho+1}{(n\rho+1)}}\phi(x) \\ &+ \Big| \int_0^x \int_x^t ((f')_x(u)du)K_n^{\rho}(x,t)dt + \int_x^1 \int_x^t \Big((f')_x(u)du\Big)K_n^{\rho}(x,t)dt \Big|. \end{aligned} \tag{16}$$

Now, let

$$A_n^{\rho}(f',x) = \int_0^x \int_x^t ((f')_x(u)du)K_n^{\rho}(x,t)dt,$$

and

$$B_n^{\rho}(f',x) = \int_x^1 \int_x^t ((f')_x(u)du)K_n^{\rho}(x,t)dt.$$

Thus our problem is reduced to calculate the estimates of the terms $A_n^{\rho}(f',x)$ and $B_n^{\rho}(f',x)$. Since $\int_a^b d_t\xi_n^{\rho}(x,t) \leq 1$ for all $[a,b] \subseteq [0,1]$, using integration by parts and applying Lemma 2.6 with $y = x - x/\sqrt{n}$, we have

$$\begin{aligned} |A_n^{\rho}(f',x)| &= \Big| \int_0^x \int_x^t \Big((f')_x(u)du\Big)d_t\xi_n^{\rho}(x,t) \Big| \\ &= \Big| \int_0^x \xi_n^{\rho}(x,t)(f')_x(t)dt \Big| \\ &\leq \int_0^y |(f')_x(t)| \, |\xi_n^{\rho}(x,t)|dt + \int_y^x |(f')_x(t)| \, |\xi_n^{\rho}(x,t)|dt \\ &\leq \frac{2\rho+1}{n\rho+1}\phi^2(x) \int_0^y \bigvee_t^x ((f')_x)(x-t)^{-2}dt + \int_y^x \bigvee_t^x ((f')_x)dt \end{aligned}$$

$$\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_0^y \bigvee_t^x((f^{'})_x)(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}((f')_x)$$

$$= \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x((f^{'})_x)(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}((f')_x)$$

Substituting $u = x/(x-t)$, we get

$$\begin{aligned}\frac{2\rho+1}{n\rho+1}\phi^2(x)\int_0^{x-x/\sqrt{n}}(x-t)^{-2}\bigvee_t^x(f^{'})_x)dt &= \frac{2\rho+1}{n\rho+1}\phi^2(x)x^{-1}\int_1^{\sqrt{n}}\bigvee_{x-x/u}^{x}((f^{'})_x)du\\ &\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)x^{-1}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-x/k}^{x}((f^{'})_x)du\\ &\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)x^{-1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x}((f')_x).\end{aligned}$$

Thus,

$$|A_n^{\rho}(f',x)| \leq \frac{2\rho+1}{n\rho+1}\phi^2(x)x^{-1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x}((f^{'})_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}((f^{'})_x). \tag{17}$$

Again, using integration by parts in $B_n^{\rho}(f^{'},x)$ and applying Lemma 2.6 with $z = x+(1-x)/\sqrt{n}$, we have

$$\begin{aligned}|B_n^{\rho}(f^{'},x)| &= \left|\int_x^1\int_x^t((f')_x(u)du)K_n^{\rho}(x,t)dt\right|\\ &= \left|\int_x^z\int_x^t((f')_x(u)du)d_t(1-\xi_n^{\rho}(x,t))+\int_z^1\int_x^t((f')_x(u)du)d_t(1-\xi_n^{\rho}(x,t))\right|\\ &= \left|\left[\int_x^t((f')_x(u)du)(1-\xi_n^{\rho}(x,t))\right]_x^z-\int_x^z(f')_x(t)(1-\xi_n^{\rho}(x,t))dt\right.\\ &\quad\left.+\int_z^1\int_x^t((f')_x(u)du)d_t(1-\xi_n^{\rho}(x,t))\right|\\ &= \left|\int_x^z((f')_x(u)du)(1-\xi_n^{\rho}(x,z))-\int_x^z(f')_x(t)(1-\xi_n^{\rho}(x,t))dt+\left[\int_x^t((f')_x(u)du)(1-\xi_n^{\rho}(x,t))\right]_z^1\right.\\ &\quad\left.-\int_z^1(f')_x(t)(1-\xi_n^{\rho}(x,t))dt\right|\\ &= \left|\int_x^z(f')_x(t)(1-\xi_n^{\rho}(x,t))dt+\int_z^1(f')_x(t)(1-\xi_n^{\rho}(x,t))dt\right|\\ &\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_z^1\bigvee_x^t(f')_x(t-x)^{-2}dt+\int_x^z\bigvee_x^t(f')_xdt\\ &= \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_{x+(1-x)/\sqrt{n}}^1\bigvee_x^t(f')_x(t-x)^{-2}dt+\frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+(1-x)/\sqrt{n}}(f')_x.\end{aligned}$$

By substituting $u = (1-x)/(t-x)$, we get

$$\begin{aligned}
|B_n^{\rho}(f^{'},x)| &\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_1^{\sqrt{n}} \bigvee_x^{x+(1-x)/u}(f')_x(1-x)^{-1}du + \frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+(1-x)/\sqrt{n}}(f')_x \\
&\leq \frac{2\rho+1}{n\rho+1}\frac{\phi^2(x)}{(1-x)}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_x^{x+(1-x)/k}(f')_x du + \frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+(1-x)/\sqrt{n}}(f')_x \\
&= \frac{2\rho+1}{n\rho+1}\frac{\phi^2(x)}{(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+(1-x)/k}((f')_x) + \frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+(1-x)/\sqrt{n}}((f')_x). \qquad (18)
\end{aligned}$$

Collecting the estimates (16- 18), we get the required result. This completes the proof. □

Acknowledgement: The author are extremely grateful to the reviewers for a very careful reading of the manuscript and making valuable suggestions leading to a better presentation of the paper. The author are also thankful to Prof. H. M. Srivastava for sending the report of the paper timely.
The first author is thankful to the "Ministry of Human Resource Development" India for financial support to carry out the above research work.

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