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The Drazin Inverse of the Sum of Two Bounded Linear Operators and it's Applications

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Abstract. Let P and Q be bounded linear operators on a Banach space. The existence of the Drazin inverse of $P+Q$ is proved under some assumptions, and the representations of $(P+Q)^D$ are also given. The results recover the cases $P^2Q = 0, QPQ = 0$ studied by Yang and Liu in [19] for matrices, $Q^2P = 0, PQP = 0$ studied by Cvetković and Milovanović in [7] for operators and $P^2Q + QPQ = 0$, $P^3Q = 0$ studied by Shakoor, Yang and Ali in [16] for matrices. As an application, we give representations for the Drazin inverse of the operator matrix $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

1. Introduction

Let $\mathcal{X}$ be a Banach space. The set $\mathcal{B}(\mathcal{X})$ consists of all bounded linear operators on $\mathcal{X}$. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be Drazin invertible, if there exists an operator $T^D \in \mathcal{B}(\mathcal{X})$ such that

$$TT^D = T^DT, \quad T^D = T(T^D)^2, \quad T^{k+1}T^D = T^k \text{ for some integer k} \geq 0,$$

where T^D is called the Drazin inverse of T. The smallest integer k satisfying the previous system of equations is called the index of T, and is denoted by $\text{ind}(T)$. In particular, if $\text{ind}(T) = 1$, T^D is called the group inverse of T; if $\text{ind}(T) = 0$, it can be seen that T is invertible and $T^D = T^{-1}$. Note that T^D may not exist, but T^D must be unique if it exists. Moreover, if T is nilpotent, then T is Drazin invertible, and $T^D = 0$.

The Drazin inverse has become a useful tool in the researches of Markov chains, differential and difference equations, optimal control and iterative methods[1, 3].

In [11], M. P. Drazin proves that $(P+Q)^D = P^D + Q^D$ if $PQ = QP = 0$ in an associative ring. In the sequel, many authors begin to consider this problem for matrices and operators, and present explicit representations of $(P+Q)^D$ under the conditions such as

(1) $PQ = QP = 0$ (see [11]),
(2) $PQ = 0$ (see [9, 12]),
(3) $P^2Q = PQ^2 = 0$ (see [5]),

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(4) $P^2Q + PQ^2 = 0,\ P^3Q = PQ^3 = 0$ (see [13]),
(5) $PQP = 0, Q^2P = 0$ (or $QPQ = 0, P^2Q = 0$) (see [7, 19]),
(6) $P^2Q + QPQ = 0, P^3Q = 0$ (see [16]),
(7) $P^2QP = P^2Q^2 = PQ^2P = PQ^3 = 0$ (see [17]),
(8) $P^DQ = PQ^D = 0, Q^\pi PQP^\pi = 0$ (see [6]).

For more general Drazin inverse problems, we refer the reader to [2, 4, 14] and their references. Note that the representation of $(P + Q)^D$ by P, Q, P^D and Q^D is very difficult without any conditions.

In this paper, using the technique of the resolvent expansion, we investigate the existence of the Drazin inverse of $P + Q$ for bounded linear operators P and Q and the explicit representations of $(P + Q)^D$ in term of P, P^D, Q and Q^D under the conditions (1) $P^2Q + QPQ = 0, P^nQ = 0$, (2) $PQ^2 + PQP = 0\ PQ^n = 0$ for some integer n, respectively, which extend the relevant results in [7, 12, 16, 19]. Then, we apply these results to establish representations of the Drazin inverse of the operator matrix, which can be regarded as the generalizations of some results given in [10, 16]. Actually, the proof of the main results show the efficiency of the method employed to some extent.

Throughout this paper, we write $\rho(T)$, $\sigma(T)$ and $\mathrm{r}(T)$ for the resolvent set, the spectrum and the spectral radius of the operator T. Write $T^\pi = I - TT^D$.

Before giving our main results, we state some auxiliary lemmas as follows.

Lemma 1.1. [4] *Let $T \in \mathcal{B}(\mathcal{X})$, then T is Drazin invertible if and only if $0 \notin \overline{\sigma(T) \setminus \{0\}}$ and the point zero, provided $0 \in \sigma(T)$, is a pole of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$, and in this case the following representation holds:*

$$R(\lambda, T) = \sum_{k=1}^{\mathrm{ind}(T)} \lambda^{-k} T^{k-1} T^\pi - \sum_{k=0}^{\infty} \lambda^k (T^D)^{k+1}, \tag{1}$$

where $0 < |\lambda| < (\mathrm{r}(T^D))^{-1}$.

Remark 1.2. *From* Lemma 1.1, *T^D can be obtained by the coefficient at λ^0 in the Laurent expansion of the resolvent $R(\lambda, T)$ in a punctured neighborhood of* 0, *i.e,*

$$T^D = -\frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda} R(\lambda, T) d\lambda, \tag{2}$$

where $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = \varepsilon\}$ *with ε being sufficiently small such that* $\{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\} \cap \sigma(T) = \{0\}$.

Lemma 1.3. [18] *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. If BA is Drazin invertible, then AB is also Drazin invertible. Moreover,*

$$(AB)^D = A((BA)^D)^2 B, \quad \mathrm{ind}(AB) \leq \mathrm{ind}(BA) + 1. \tag{3}$$

Lemma 1.4. *For the operator matrix $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{B}(\mathcal{Y})$. If A is invertible, then $\mathcal{A}$ is invertible if and only if $D - CA^{-1}B$ is invertible.*

Remark 1.5. *The Lemma above is well known, see, e.g.,* [15, Lemma 2.1].

2. Main Results

In this section, we investigate the Drazin inverse of the sum of two operators $P, Q \in \mathcal{B}(\mathcal{X})$. It is interesting that the conditions when $n \geq 2$ will share the same representation of the Drazin inverse of $P + Q$.

In order to show that $P + Q$ is Drazin invertible, we need to find out the resolvent of the operator matrix $M = \begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}$ defined on the Banach space $\mathcal{X} \times \mathcal{X}$. Write $\Delta(\lambda) = \lambda I - Q - R(\lambda, P)PQ$. Then, the following two lemmas are necessary.

Lemma 2.1. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible, $r = \text{ind}(P)$ and $s = \text{ind}(Q)$. If $P^2Q + QPQ = 0$ and $P^nQ = 0$ for some integer $n > 0$, then*

$$\Delta(\lambda)^{-1} = \lambda^{-2}(\lambda^2 I + PQ)R(\lambda, Q), \tag{4}$$

where $0 < |\lambda| < \min\{(r(P^D))^{-1}, (r(Q^D))^{-1}\}$.

Proof. From $P^nQ = 0$ and $P^D = (P^D)^2P$, it follows that $P^DQ = 0$, then $P^mQ = 0$ if the integer $m \geq r$. Moreover, $P^\pi PQ = PQ$. By $P^2Q + QPQ = 0$, we have

$$P^{2k-1}Q = (-1)^{k-1}(PQ)^k, \quad P^{2k}Q = (-1)^kQ(PQ)^k, \quad k = 1, 2, \cdots. \tag{5}$$

Since there always exists an integer k_0 such that $2^{k_0} \leq n \leq 2^{k_0+1} - 1$ for each n, we deduce $P^{2^{k_0+1}-1}Q = 0$ from $P^nQ = 0$. This together with Eq.(5) shows that PQ is Drazin invertible, $(PQ)^D = 0$ and $\text{ind}(PQ) \leq 2^{k_0}$. Thus, using Lemma 1.1, we conclude that

$$\begin{aligned} R(\lambda, P)PQ &= \left(\sum_{k=1}^{r} \lambda^{-k}P^{k-1}P^\pi - \sum_{k=0}^{\infty} \lambda^k (P^D)^{k+1}\right)PQ \\ &= \sum_{k=1}^{r} \lambda^{-k}P^kQ \\ &= \sum_{k=1}^{2^{k_0+1}-2} \lambda^{-k}P^kQ \\ &= (\lambda I - Q)\sum_{k=1}^{2^{k_0}-1}(-1)^{k-1}\lambda^{-2k}(PQ)^k \\ &= (\lambda I - Q)PQR(\lambda^2, -PQ), \end{aligned} \tag{6}$$

where $0 < |\lambda| < (\text{r}(P^D))^{-1}$. Then,

$$\begin{aligned} \Delta(\lambda) &= \lambda I - Q - R(\lambda, P)PQ \\ &= (\lambda I - Q)(I - PQR(\lambda^2, -PQ)) \\ &= \lambda^2(\lambda I - Q)R(\lambda^2, -PQ). \end{aligned}$$

Therefore, we have

$$\Delta(\lambda)^{-1} = \lambda^{-2}(\lambda^2 I + PQ)R(\lambda, Q),$$

where $0 < |\lambda| < \min\{(\text{r}(P^D))^{-1}, (\text{r}(Q^D))^{-1}\}$. □

Lemma 2.2. *Under the assumptions of Lemma 2.1, the representation of the resolvent for the operator matrix* $M = \begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}$ *is given by*

$$R(\lambda, M) = \begin{pmatrix} \lambda^{-2}(\lambda I - Q)(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P) & \lambda^{-2}(\lambda I - Q)PQR(\lambda, Q) \\ \lambda^{-2}(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P) & \lambda^{-2}(\lambda^2 I + PQ)R(\lambda, Q) \end{pmatrix}, \tag{7}$$

where $0 < |\lambda| < \min\{(\text{r}(P^D))^{-1}, (\text{r}(Q^D))^{-1}\}$.

Proof. Let $\rho(\Delta)$ denote the set of all $\lambda \in \mathbb{C}$ such that $\Delta(\lambda)$ is invertible in $\mathcal{B}(\mathcal{X})$. By Lemma 1.4, we obtain $\rho(M) \cap \rho(P) = \rho(P) \cap \rho(\Delta)$. If $\lambda \in \rho(M) \cap \rho(P)$, then

$$R(\lambda, M) = \begin{pmatrix} R(\lambda, P) + R(\lambda, P)PQ\Delta(\lambda)^{-1}R(\lambda, P) & R(\lambda, P)PQ\Delta(\lambda)^{-1} \\ \Delta(\lambda)^{-1}R(\lambda, P) & \Delta(\lambda)^{-1} \end{pmatrix},$$

where $0 < |\lambda| < \min\{(\mathrm{r}(P^D))^{-1}, (\mathrm{r}(Q^D))^{-1}\}$. By (4) and (6), we immediately have the expression

$$\begin{aligned} R(\lambda, P)PQ\Delta(\lambda)^{-1} &= (\lambda I - Q)PQR(\lambda^2, -PQ)\Delta(\lambda)^{-1} \\ &= \lambda^{-2}(\lambda I - Q)PQR(\lambda, Q). \end{aligned}$$

Then, we further have

$$\begin{aligned} & R(\lambda, P) + R(\lambda, P)PQ\Delta(\lambda)^{-1}R(\lambda, P) \\ = \ & (I + R(\lambda, P)PQ\Delta(\lambda)^{-1})R(\lambda, P) \\ = \ & (I + \lambda^{-2}(\lambda I - Q)PQR(\lambda, Q))R(\lambda, P) \\ = \ & \lambda^{-2}(\lambda I - Q)(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P). \end{aligned}$$

Moreover,

$$\Delta(\lambda)^{-1}R(\lambda, P) = \lambda^{-2}(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P).$$

The proof is completed. □

We will give other two necessary lemmas in order to obtain the representation of $(P + Q)^D$.

Lemma 2.3. *Under the assumptions of Lemma* 2.1*, the following statements are true:*
(1) *The coefficients α_i at λ^i ($i = -1, 0, 1, 2$) of $R(\lambda, Q)R(\lambda, P)$ are given by*

$$\begin{aligned} \alpha_{-1} &= -(Q^\pi \delta P^D + Q^D \tau P^\pi), \\ \alpha_0 &= -(Q^\pi \delta (P^D)^2 + (Q^D)^2 \tau P^\pi) + Q^D P^D, \\ \alpha_1 &= -(Q^\pi \delta (P^D)^3 + (Q^D)^3 \tau P^\pi) + Q^D (P^D)^2 + (Q^D)^2 P^D, \\ \alpha_2 &= -(Q^\pi \delta (P^D)^4 + (Q^D)^4 \tau P^\pi) + Q^D (P^D)^3 + (Q^D)^2 (P^D)^2 + (Q^D)^3 P^D, \end{aligned} \tag{8}$$

where $\delta = \sum_{k=0}^{s-1} Q^k (P^D)^k$, $\tau = \sum_{k=0}^{r-1} (Q^D)^k P^k$.

(2) *$\alpha_{-1} - Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2$ and $\alpha_0 + PQ\alpha_2$ are the coefficients at λ^2 of $(\lambda I - Q)(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P)$ and $(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P)$, respectively.*

(3) *$-PQ^D - P^2(Q^D)^2$ and $-Q^D - P(Q^D)^2$ are the coefficients at λ^2 of $(\lambda I - Q)PQR(\lambda, Q)$ and $(\lambda^2 I + PQ)R(\lambda, Q)$, respectively.*

Proof. (1) Note that P, Q are Drazin invertible. Applying Eq. (1) for P, Q in a punctured neighborhood of 0, we have

$$R(\lambda, Q) = \sum_{k=1}^{s} \lambda^{-k} Q^{k-1} Q^\pi - \sum_{k=0}^{\infty} \lambda^k (Q^D)^{k+1}$$

and

$$R(\lambda, P) = \sum_{k=1}^{r} \lambda^{-k} P^{k-1} P^\pi - \sum_{k=0}^{\infty} \lambda^k (P^D)^{k+1}.$$

Then the coefficients α_i at λ^i ($i = -1, 0, 1, 2$) of $R(\lambda, Q)R(\lambda, P)$ can be easily obtained.

(2) Since

$$(\lambda I - Q)(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P) = (\lambda^3 I - \lambda^2 Q + \lambda PQ - QPQ)R(\lambda, Q)R(\lambda, P).$$

Thus, by Lemma 2.3 (1), $\alpha_{-1} - Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2$ is the coefficient at λ^2 of $(\lambda I - Q)(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P)$.

Analogously, (3) can be proved. □

Lemma 2.4. *Under the assumptions of Lemma* 2.1*, the following statements are valid:*

(1) $\tau Q = Q$, *and hence* $\tau P^2 Q^2 = P^2 Q^2$.

(2) $\tau PQ = PQ + Q^D P^2 Q$, *and hence* $\tau PQ^D = PQ^D + Q^D P^2 Q^D$.

(3) $\tau\delta = \tau + \delta - I$.

(4) $\alpha_{-1}PQ = \alpha_0 PQ = \alpha_1 PQ = Q\alpha_2 PQ = 0$.

(5) $\alpha_{-1}Q = -Q^D Q$, $\alpha_i Q = -(Q^D)^{i+1}$, $i = 0, 1, 2, 3$.

(6) $\alpha_i \alpha_{-1} = -\alpha_{i+1}$, $i = -1, 0, 1, 2$.

(7) $\alpha_i P^2 (Q^D)^2 = -(Q^D)^{i+2} P^2 (Q^D)^2$, $i = -1, 0, 1, 2$.

Here

$$\begin{aligned} \alpha_3 &= -(Q^\pi \delta (P^D)^5 + (Q^D)^5 \tau P^\pi) \\ &\quad + Q^D (P^D)^4 + (Q^D)^2 (P^D)^3 + (Q^D)^4 P^D + (Q^D)^3 (P^D)^2, \end{aligned} \tag{9}$$

and δ, τ *are defined as in Lemma* 2.3.

Proof. (1) By $\tau = \sum_{k=0}^{r-1} (Q^D)^k P^k$, we have $\tau Q = \sum_{k=0}^{r-1} (Q^D)^k P^k Q$. If r is odd, then, by Eq.(5), we get

$$\begin{aligned} \tau Q &= Q + \sum_{k=1}^{\frac{r-1}{2}} ((Q^D)^{2k-1} P^{2k-1} Q + (Q^D)^{2k} P^{2k} Q) \\ &= Q + \sum_{k=1}^{\frac{r-1}{2}} ((-1)^{k-1} (Q^D)^{2k-1} (PQ)^k + (-1)^k (Q^D)^{2k} Q (PQ)^k) \\ &= Q + \sum_{k=1}^{\frac{r-1}{2}} ((-1)^{k-1} (Q^D)^{2k-1} (PQ)^k + (-1)^k (Q^D)^{2k-1} (PQ)^k) \\ &= Q. \end{aligned}$$

If r is even, then

$$\begin{aligned} \tau Q &= Q + \sum_{k=1}^{\frac{r-2}{2}} ((Q^D)^{2k-1} P^{2k-1} Q + (Q^D)^{2k} P^{2k} Q) + (Q^D)^{r-1} P^{r-1} Q \\ &= Q + (Q^D)^{r-1} P^{r-1} Q \\ &= Q + (-1)^{\frac{r}{2}-1} (Q^D)^{r-1} (PQ)^{\frac{r}{2}} \\ &= Q + (-1)^{\frac{r}{2}-1} (Q^D)^r Q (PQ)^{\frac{r}{2}} \\ &= Q - (Q^D)^r P^r Q \\ &= Q \end{aligned}$$

since $P^r Q = P^{r+1} P^D Q = 0$, and hence $\tau Q = Q$. Thus, (1) is proved.

(2) Obviously, $\tau PQ = \sum_{n=0}^{r-1}(Q^D)^n P^n PQ$. If r is even, then

$$\begin{aligned}
\tau PQ &= PQ + Q^D P^2 Q + \sum_{k=1}^{\frac{r}{2}-1}((Q^D)^{2k}P^{2k+1}Q + (Q^D)^{2k+1}P^{2k+2}Q)\\
&= PQ + Q^D P^2 Q + \sum_{k=1}^{\frac{r}{2}-1}((-1)^k(Q^D)^{2k}(PQ)^{k+1} + (-1)^{k+1}(Q^D)^{2k+1}Q(PQ)^{k+1})\\
&= PQ + Q^D P^2 Q + \sum_{k=1}^{\frac{r}{2}-1}((-1)^k(Q^D)^{2k}(PQ)^{k+1} + (-1)^{k+1}(Q^D)^{2k}(PQ)^{k+1})\\
&= PQ + Q^D P^2 Q.
\end{aligned}$$

Similarly, if r is odd, then

$$\begin{aligned}
\tau PQ &= PQ + Q^D P^2 Q + \sum_{k=1}^{\frac{r-3}{2}}((Q^D)^{2k}P^{2k+1}Q + (Q^D)^{2k+1}P^{2k+2}Q) + (Q^D)^{r-1}P^r Q\\
&= PQ + Q^D P^2 Q + (Q^D)^{r-1}P^r Q\\
&= PQ + Q^D P^2 Q.
\end{aligned}$$

Therefore, the relation $\tau PQ = PQ + Q^D P^2 Q$ is proved.

On the other hand, by $P^2Q = -QPQ$, it is obvious that

$$\tau P^2 Q^2 = -\tau QPQ^2 = -QPQ^2 = P^2Q^2.$$

(3) In view of $\tau Q = Q$, we clearly have

$$\begin{aligned}
\tau\delta &= \tau\sum_{k=0}^{s-1}Q^k(P^D)^k\\
&= \tau + (QP^D + Q^2(P^D)^2 + \cdots + Q^{s-1}(P^D)^{s-1})\\
&= \tau + \delta - I.
\end{aligned}$$

(4) We only prove $\alpha_{-1}PQ = 0$, and the proof of others are similar.

Since $P^\pi PQ = PQ$, $\tau PQ = PQ + Q^D P^2 Q$ and $P^2Q + QPQ = 0$, it follows that

$$\begin{aligned}
\alpha_{-1}PQ &= -Q^D\tau PQ\\
&= -Q^D(PQ + Q^D P^2 Q)\\
&= -Q^D PQ + (Q^D)^2 QPQ\\
&= 0.
\end{aligned}$$

(5) The conclusion can be immediately obtained from $P^D Q = 0$, $P^\pi Q = Q$ and $\tau Q = Q$.

(6) We only prove the case $i = -1$, and other cases are similar.

Note that $P^D Q^\pi = P^D$, $P^\pi Q^D = Q^D$, $P^D Q^D = 0$ and $P^\pi Q^\pi = P^\pi - QQ^D$, so

$$\begin{aligned}
\alpha_{-1}\alpha_{-1} &= (Q^\pi\delta P^D + Q^D\tau P^\pi)^2,\\
&= Q^\pi\delta P^D\delta P^D + (Q^D)\tau Q^D\tau P^\pi + Q^D\tau P^\pi\delta P^D - Q^D\tau QQ^D\delta P^D.
\end{aligned}$$

On the other hand, the relation $P^DQ = 0$ implies $P^D\delta = P^D$ and $P^\pi\delta = \delta - PP^D$. Also, $\tau Q^D = Q^D$ can be obtained based on $\tau Q = Q$. Therefore, we have

$$\begin{aligned}\alpha_{-1}\alpha_{-1} &= Q^\pi\delta(P^D)^2 + (Q^D)^2\tau P^\pi + Q^D\tau(\delta - PP^D)P^D - Q^DQQ^D\delta P^D\\ &= Q^\pi\delta(P^D)^2 + (Q^D)^2\tau P^\pi - Q^DP^D\\ &= -\alpha_0,\end{aligned}$$

since, by Lemma 2.4 (3),

$$\begin{aligned}Q^D\tau(\delta - PP^D)P^D &= Q^D(\tau\delta - \tau PP^D)P^D\\ &= Q^D(\tau + \delta - I - \tau PP^D)P^D\\ &= Q^D(\tau + \delta - I)P^D - Q^D\tau P^D\\ &= Q^D(\delta - I)P^D.\end{aligned}$$

(7) Note that $\tau P^2Q^2 = P^2Q^2$. Then, the claim follows from $P^DQ^D = 0$ and $P^\pi P^2(Q^D)^2 = P^2(Q^D)^2$. □

The following is the main result of this section.

Theorem 2.5. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible, $r = \mathrm{ind}(P)$ and $s = \mathrm{ind}(Q)$. If $P^2Q + QPQ = 0$ and $P^nQ = 0$ for some integer $n > 0$, then $P + Q$ is Drazin invertible, and*

$$(P + Q)^D = -\alpha_0P - PQ\alpha_2P + P(Q^D)^2 + Q^D, \tag{10}$$

i.e.,

$$\begin{aligned}(P + Q)^D &= Q^\pi\sum_{i=0}^{s-1}Q^i(P^D)^{i+1} + \sum_{i=0}^{r-1}(Q^D)^{i+1}P^iP^\pi + P\sum_{i=0}^{r-1}(Q^D)^{i+2}P^iP^\pi\\ &\quad + PQ^\pi\sum_{i=0}^{s-2}Q^{i+1}(P^D)^{i+3} - PQ^DP^D - PQQ^D(P^D)^2.\end{aligned} \tag{11}$$

Moreover, $\mathrm{ind}(P + Q) \leq r + s + 3$.

Proof. Let $A = (I\ \ Q) : \mathcal{X} \oplus \mathcal{X} \to \mathcal{X}$ and $B = \binom{P}{I} : \mathcal{X} \to \mathcal{X} \oplus \mathcal{X}$. Then $P + Q = AB$ and $BA = M$, where M is defined as in Lemma 2.2. By Lemma 2.2, we obtain

$$R(\lambda, BA) = \begin{pmatrix} \lambda^{-2}(\lambda I - Q)(\lambda^2I + PQ)R(\lambda, Q)R(\lambda, P) & \lambda^{-2}(\lambda I - Q)PQR(\lambda, Q)\\ \lambda^{-2}(\lambda^2I + PQ)R(\lambda, Q)R(\lambda, P) & \lambda^{-2}(\lambda^2I + PQ)R(\lambda, Q)\end{pmatrix} \tag{12}$$

for λ belonging to a punctured neighborhood of 0, which shows that $R(\lambda, BA)$ has a pole at $\lambda = 0$ of order at most $r + s + 2$. So, according to Lemma 1.1, BA is Drazin invertible and $R(\lambda, BA)$ has the Laurent series

$$R(\lambda, BA) = \sum_{k=1}^{r+s+2}\lambda^{-k}(BA)^{k-1}(BA)^\pi - \sum_{k=0}^{\infty}\lambda^k((BA)^D)^{k+1}$$

in a punctured neighborhood of 0. Thus, by Lemma 2.1, AB is Drazin invertible, i.e., $P + Q$ is Drazin invertible. In addition, we have

$$(P + Q)^D = (AB)^D = A((BA)^D)^2B \tag{13}$$

and $\mathrm{ind}(P + Q) \leq \mathrm{ind}(BA) + 1 \leq r + s + 3$.

According to Lemma 2.3 and the expression (12) for $R(\lambda, BA)$, $\alpha_{-1} - Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2$, $\alpha_0 + PQ\alpha_2$, $-PQ^D - P^2(Q^D)^2$ and $-Q^D - P(Q^D)^2$ are the coefficients at λ^0 of $\lambda^{-2}(\lambda I - Q)(\lambda^2I + PQ)R(\lambda, Q)R(\lambda, P)$, $\lambda^{-2}(\lambda^2I +$

$PQ)R(\lambda, Q)R(\lambda, P)$, $\lambda^{-2}(\lambda I - Q)PQR(\lambda, Q)$ and $\lambda^{-2}(\lambda^2 I + PQ)R(\lambda, Q)$, respectively. Thus, applying Eq.(2), we obtain that

$$\begin{aligned}(BA)^D &= -\frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda}R(\lambda, BA)\mathrm{d}\lambda \\ &= -\begin{pmatrix} \alpha_{-1} - Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2 & -PQ^D - P^2(Q^D)^2 \\ \alpha_0 + PQ\alpha_2 & -Q^D - P(Q^D)^2 \end{pmatrix}.\end{aligned}$$

Then

$$((BA)^D)^2 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned}C_{11} &= (\alpha_{-1} - Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2)^2 - (PQ^D + P^2(Q^D)^2)(\alpha_0 + PQ\alpha_2), \\ C_{12} &= -(\alpha_{-1} - Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2)(PQ^D + P^2(Q^D)^2) \\ &\quad +(PQ^D + P^2(Q^D)^2)(Q^D + P(Q^D)^2), \\ C_{21} &= (\alpha_0 + PQ\alpha_2)(\alpha_{-1} - Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2) - (Q^D + P(Q^D)^2)(\alpha_0 + PQ\alpha_2), \\ C_{22} &= -(\alpha_0 + PQ\alpha_2)(PQ^D + P^2(Q^D)^2) + (Q^D + P(Q^D)^2)^2.\end{aligned}$$

By Lemma 2.3 and Lemma 2.4, together with $P^2Q + QPQ = 0$, $Q^D = Q(Q^D)^2$ and $(Q^D)^2P^2(Q^D)^2 = -Q^DP(Q^D)^2$, we can deduce that

$$\begin{aligned}C_{11} &= -\alpha_0 + Q^DQ\alpha_0 + Q^DQPQ\alpha_2 - Q^DQ\alpha_0 - QQ^DPQ\alpha_2 + PQ(Q^D)^2\alpha_0 \\ &\quad +PQ^DPQ\alpha_2 + QPQ\alpha_3 - QPQ(Q^D)^3\alpha_0 - QPQ(Q^D)^3PQ\alpha_2 + Q\alpha_1 \\ &\quad -PQ\alpha_2 - PQ^D\alpha_0 - PQ^DPQ\alpha_2 - P^2(Q^D)^2\alpha_0 - P^2(Q^D)^2PQ\alpha_2 \\ &= -\alpha_0 + Q\alpha_1 - PQ\alpha_2 + QPQ\alpha_3, \\ C_{12} &= Q^DP^2(Q^D)^2 - Q(Q^D)^2P^2(Q^D)^2 + PQ(Q^D)^3P^2(Q^D)^2 - QPQ(Q^D)^4P^2(Q^D)^2 \\ &\quad +P(Q^D)^2 + PQ^DP(Q^D)^2 + P^2(Q^D)^3 + P^2(Q^D)^2P(Q^D)^2 \\ &= P(Q^D)^2 + P^2(Q^D)^3, \\ C_{21} &= -\alpha_1 + Q^D\alpha_0 + Q^DPQ\alpha_2 - PQ\alpha_3 + PQ(Q^D)^3Q\alpha_0 + PQ(Q^D)^3PQ\alpha_2 \\ &\quad -Q^D\alpha_0 - Q^DPQ\alpha_2 - P(Q^D)^2\alpha_0 - P(Q^D)^2PQ\alpha_2 \\ &= -\alpha_1 - PQ\alpha_3, \\ C_{22} &= (Q^D)^2P^2(Q^D)^2 + PQ(Q^D)^4P^2(Q^D)^2 + (Q^D)^2 + Q^DP(Q^D)^2 \\ &\quad +P(Q^D)^3 + P(Q^D)^2P(Q^D)^2 \\ &= (Q^D)^2 + P(Q^D)^3.\end{aligned}$$

Thus,

$$((BA)^D)^2 = \begin{pmatrix} -\alpha_0 + Q\alpha_1 - PQ\alpha_2 + QPQ\alpha_3 & P(Q^D)^2 + P^2(Q^D)^3 \\ -\alpha_1 - PQ\alpha_3 & (Q^D)^2 + P(Q^D)^3 \end{pmatrix}.$$

Therefore, from Eq.(13), we obtain

$$\begin{aligned}(P+Q)^D &= (I\ \ Q)\begin{pmatrix} -\alpha_0 + Q\alpha_1 - PQ\alpha_2 + QPQ\alpha_3 & P(Q^D)^2 + P^2(Q^D)^3 \\ -\alpha_1 - PQ\alpha_3 & (Q^D)^2 + P(Q^D)^3 \end{pmatrix}\begin{pmatrix} P \\ I \end{pmatrix} \\ &= -\alpha_0P - PQ\alpha_2P + P(Q^D)^2 + P^2(Q^D)^3 + Q(Q^D)^2 + QP(Q^D)^3 \\ &= -\alpha_0P - PQ\alpha_2P + P(Q^D)^2 + Q^D.\end{aligned} \tag{15}$$

Instituting the expression (8) of α_0, α_2 into Eq.(15), then we have

$$
\begin{aligned}
(P+Q)^D &= Q^\pi \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P^i P^\pi + P \sum_{i=0}^{r-1} (Q^D)^{i+2} P^i P^\pi \\
&\quad + PQ^\pi \sum_{i=0}^{s-2} Q^{i+1} (P^D)^{i+3} - PQ^D P^D - PQQ^D (P^D)^2
\end{aligned}
$$

from $Q^\pi Q^s = 0$, $P^r P^\pi = 0$, $Q^D - Q^D P^D P = Q^D P^\pi$ and $P(Q^D)^2 - P(Q^D)^2 P^Q P = P(Q^D)^2 P^\pi$. □

Remark 2.6. *In* Theorem 2.5, we find that the representation (11) of $(P+Q)^D$ is the same when $n \geq 2$.

If let $A = (Q \;\; I) : \mathcal{X} \oplus \mathcal{X} \to \mathcal{X}$ and $B = \binom{I}{P} : \mathcal{X} \to \mathcal{X} \oplus \mathcal{X}$, then $P + Q = AB$, and we have

Theorem 2.7. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible, $r = \operatorname{ind}(P)$ and $s = \operatorname{ind}(Q)$. If $PQ^2 + PQP = 0$ and $PQ^n = 0$ for some integer $n > 0$, then $P + Q$ is Drazin invertible, and*

$$
\begin{aligned}
(P+Q)^D &= Q^\pi \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P^i P^\pi + \sum_{i=0}^{r-2} (Q^D)^{i+3} P^{i+1} P^\pi Q \\
&\quad + Q^\pi \sum_{i=0}^{s-1} Q^i (P^D)^{i+2} Q - Q^D P^D Q - (Q^D)^2 P P^D Q.
\end{aligned}
$$

The following corollary is the case when $n = 1$ of Theorem 2.5.

Corollary 2.8. [9, 12] *Let $P, Q \in \mathcal{B}(\mathcal{X})$ is Drazin invertible, $r = \operatorname{ind}(P)$ and $s = \operatorname{ind}(Q)$. If $PQ = 0$. Then $P + Q$ is Drazin invertible, and*

$$
(P+Q)^D = Q^\pi \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P^i P^\pi.
$$

Remark 2.9. *When $n = 2$ in* Theorem 2.5 *and* Theorem 2.7, *we obtain the results of* [19, Theorem 2.1, Theorem 2.2] *and* [7, Lemma 4]. *When $n = 3$ in* Theorem 2.5, *we get the result of* [16, Theorem 5].

In fact, the condition $PQPQ = 0$ in [16, Theorem 5] *can be obtained from $P^2Q + QPQ = 0$ and $P^3Q = 0$. On the other hand, since $\operatorname{ind}(P^2) = [\frac{\operatorname{ind}(P)+1}{2}]$ and $P^k P^\pi = 0$ $(k \geq \operatorname{ind}(P))$, X in* [16, Theorem 5] *can be simplified as $X = \sum_{i=0}^{r-1} (Q^D)^{i+3} P^i P^\pi + \sum_{i=0}^{s-1} Q^\pi Q^i (P^D)^{i+3} - (Q^D)^2 P^D - Q^D (P^D)^2$, where $r = \operatorname{ind}(P), s = \operatorname{ind}(Q)$. Thus, the representation of $(P+Q)^D$ in* [16, Theorem 5] *is reduced to the formula of (11).*

3. Application to Bounded Operator Matrices

Let $\mathcal{Y}, \mathcal{Z}$ be Banach spaces, and let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a bounded linear operator matrix on $\mathcal{Y} \times \mathcal{Z}$. In the following, we illustrate an application of our results to establish representations for $\mathcal{A}^D$ under some conditions.

Lemma 3.1.[8] *Let* $M_1 = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, M_2 = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ *be operator matrices. If* $\mathrm{ind}(A) = a, \mathrm{ind}(D) = d$, *then* M_1 *and* M_2 *are Drazin invertible, and*

$$M_1^D = \begin{pmatrix} A^D & 0 \\ X_1 & D^D \end{pmatrix}, \quad M_2^D = \begin{pmatrix} A^D & X_2 \\ 0 & D^D \end{pmatrix},$$

where $X_1 = D^\pi \sum_{i=0}^{d-1} D^i C (A^D)^{i+2} + \sum_{i=0}^{a-1} (D^D)^{i+2} C A^i A^\pi - D^D C A^D$,

$X_2 = A^\pi \sum_{i=0}^{d-1} A^i B (D^D)^{i+2} + \sum_{i=0}^{a-1} (A^D)^{i+2} B D^i D^\pi - A^D B D^D$.

The case $BC = 0, BDC = 0$ and $BD^2 = 0$ has been studied in [10] and the case $ABC = 0, BDC = 0, CBC = 0$ and $D^2C = 0$ in [16] for matrices. We focus our attention in the generalization of the mentioned results.

Theorem 3.2. *Let* $A \in \mathcal{B}(\mathcal{Y}), \mathcal{D} \in \mathcal{B}(\mathcal{Z})$ *be Drazin invertible,* $a = \mathrm{ind}(A)$, $d = \mathrm{ind}(D)$. *Assume that one of the following holds:*

(1) $ABC + BDC = 0, CBC + D^2C = 0$ *and* $D^nC = 0$ *for some integer* $n > 0$. *further,* $BD^{n-1}C = 0$ *if n is odd;*

(2) $CAB + CBD = 0, CBC + CA^2 = 0$ *and* $CA^n = 0$ *for some integer* $n > 0$. *further,* $CA^{n-1}B = 0$ *if n is odd.*

Then the operator matrix $\mathcal{A}$ *is Drazin invertible, and*

$$\mathcal{A}^D = \begin{pmatrix} A^D + BC(A^D)^3 & X + BC(A^D)^2 X + BCA^D X D^D + BCX(D^D)^2 \\ C(A^D)^2 + DC(A^D)^3 & D^D + CA^D X + CXD^D + DC((A^D)^2 X + A^D X D^D + X(D^D)^2) \end{pmatrix}.$$

where $X = A^\pi \sum_{i=0}^{a-1} A^i B (D^D)^{i+2} + \sum_{i=0}^{d-1} (A^D)^{i+2} B D^i D^\pi - A^D B D^D$.

Proof. We consider the splitting $\mathcal{A} = P + Q$, where $P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$. Then

$$P^n Q = \begin{pmatrix} \sum_{k=0}^{n-1} A^k B D^{n-1-k} C & 0 \\ D^n C & 0 \end{pmatrix}.$$

If (1) holds, then $D^{2k}C = C(-BC)^k$ by $CBC + D^2C = 0$. Thus, using $ABC + BDC = 0$, we have

$$\begin{aligned} \sum_{k=0}^{n-1} A^k B D^{n-1-k} C &= \sum_{k=0}^{\frac{n}{2}-1} (A^{2k+1} B D^{n-2-2k} C + A^{2k} B D^{n-1-2k} C) \\ &= \sum_{k=0}^{\frac{n}{2}-1} (A^{2k+1} BC(-BC)^{\frac{n-2-2k}{2}} + A^{2k} BDC(-BC)^{\frac{n-2-2k}{2}}) \\ &= \sum_{k=0}^{\frac{n}{2}-1} A^{2k}(ABC + BDC)(-BC)^{\frac{n-2-2k}{2}} \\ &= 0 \end{aligned}$$

when n is even, and

$$\begin{aligned}\sum_{k=0}^{n-1} A^kBD^{n-1-k}C &= BD^{n-1}C + \sum_{k=1}^{n-1} A^kBD^{n-1-k}C \\ &= \sum_{k=1}^{\frac{n-1}{2}} (A^{2k}BD^{n-1-2k}C + A^{2k-1}BD^{n-2k}C) \\ &= \sum_{k=1}^{\frac{n-1}{2}} (A^{2k}BC(-BC)^{\frac{n-1-2k}{2}} + A^{2k-1}BDC(-BC)^{\frac{n-1-2k}{2}}) \\ &= \sum_{k=1}^{\frac{n-1}{2}} A^{2k-1}(ABC + BDC)(-BC)^{\frac{n-1-2k}{2}} \\ &= 0\end{aligned}$$

when n is odd. So, $P^nQ = 0$ according to $D^nC = 0$. On the other hand, a straightforward calculation shows that $P^2Q + QPQ = 0$. The desired result follows from Theorem 2.5 and Lemma 3.1.

Similarly, if (2) holds, then we conclude that $QP^2 + QPQ = 0$ and $QP^n = 0$. Therefore, the claim follows from Theorem 2.7. □

If we consider the splitting $M = P + Q$, where $P = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, $Q = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, then we obtain the following result.

Theorem 3.3. *Let $A \in \mathcal{B}(\mathcal{Y}), \mathcal{D} \in \mathcal{B}(\mathcal{Z})$ be Drazin invertible, $a = \mathrm{ind}(A), d = \mathrm{ind}(D)$. Assume that one of the following holds:*

(1) $CAB + DCB = 0, BCB + A^2B = 0$ and $A^nB = 0$ for some integer $n > 0$. further, $CA^{n-1}B = 0$ if n is odd;

(2) $BCA + BDC = 0, BCB + BD^2 = 0$ and $BD^n = 0$ for some integer $n > 0$. further, $BD^{n-1}C = 0$ if n is odd.

Then the operator matrix $\mathcal{A}$ is Drazin invertible, and

$$\mathcal{A}^D = \begin{pmatrix} A^D+BXA^D+BD^DX+AB((D^D)^2X+D^DXA^D+X(A^D)^2) & B(D^D)^2+AB(D^D)^3 \\ X+CBX(A^D)^2+CBD^DXA^D+CB(D^D)^2X & D^D+CB(D^D)^3) \end{pmatrix}.$$

where $X = D^\pi \sum_{i=0}^{d-1} D^iC(A^D)^{i+2} + \sum_{i=0}^{a-1} (D^D)^{i+2}CA^iA^\pi - D^DCA^D.$

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