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Further Generalizations of Some Operator Inequalities Involving Positive Linear Map

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Abstract. We obtain a generalized conclusion based on an α-geometric mean inequality. The conclusion is presented as follows: If m_1, M_1, m_2, M_2 are positive real numbers, $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$ for $m_1 < M_1$ and $m_2 < M_2$, then for every unital positive linear map Φ and $\alpha \in (0, 1]$, the operator inequality below holds:
$(\Phi(A)\sharp_\alpha\Phi(B))^p \leq \frac{1}{16}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha})}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^p \Phi^p(A\sharp_\alpha B), \quad p \geq 2.$
Likewise, we give a second powering of the Diaz-Metcalf type inequality. Finally, we present $p-$th powering of some reversed inequalities for n operators related to Karcher mean and power mean involving positive linear maps.

1. Introduction

Let $B(\mathcal{H})$ stand for the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$ and I denote the identity operator. $\|\cdot\|$ denote the operator norm. An operator A is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and we write $A \geq 0$. We identify $T \geq S$(the same as $S \leq T$) with $T - S \geq 0$. A positive invertible operator T is denoted by $T > 0$. The absolute value of T is denoted by $|T| = (T^*T)^{\frac{1}{2}}$, where T^* stands for the adjoint of T.

A linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is called positive (strictly positive) if $\Phi(A) \geq 0$ ($\Phi(A) > 0$) whenever $A \geq 0$ ($A > 0$), and Φ is said to be unital if $\Phi(I) = I$. Take $A, B > 0$ and $\alpha \in [0, 1]$, the α-geometric mean $A\sharp_\alpha B$ is defined by $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$, when $\alpha = \frac{1}{2}$, $A\sharp_{\frac{1}{2}}B = A\sharp B$ is said to be the geometric mean between A and B.

The geometric mean $A\sharp B$ of two positive operators $A, B \in B(\mathcal{H})$ is characterized by Ando [2]

$$A\sharp B = max\left\{X = X^* \in B(\mathcal{H}) : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\right\}.$$

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The definition above reveals the maximal characterization of geometric mean [16]. The 2×2 operator matrix mentioned in this work is naturally understood as an operator acting on $\mathcal{H}\bigoplus\mathcal{H}$.

In 1948, Kantorovich [9] introduced the famous Kantorovich inequality. In 1990, Marshall and Olkin [13] established an operator Kantorovich inequality. It is well known that t^α is operator monotone function on $[0,\infty)$ if and only if $\alpha\in[0,1]$. Since t^2 is not an operator monotone function, we can not obtain $A^2\geq B^2$ directly by $A\geq B\geq 0$. In 2013, Lin [12] proved that the operator Kantorovich inequality is order preserving under squaring. In 2011, Seo gave an α-geometric mean inequality for positive linear map as follows:

Theorem 1.1. *[17] Let $\Phi: B(\mathcal{H})\to B(\mathcal{K})$ be a unital positive linear map and let A and B be positive operators such that $0<m_1^2\leq A\leq M_1^2$ and $0<m_2^2\leq B\leq M_2^2$ for some positive real numbers $m_1<M_1$ and $m_2<M_2$. Then for $\alpha\in[0,1]$*

$$\Phi(A)\sharp_\alpha\Phi(B)\leq K(m,M,\alpha)^{-1}\Phi(A\sharp_\alpha B), \tag{1}$$

where we suppose $(\frac{m_2}{M_1})^2=m$, $(\frac{M_2}{m_1})^2=M$ and the generalized Kantorovich constant $K(M,m,\alpha)$ [7] is defined by

$$K(m,M,\alpha)=\frac{mM^\alpha-Mm^\alpha}{(\alpha-1)(M-m)}\Big(\frac{\alpha-1}{\alpha}\frac{M^\alpha-m^\alpha}{mM^\alpha-Mm^\alpha}\Big)^\alpha$$

for any real number $\alpha\in\mathbb{R}$.

Motived by Lin's idea [12], Fu obtained a second powering of the operator inequality (1):

Theorem 1.2. *[6] Let $\Phi: B(\mathcal{H})\to B(\mathcal{K})$ be a unital positive linear map and let A and B be positive operators such that $0<m_1\leq A\leq M_1$ and $0<m_2\leq B\leq M_2$ for some positive real numbers $m_1<M_1$ and $m_2<M_2$. Then for $\alpha\in[0,1]$*

$$(\Phi(A)\sharp_\alpha\Phi(B))^2\leq\Big(\tfrac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{4(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\Big)^2\Phi^2(A\sharp_\alpha B). \tag{2}$$

Next we present the Diaz-Metcalf type inequality.

Theorem 1.3. *[14] Let Φ be a positive linear map. If $0<m_1^2\leq A\leq M_1^2$ and $0<m_2^2\leq B\leq M_2^2$ for some positive real numbers $m_1\leq M_1$ and $m_2\leq M_2$, then the following inequality holds:*

$$\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\leq\Big(\frac{M_2}{m_1}+\frac{m_2}{M_1}\Big)\Phi(A\sharp B).$$

Let $A_1,\cdots,A_n>0$ and $\omega=(w_1,\dots,w_n)$ is a probability vector: $\sum_{i=1}^n w_i=1$ and $w_i>0$ for $i=1,\dots,n$. For $t\in(0,1]$, the $\omega-$weighted power mean $P_t(\omega;A_1,\dots,A_n)$ (or $P_t(\omega;\mathbb{A})$)is defined as the unique solution of

$$X=\sum_{i=1}^n\omega_i(X\sharp_t A_i).$$

For $t\in[-1,0)$, we define $P_t(\omega;A_1,\dots,A_n)=P_{-t}(\omega;A_1^{-1},\dots,A_n^{-1})^{-1}$. Next we bring some basic properties of the power mean that is useful to obtain the results in this work, for more details about power mean, see [11].

Proposition 1.4. *[11] The power mean satisfies the following properties:*

(i) $(self\ duality)$ $P_{-t}(\omega; A_1^{-1}, \cdots, A_n^{-1})^{-1} = P_t(\omega; A_1, \cdots, A_n)$.

(ii) $(AGH\ weighted\ mean\ inequalities)$ $(\sum_{i=1}^{n} \omega_i A_i^{-1})^{-1} \leq P_t(\omega; A_1, \cdots, A_n) \leq \sum_{i=1}^{n} \omega_i A_i$.

(iii) $t \in (0,1]$, $\Phi(P_t(\omega; \mathbb{A})) \leq P_t(\omega; \Phi(\mathbb{A}))$ *for any positive unital linear map* Φ; $t \in [-1,0)$, $\Phi(P_t(\omega; \mathbb{A})) \geq P_t(\omega; \Phi(\mathbb{A}))$ *for any strictly positive unital linear map* Φ.

(iv) $P_t(\omega; \mathbb{A}) \leq P_t(\omega; \mathbb{B})$ *if* $A_i \leq B_i$ *for all* $i = 1, 2, \cdots, n$.

The $\omega-$weighted Karcher mean $\Lambda(\omega; A_1, \ldots, A_n)$(or$\Lambda(\omega; \mathbb{A})$) of $A_1, \cdots, A_n > 0$ is defined to be the unique positive definite solution of equation

$$\sum_{i=1}^{n} \omega_i log(X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}}) = 0,$$

where $\omega = (w_1, \ldots, w_n)$ is a probability vector. Next we cite some basic properties of the Karcher mean as follows, for more details about Karcher mean, see [11].

Proposition 1.5. *[11] The Karcher mean satisfies the following properties:*

(i) $(self\ duality)$ $\Lambda(\omega; A_1^{-1}, \cdots, A_n^{-1})^{-1} = \Lambda(\omega; A_1, \cdots, A_n)$.

(ii) $(AGH\ weighted\ mean\ inequalities)$ $(\sum_{i=1}^{n} \omega_i A_i^{-1})^{-1} \leq \Lambda(\omega; A_1, \cdots, A_n) \leq \sum_{i=1}^{n} \omega_i A_i$.

(iii) $\Phi(\Lambda(\omega; \mathbb{A})) \leq \Lambda(\omega; \Phi(\mathbb{A}))$ *for any positive unital linear map* Φ.

(iv) $(monotonicity)$ *If* $B_i \leq A_i$ *for all* $1 \leq i \leq n$, *then* $\Lambda(\omega; \mathbb{B}) \leq \Lambda(\omega; \mathbb{A})$.

As mentioned in the abstract, we shall give a further generalization of Theorem 1.2 and Diaz-Metcalf type inequality in the following section, along with presenting $p-$th powering of some reversed inequalities for n operators related to Karcher mean and power mean involving positive linear maps.

2. Main Results

Before giving our main results, let us first consider the following lemmas.

Lemma 2.1. **(Choi inequality.)** *[5, 7] Let* Φ *be a unital positive linear map, then*
(C_1) when $A > 0$ *and* $-1 \leq p \leq 0$, *then* $\Phi(A)^p \leq \Phi(A^p)$, *in particular,*$\Phi(A)^{-1} \leq \Phi(A^{-1})$;
(C_2) when $A \geq 0$ *and* $0 \leq p \leq 1$, *then* $\Phi(A)^p \geq \Phi(A^p)$;
(C_3) when $A \geq 0$ *and* $1 \leq p \leq 2$, *then* $\Phi(A)^p \leq \Phi(A^p)$.

Lemma 2.2. *[10] Let* Φ *be a unital positive linear map and* A, B *be positive operators. Then for* $\alpha \in [0,1]$

$$\Phi(A \sharp_\alpha B) \leq \Phi(A) \sharp_\alpha \Phi(B).$$

Lemma 2.3. *[4] Let* $A, B \geq 0$. *Then the following norm inequality holds:*

$$\|AB\| \leq \frac{1}{4} \|A + B\|^2.$$

Lemma 2.4. *[3] Let $A, B \geq 0$. Then for $1 \leq r < +\infty$,*

$$\|A^r + B^r\| \leq \|(A + B)^r\|.$$

Now we are going to present the first main theorem of this paper.

Theorem 2.5. *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a unital positive linear map and let A and B be positive operators such that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$. Then for $\alpha \in [0, 1]$, $p \geq 2$*

$$(\Phi(A)\sharp_\alpha\Phi(B))^p \leq \frac{1}{16}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^p \Phi^p(A\sharp_\alpha B) \tag{3}$$

Proof. The desired inequality is equivalent to

$$\|(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)\| \leq \frac{1}{4}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^{\frac{p}{2}}. \tag{4}$$

Note that

$$(M_1 - A)(m_1 - A)A^{-1} \leq 0,$$

then

$$M_1m_1A^{-1} + A \leq M_1 + m_1,$$

and therefore

$$M_1m_1\Phi(A^{-1}) + \Phi(A) \leq M_1 + m_1. \tag{5}$$

Likewise, we can get

$$M_2m_2\Phi(B^{-1}) + \Phi(B) \leq M_2 + m_2. \tag{6}$$

Hence, by the property of weighted geometric mean, through (5) and (6), we obtain

$$(M_1m_1\Phi(A^{-1}) + \Phi(A))\sharp_\alpha(M_2m_2\Phi(B^{-1}) + \Phi(B)) \leq (M_1 + m_1)\sharp_\alpha(M_2 + m_2).$$

Using the subadditivity of weighted geometric mean, we get

$$\begin{aligned}
&(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}\Phi^{-1}(A\sharp_\alpha B) + \Phi(A)\sharp_\alpha\Phi(B)\\
&\leq (m_2M_2)^\alpha(m_1M_1)^{1-\alpha}(\Phi(A^{-1}\sharp_\alpha B^{-1})) + \Phi(A)\sharp_\alpha\Phi(B)\\
&\leq (m_2M_2)^\alpha(m_1M_1)^{1-\alpha}(\Phi(A^{-1})\sharp_\alpha\Phi(B^{-1})) + \Phi(A)\sharp_\alpha\Phi(B)\\
&= (m_1M_1\Phi(A^{-1}))\sharp_\alpha(m_2M_2\Phi(B^{-1})) + \Phi(A)\sharp_\alpha\Phi(B)\\
&\leq (m_1M_1\Phi(A^{-1}) + \Phi(A))\sharp_\alpha(m_2M_2\Phi(B^{-1}) + \Phi(B)),
\end{aligned}$$

where the first inequality is obtained by Lemma 2.1, the second one is obtained by Lemma 2.2.
Since

$$\begin{aligned}
&\|(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}(m_2M_2)^{\frac{\alpha p}{2}}(m_1M_1)^{\frac{(1-\alpha)p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)\|\\
&\leq \frac{1}{4}\|(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}} + (m_2M_2)^{\frac{\alpha p}{2}}(m_1M_1)^{\frac{(1-\alpha)p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)\|^2\\
&\leq \frac{1}{4}\|(\Phi(A)\sharp_\alpha\Phi(B) + (m_2M_2)^\alpha(m_1M_1)^{1-\alpha}\Phi^{-1}(A\sharp_\alpha B))^{\frac{p}{2}}\|^2\\
&= \frac{1}{4}\|\Phi(A)\sharp_\alpha\Phi(B) + (m_2M_2)^\alpha(m_1M_1)^{1-\alpha}\Phi^{-1}(A\sharp_\alpha B)\|^p\\
&\leq \frac{1}{4}\|(\Phi(A) + m_1M_1\Phi(A^{-1}))\sharp_\alpha(\Phi(B) + m_2M_2\Phi(B^{-1}))\|^p\\
&\leq \frac{1}{4}\|(M_1 + m_1)\sharp_\alpha(M_2 + m_2)\|^p\\
&= \frac{((M_1+m_1)((M_1+m_1)^{-1}(M_2+m_2))^\alpha)^p}{4},
\end{aligned}$$

where the first inequality is obtained by Lemma 2.3, the second one is obtained by Lemma 2.4., so we have

$$\begin{aligned}
&\|(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)\| \\
&\le \frac{((M_1+m_1)((M_1+m_1)^{-1}(M_2+m_2))^\alpha)^p}{4(m_2M_2)^{\frac{\alpha p}{2}}(m_1M_1)^{\frac{(1-\alpha)p}{2}}} \\
&= \frac{1}{4}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^{\frac{p}{2}}.
\end{aligned}$$

Hence the inequality (4) has been obtained. □

Remark 2.6. *Inequality (2) is a special case of Theorem 2.5 by taking $p = 2$.*

Lemma 2.7. *[1] Let $A, B \ge 0$. Then*

$$A\sharp B \le \frac{A+B}{2}.$$

Inspired by Theorem 2.5, we give a further generalization related to the Diaz-Metcalf type inequality as follows.

Theorem 2.8. *Let Φ be a unital positive linear map. If $0 < m_1^2 \le A \le M_1^2$ and $0 < m_2^2 \le B \le M_2^2$ for some positive real numbers $m_1 \le M_1$ and $m_2 \le M_2$, then the following inequality holds:*

$$\left(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\right)^p \le \frac{1}{16}\left\{\frac{(M_1m_1(M_2^2+m_2^2)+M_2m_2(M_1^2+m_1^2))^2}{2\sqrt{M_2M_1m_1m_2}M_1^2m_1^2M_2m_2}\right\}^p\Phi^p(A\sharp B), \quad p\ge 2. \tag{7}$$

Proof. Obviously (7) is equivalent to

$$\left\|\left(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\right)^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp B)\right\| \le \frac{1}{4}\left\{\frac{(M_1m_1(M_2^2+m_2^2)+M_2m_2(M_1^2+m_1^2))^2}{2\sqrt{M_2M_1m_1m_2}M_1^2m_1^2M_2m_2}\right\}^{\frac{p}{2}}.$$

By the proof of (5), we have

$$M_1^2m_1^2\Phi(A^{-1})+\Phi(A) \le M_1^2+m_1^2,$$

which equals to

$$M_2m_2m_1M_1\Phi(A^{-1})+\frac{M_2m_2}{M_1m_1}\Phi(A) \le \frac{M_2m_2}{M_1m_1}(M_1^2+m_1^2). \tag{8}$$

Similarly, we have

$$M_2^2m_2^2\Phi(B^{-1})+\Phi(B) \le M_2^2+m_2^2. \tag{9}$$

By (8) and (9) we have

$$\begin{aligned}
&2\sqrt{M_2M_1m_1m_2}M_2m_2\Phi(A\sharp B)^{-1}+\left(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\right) \\
&\le 2\sqrt{M_2M_1m_1m_2}M_2m_2\Phi(A^{-1}\sharp B^{-1})+\left(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\right) \\
&\le 2\sqrt{M_2M_1m_1m_2}M_2m_2\Phi(A^{-1})\sharp\Phi(B^{-1})+\left(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\right) \\
&\le M_2M_1m_1m_2\Phi(A^{-1})+M_2^2m_2^2\Phi(B^{-1})+\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B) \\
&\le \frac{M_2m_2}{M_1m_1}(M_1^2+m_1^2)+M_2^2+m_2^2,
\end{aligned}$$

where the first inequality is obtained by Lemma 2.1, the second one is obtained by Lemma 2.2, the third one is obtained by Lemma 2.7, the last one is the result of combining (8) and (9).
Since

$$\begin{aligned}
&\|\Big(\tfrac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\Big)^{\frac{p}{2}}(2\sqrt{M_2M_1m_1m_2}M_2m_2)^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp B)\|\\
&\le \tfrac{1}{4}\|\Big(\tfrac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\Big)^{\frac{p}{2}}+(2\sqrt{M_2M_1m_1m_2}M_2m_2)^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp B)\|^2\\
&\le \tfrac{1}{4}\|\Big(\tfrac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)+2\sqrt{M_2M_1m_1m_2}M_2m_2\Phi^{-1}(A\sharp B)\Big)^{\frac{p}{2}}\|^2\\
&= \tfrac{1}{4}\|\tfrac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)+2\sqrt{M_2M_1m_1m_2}M_2m_2\Phi^{-1}(A\sharp B)\|^p\\
&\le \tfrac{1}{4}\left(\tfrac{M_2m_2(M_1^2+m_1^2)+M_1m_1(M_2^2+m_2^2)}{M_1m_1}\right)^p,
\end{aligned}$$

we have

$$\|\Big(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\Big)^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp B)\|\le \frac{1}{4}\left\{\frac{(M_1m_1(M_2^2+m_2^2)+M_2m_2(M_1^2+m_1^2))^2}{2\sqrt{M_2M_1m_1m_2}M_1^2m_1^2M_2m_2}\right\}^{\frac{p}{2}}.$$

□

Corollary 2.9. *In Theorem 2.8, put $p=2$, we get*

$$\Big(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\Big)^2\le\left\{\frac{(M_1m_1(M_2^2+m_2^2)+M_2m_2(M_1^2+m_1^2))^2}{8\sqrt{M_2M_1m_1m_2}M_1^2m_1^2M_2m_2}\right\}^2\Phi^2(A\sharp B),$$

which can be seen as a squared operator inequality related to Diaz-Metcalf type inequality.

Lemma 2.10. *[8] For any bounded operator X,*

$$|X|\le tI\Leftrightarrow\|X\|\le t\Leftrightarrow\begin{bmatrix} tI & X\\ X^* & tI\end{bmatrix}\ge 0\quad(t\ge 0)$$

Theorem 2.11. *Let A and B be positive operators on a Hilbert space $\mathcal{H}$ with $0<m_1\le A\le M_1$ and $0<m_2\le B\le M_2$, Φ be a unital positive linear map on $B(\mathcal{H})$. Then for $0\le\alpha\le 1$ and $p\ge 2$,*

$$\begin{aligned}
&(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)+\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\\
&\le\tfrac{1}{2}\left\{\tfrac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^{\frac{p}{2}}.
\end{aligned}\tag{10}$$

Proof. By (4) and Lemma 2.10, we obtain

$$\begin{bmatrix}\frac{1}{4}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^{\frac{p}{2}}I & (\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)\\ \Phi^{-\frac{p}{2}}(A\sharp_\alpha B)(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}} & \frac{1}{4}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^{\frac{p}{2}}I\end{bmatrix}\ge 0$$

and

$$\begin{bmatrix}\frac{1}{4}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^{\frac{p}{2}}I & \Phi^{-\frac{p}{2}}(A\sharp_\alpha B)(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\\ (\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B) & \frac{1}{4}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^{\frac{p}{2}}I\end{bmatrix}\ge 0$$

Summing up these two operator matrices above, and putting

$$t=\frac{1}{4}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^{\frac{p}{2}},$$

$$X = \Phi^{-\frac{p}{2}}(A\sharp_\alpha B)(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}} + (\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B),$$

we have

$$\begin{bmatrix} 2tI & X \\ X & 2tI \end{bmatrix} \geq 0$$

Since $\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}} + (\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B)$ is self-adjoint, (10) follows from the maximal characterization of geometric mean. □

Theorem 2.12. *Let Φ be a unital positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, $p \geq 2$, then the following inequalities holds:*

$$\begin{aligned} &(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\alpha B) + \Phi^{-\frac{p}{2}}(A\sharp_\alpha B)(\Phi(A)\sharp_\alpha\Phi(B))^{\frac{p}{2}} \\ &\leq \tfrac{1}{2}\left\{\tfrac{(M_1m_1(M_2^2+m_2^2)+M_2m_2(M_1^2+m_1^2))^2}{2\sqrt{M_2M_1m_1m_2}M_1^2m_1^2M_2m_2}\right\}^{\frac{p}{2}} \end{aligned} \tag{11}$$

Proof. By the method of proving Theorem 2.11, we can easily get (11). □

Next we present a $p-$th powering of a reversed HM-PM inequality for n operators.

Theorem 2.13. *Let $0 < m \leq A_i \leq M$ for $i = 1,\cdots n$, $M \geq m$, $\omega = (\omega_1,\cdots,\omega_n)$ be a probability vector, $t \in [-1,0)\cup(0,1]$. Then we have*

$$P_t(\omega; A_1,\cdots,A_n)^p \leq \tfrac{1}{16}\left\{\tfrac{(M+m)^2}{Mm}\right\}^p (\sum_{i=1}^n \omega_i A_i^{-1})^{-p}, \quad p \geq 2. \tag{12}$$

Proof. The inequality (12) is equivalent to

$$\|P_t(\omega; A_1,\cdots,A_n)^{\frac{p}{2}}(\sum_{i=1}^n \omega_i A_i^{-1})^{\frac{p}{2}}\| \leq \frac{1}{4}\left\{\frac{(M+m)^2}{Mm}\right\}^{\frac{p}{2}}.$$

Since (5) implies

$$Mm(\sum_{i=1}^n \omega_i A_i^{-1}) + \sum_{i=1}^n \omega_i A_i \leq M + m. \tag{13}$$

Then by Proposition 1.4, we have

$$\begin{aligned} &P_t(\omega; A_1,\cdots,A_n) + Mm(\sum_{i=1}^n \omega_i A_i^{-1}) \\ &\leq \sum_{i=1}^n \omega_i A_i + Mm(\sum_{i=1}^n \omega_i A_i^{-1}) \\ &\leq M + m. \end{aligned}$$

Thus

$$\begin{aligned} &\|P_t\omega; A_1,\cdots,A_n)^{\frac{p}{2}}(Mm)^{\frac{p}{2}}(\omega_i A_i^{-1})^{\frac{p}{2}}\| \\ &\leq \tfrac{1}{4}\|P_t(\omega; A_1,\cdots,A_n)^{\frac{p}{2}} + (Mm)^{\frac{p}{2}}(\omega_i A_i^{-1})^{\frac{p}{2}}\|^2 \\ &\leq \tfrac{1}{4}\|(P_t(\omega; A_1,\cdots,A_n) + Mm(\omega_i A_i^{-1}))^{\frac{p}{2}}\|^2 \\ &= \tfrac{1}{4}\|P_t(\omega; A_1,\cdots,A_n) + Mm(\omega_i A_i^{-1})\|^p \\ &\leq \tfrac{1}{4}(M+m)^p. \end{aligned}$$

Therefore

$$\|P_t(\omega; A_1, \cdots, A_n)^{\frac{p}{2}} (\sum_{i=1}^{n} \omega_i A_i^{-1})^{\frac{p}{2}} \| \leq \frac{1}{4} \left\{ \frac{(M+m)^2}{Mm} \right\}^{\frac{p}{2}} .$$

□

From [11] we know, power mean is increasing from $[-1, 0) \cup (0, 1]$. Furthermore, $\lim_{t \to 0} P_t(\omega; \Phi(\mathbb{A})) = \Lambda(\omega; \Phi(\mathbb{A}))$. Therefore in Theorem 2.13 put $t \to 0$, we obtain:

Corollary 2.14. *Let $0 < m \leq A_i \leq M$ for $i = 1, \cdots n, M \geq m, \omega = (\omega_1, \cdots, \omega_n)$ be a probability vector. Then we have*

$$\Lambda(\omega; A_1, \cdots, A_n)^p \leq \tfrac{1}{16} \left\{ \tfrac{(M+m)^2}{Mm} \right\}^p (\sum_{i=1}^{n} \omega_i A_i^{-1})^{-p}, \quad p \geq 2. \tag{14}$$

Lemma 2.15 presented a p-th powering of a reversed HM-KM inequality for n operators.

Lemma 2.15. *[7] (L-H inequality) If $0 \leq \alpha \leq 1$, $A, B \geq 0$ and $A \geq B$, then $A^\alpha \geq B^\alpha$.*

Lemma 2.16. *Let $0 < m \leq A_i \leq M$ for $i = 1, \cdots n, M \geq m, \omega = (\omega_1, \cdots, \omega_n)$ be a probability vector. Then we obtain*

$$\sum_{i=1}^{n} \omega_i A_i \leq \tfrac{(m+M)^2}{4mM} (\sum_{i=1}^{n} \omega_i A_i^{-1})^{-1}.$$

Proof. First we proof $(\sum_{i=1}^{n} \omega_i A_i)^2 \leq (\frac{(m+M)^2}{4mM})^2 (\sum_{i=1}^{n} \omega_i A_i^{-1})^{-2}$. This inequality equals to

$$\|\sum_{i=1}^{n} \omega_i A_i \sum_{i=1}^{n} \omega_i A_i^{-1} \| \leq \frac{(M+m)^2}{4Mm}.$$

Now by

$$\begin{aligned} &\|(\sum_{i=1}^{n} \omega_i A_i) Mm (\sum_{i=1}^{n} \omega_i A_i^{-1})\| \\ &\leq \tfrac{1}{4} \|\sum_{i=1}^{n} \omega_i A_i + Mm(\sum_{i=1}^{n} \omega_i A_i^{-1})\|^2 \\ &\leq \tfrac{1}{4}(M+m)^2, \end{aligned}$$

where the second inequality is obtained by (13). Hence we can get desired inequality by Lemma 2.15.
We remark that the result of Lemma 2.16 is a special case of Theorem 1 in [15]. □

Theorem 2.17. *Let Φ be a unital positive linear map, $0 < m \leq A_i \leq M$ for $i = 1, \cdots n, M \geq m$. $\omega = (\omega_1, \cdots, \omega_n)$ be a probability vector, $t \in (0, 1]$. Then we have*

$$P_t(\omega; \Phi(\mathbb{A})) \leq \tfrac{(m+M)^2}{4mM} \Phi(\Lambda(\omega; \mathbb{A})). \tag{15}$$

Proof. By Proposition 1.4, 1.5 and Lemma 2.16 we get

$$P_t(\omega; \Phi(\mathbb{A})) \leq \sum_{i=1}^{n} \omega_i \Phi(A_i) = \Phi(\sum_{i=1}^{n} \omega_i A_i) \leq \frac{(m+M)^2}{4mM} \Phi((\sum_{i=1}^{n} \omega_i A_i^{-1})^{-1}) \leq \frac{(m+M)^2}{4mM} \Phi(\Lambda(\omega; \mathbb{A})).$$

From [11] we know $P_t(\omega; \mathbb{A}) \geq \Lambda(\omega; \mathbb{A})$ when $t \in (0, 1]$, and by Proposition 1.4, 1.5 we have $P_t(\omega; \Phi(\mathbb{A})) \geq \Lambda(\omega; \Phi(\mathbb{A})) \geq \Phi(\Lambda(\omega; \mathbb{A}))$. Thus (15) can be viewed as a reversed PM-KM inequality involving positive unital linear maps. □

Corollary 2.18. *Under the same conditions as in Theorem 2.13, one can obtain*

$$\Phi(P_t(\omega;\mathbb{A})) \leq \tfrac{(m+M)^2}{4mM}\Lambda(\omega;\Phi(\mathbb{A})). \tag{16}$$

Proof. By Proposition 1.4 and 1.5, we obtain

$$\Phi(P_t(\omega;\mathbb{A})) \leq P_t(\omega;\Phi(\mathbb{A}))) \leq \frac{(m+M)^2}{4mM}\Phi(\Lambda(\omega;\mathbb{A})) \leq \frac{(m+M)^2}{4mM}\Lambda(\omega;\Phi(\mathbb{A})).$$

Next we present p-th powering of (14) involving positive unital linear maps. □

Theorem 2.19. *Let Φ be a unital positive linear map, $0 < m \leq A_i \leq M$ for $i = 1,\cdots n$, $M \geq m$. $\omega = (\omega_1,\cdots,\omega_n)$ be a probability vector, $t \in (0,1]$, $p \geq 2$. Then we have*

$$P_t(\omega;\Phi(\mathbb{A}))^p \leq \tfrac{1}{16}\left\{\tfrac{(M+m)^2}{Mm}\right\}^p \Phi(\Lambda(\omega;\mathbb{A}))^p. \tag{17}$$

Proof. The inequality (17) is equivalent to

$$||P_t(\omega;\Phi(\mathbb{A}))^{\frac{p}{2}}\Phi(\Lambda(\omega;\mathbb{A}))^{-\frac{p}{2}}|| \leq \frac{1}{4}\left\{\frac{(M+m)^2}{Mm}\right\}^{\frac{p}{2}}.$$

By (13), we obtain

$$\begin{aligned}
&P_t(\omega;\Phi(\mathbb{A})) + Mm\Phi(\Lambda(\omega;\mathbb{A}))^{-1}\\
&\leq P_t(\omega;\Phi(\mathbb{A})) + Mm\Phi(\Lambda(\omega;\mathbb{A})^{-1})\\
&\leq (\sum_{i=1}^{n}\omega_i\Phi(A_i)) + Mm\Phi(\sum_{i=1}^{n}\omega_i A_i^{-1})\\
&\leq M + m.
\end{aligned}$$

Then

$$\begin{aligned}
&||P_t(\omega;\Phi(\mathbb{A}))^{\frac{p}{2}}(Mm)^{\frac{p}{2}}\Phi(\Lambda(\omega;\mathbb{A}))^{-\frac{p}{2}}||\\
&\leq \tfrac{1}{4}||P_t(\omega;\Phi(\mathbb{A}))^{\frac{p}{2}} + (Mm)^{\frac{p}{2}}\Phi(\Lambda(\omega;\mathbb{A}))^{-\frac{p}{2}}||^2\\
&\leq \tfrac{1}{4}||P_t(\omega;\Phi(\mathbb{A})) + Mm\Phi(\Lambda(\omega;\mathbb{A}))^{-1}||^p\\
&\leq \tfrac{1}{4}(M+m)^p.
\end{aligned}$$

Therefore we have the desired inequality. □

Theorem 2.20. *Let Φ be a strictly unital positive linear map, $0 < m \leq A_i \leq M$ for $i = 1,\cdots n$, $M \geq m$. $\omega = (\omega_1,\cdots,\omega_n)$ be a probability vector, $t \in [-1,0)$. Then we have*

$$\Lambda(\omega;\Phi(\mathbb{A})) \leq \tfrac{(m+M)^2}{4mM}\Phi(P_t(\omega;\mathbb{A})). \tag{18}$$

Proof. By Proposition 1.4, Proposition 1.5 and Lemma 2.16 we get

$$\Lambda(\omega;\Phi(\mathbb{A})) \leq \sum_{i=1}^{n}\omega_i\Phi(A_i) = \Phi(\sum_{i=1}^{n}\omega_i A_i) \leq \frac{(m+M)^2}{4mM}\Phi((\sum_{i=1}^{n}\omega_i A_i^{-1})^{-1}) \leq \frac{(m+M)^2}{4mM}\Phi(P_t(\omega;\mathbb{A})).$$

From [11] we know $\Lambda(\omega;\mathbb{A}) \geq P_t(\omega;\mathbb{A})$ when $t \in [-1,0)$, and by Proposition 1.4, 1.5 we have $\Lambda(\omega;\Phi(\mathbb{A})) \geq \Phi(\Lambda(\omega;\mathbb{A})) \geq \Phi(P_t(\omega;\mathbb{A}))$. Thus (18) can be viewed as a reversed PM-KM inequality as well.
Next we give a p-th powering of (18). □

Theorem 2.21. *Let Φ be a strictly unital positive linear map, $0 < m \le A_i \le M$ for $i = 1, \cdots n$, $M \ge m$. $\omega = (\omega_1, \cdots, \omega_n)$ be a probability vector, $t \in [-1, 0)$, $p \ge 2$. Then we have*

$$\Lambda(\omega; \Phi(\mathbb{A}))^p \le \frac{1}{16}\left\{\frac{(M+m)^2}{Mm}\right\}^p \Phi(P_t(\omega; \mathbb{A}))^p. \tag{19}$$

Proof. Since

$$\begin{aligned}
&\Lambda(\omega; \Phi(\mathbb{A})) + \Phi(P_t(\omega; \mathbb{A}))^{-1} \\
&\le \Lambda(\omega; \Phi(\mathbb{A})) + \Phi(P_t(\omega; \mathbb{A})^{-1}) \\
&= \Lambda(\omega; \Phi(\mathbb{A})) + \Phi(P_{-t}(\omega; \mathbb{A}^{-1})) \\
&\le (\sum_{i=1}^{n} \omega_i \Phi(A_i)) + Mm\Phi(\sum_{i=1}^{n} \omega_i A_i^{-1}) \\
&\le M + m,
\end{aligned}$$

so

$$\begin{aligned}
&\|\Lambda(\omega; \Phi(\mathbb{A}))^{\frac{p}{2}} (Mm)^{\frac{p}{2}} \Phi(P_t(\omega; \mathbb{A}))^{-\frac{p}{2}}\| \\
&\le \frac{1}{4}\|\Lambda(\omega; \Phi(\mathbb{A}))^{\frac{p}{2}} + (Mm)^{\frac{p}{2}} \Phi(P_t(\omega; \mathbb{A}))^{-\frac{p}{2}}\|^2 \\
&\le \frac{1}{4}\|\Lambda(\omega; \Phi(\mathbb{A})) + Mm\Phi(P_t(\omega; \mathbb{A}))^{-1}\|^p \\
&\le \frac{1}{4}(M + m)^p.
\end{aligned}$$

Therefore we have

$$\|\Lambda(\omega; \Phi(\mathbb{A}))^{\frac{p}{2}} \Phi(P_t(\omega; \mathbb{A}))^{-\frac{p}{2}}\| \le \frac{1}{4}\left\{\frac{(M+m)^2}{Mm}\right\}^{\frac{p}{2}},$$

which is equivalent (19). □

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