



Jungck-Type Implicit Iterative Algorithms with Numerical Examples

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Abstract. We introduce a new Jungck-type implicit iterative scheme and study its strong convergence, stability under weak parametric restrictions in generalized convex metric spaces and data dependency in generalized hyperbolic spaces. We show that new introduced iterative scheme has better convergence rate as compared to well known Jungck implicit Mann, Jungck implicit Ishikawa and Jungck implicit Noor iterative schemes. It is also shown that Jungck implicit iterative schemes converge faster than the corresponding Jungck explicit iterative schemes. Validity of our analytic proofs is shown through numerical examples. Our results are improvements and generalizations of some recent results of Khan et al.[21], Chugh et al.[8] and many others in fixed point theory.

1. Introduction

For the last 25 years, considerable efforts have been devoted to introduce various implicit as well as explicit iterative schemes and study its qualitative features like convergence, stability, convergence rate, data dependency etc. [1–14, 21, 22, 24–29, 31, 32, 34–42]. Implicit iterative schemes are of great importance from numerical stand point as they provide accurate approximation as compared to explicit iterative schemes. A very common iterative scheme is due to Jungck [16] which involves the use of two coupled mappings and is useful in fixed point theory to approximate common fixed point of the mappings. Convergence analysis has a key role in the study of iterative approximation, convergence speed and stability which is of theoretical and practical interest [2, 7–9, 11–14, 19–23, 30, 31, 34, 38, 39, 41, 43]. Data dependence of fixed points is a concept related to iterative schemes which has become an important subject for research now a days. Very recently, Razani and Bagherboum [34], Sahin and Basarir [36], Olatinwo and Postolache [27], Chugh et al. [8] etc. have done remarkable work on iterative schemes in extended metric spaces.

Keeping in mind the above facts, our aim is to:

1. introduce a more general Jungck-type implicit iterative scheme and study its convergence as well as stability for contractive mappings under weak parametric restrictions
2. show that newly introduced iterative scheme has better convergence rate as compared to other Jungck-type implicit iterative schemes

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3. compare Jungck-type implicit iterative schemes with their corresponding Jungck-type explicit iterative schemes
4. prove data dependence results for newly introduced iterative scheme
5. solve a quadratic equation by applying the newly introduced Jungck-type implicit iterative scheme.

Definition 1.1 ([42]). Let (X, g) be a metric space. A map $W : X^2 \times [0, 1] \rightarrow X$ is a convex structure on X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y, u \in X$ and $\lambda \in [0, 1]$. A metric space together with a convex structure W is known as convex metric space and is denoted by (X, d, W) . A nonempty subset C of a convex metric space is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

All normed spaces and their subsets are the examples of convex metric spaces. There are many examples of convex metric spaces which can not be embedded in any normed space (see [42]).

Let X be a convex metric space, Y an arbitrary set and $S, T : Y \rightarrow X$ such that $T(Y) \subseteq S(Y)$. Let $C(S; T)$ be the set of the coincidence points of S and T . For $\alpha_n \in [0, 1]$, Singh et al. [39] defined the Jungck-Mann iterative scheme as follows:

$$Sx_{n+1} = W(Sx_n, Tx_n, \alpha_n) \tag{1.1}$$

For $\alpha_n, \beta_n, \gamma_n \in [0, 1]$, Olatinwo defined the Jungck-Ishikawa [25, 26] and Jungck-Noor [24] iterative schemes as under;

$$\begin{aligned} Sx_{n+1} &= W(Sx_n, Ty_n, \alpha_n) \\ Sy_n &= W(Sx_n, Tx_n, \beta_n) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} Sx_{n+1} &= W(Sx_n, Ty_n, \alpha_n) \\ Sy_n &= W(Sx_n, Tz_n, \beta_n) \\ Sz_n &= W(Sx_n, Tx_n, \gamma_n) \end{aligned} \tag{1.3}$$

respectively.

Very recently, for $\alpha_n + \alpha'_n, \beta_n + \beta'_n, \gamma_n \in [0, 1]$, the following Jungck-Khan [21] iterative scheme was introduced:

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n - \alpha'_n)Sx_n + \alpha_n Ty_n + \alpha'_n Tx_n \\ Sy_n &= (1 - \beta_n - \beta'_n)Sz_n + \beta_n Ty_n + \beta'_n Tx_n \\ Sz_n &= (1 - \gamma_n)Sx_n + \gamma_n Tz_n \end{aligned} \tag{1.4}$$

Remark 1.2. If $X = Y$ and $S = I_d$ (identity mapping), then the Jungck-Noor (1.3), Jungck-Ishikawa (1.2) and the Jungck-Mann (1.1) iterative schemes, respectively, become Noor [23], Ishikawa [15] and Mann [22] iterative schemes.

Jungck and Hussain [17] used the iterative scheme (1.1) to approximate common fixed points of the mappings S and T satisfying the following Jungck-contraction

$$d(Tx, Ty) \leq \alpha d(Sx, Sy), \quad 0 \leq \alpha < 1. \tag{1.5}$$

Singh et al. [39] established some stability results for Jungck and Jungck-Mann iterative schemes for the contractive conditions (1.5) and (1.6) below: for some $a \in [0, 1)$ and $0 \leq L$

$$d(Tx, Ty) \leq \alpha d(Sx, Sy) + Ld(Sx, Tx). \tag{1.6}$$

A mapping T is said to be a contractive if it satisfies (1.7) below: there exists a constant $q \in [0, 1)$ such that for any $x, y \in Y$

$$d(Tx, Ty) \leq q \max \left\{ d(Sx, Sy), \frac{1}{2}[d(Sx, Tx) + d(Sy, Ty)], d(Sx, Ty), d(Sy, Tx) \right\}. \tag{1.7}$$

Olatinwo and Imoru [26], studied the generalized Zamfirescu operators more general than Zamfirescu operators [44] for the pair (S, T) , satisfying the following condition: for each pair of points x, y in Y at least one of the following is true:

- (i) $d(Tx, Ty) \leq ad(Sx, Sy)$
 - (ii) $d(Tx, Ty) \leq b[d(Sx, Tx) + d(Sy, Ty)]$
 - (iii) $d(Tx, Ty) \leq c[d(Sx, Ty) + d(Sy, Tx)],$
- (1.8)

where a, b, c are nonnegative constants satisfying $0 \leq a \leq 1, 0 \leq b, c \leq \frac{1}{2}$.

Any mapping satisfying (1.8)(ii) is called a Kannan mapping while a mapping satisfying (1.8)(iii) is called Chatterjea mapping.

The contractive condition (1.8) implies

$$d(Tx, Ty) \leq 2\delta d(Sx, Tx) + \delta d(Sx, Sy), \quad \forall x, y \in Y, \tag{1.9}$$

where $\delta = \max \left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\}$ (see Berinde [4, 5]).

Hussain et al. [13] and Olatinwo [24, 25], respectively, used the following more general contractive conditions than (1.9) to prove stability and strong convergence results for various iterative schemes: there exists $a \in [0, 1)$ and a monotone increasing function $\phi : R^+ \rightarrow R^+$ with $\phi(0) = 0$, such that

$$\|Tx - Ty\| \leq \phi(\|Sx - Tx\|) + a\|Sx - Sy\|. \tag{1.10}$$

Definition 1.3 ([33]). A map $W : X \times X \times X \times [0, 1] \times [0, 1] \times [0, 1] \rightarrow X$ is said to be a generalized convex structure on X if for each $(x, y, z; a, b, c) \in X \times X \times X \times [0, 1] \times [0, 1] \times [0, 1]$ and $u \in X$,

$$d(u, W(x, y, z; a, b, c)) \leq ad(u, x) + bd(u, y) + cd(u, z); \quad a + b + c = 1.$$

A metric space (X, g) together with the above map W is known as generalized convex metric space.

Definition 1.4 ([33]). Let X be a generalized convex metric space. A nonempty subset C of X is said to be generalized convex if $W(x, y, z; a, b, c) \in C$ whenever

$$(x, y, z; a, b, c) \in C \times C \times C \times [0, 1] \times [0, 1] \times [0, 1].$$

Clearly, every generalized convex metric space is a convex metric space and every generalized convex set is a convex set. It can be easily seen that open spheres and closed spheres in a generalized convex metric space are generalized convex subsets. All normed spaces and their generalized convex subsets are generalized convex metric spaces.

Clearly, a Banach space, or any of it generalized convex subset, is a generalized convex metric space with $W(x, y, z; a, b, c) = ax + by + cz$. More generally, if X is a linear space with a translation invariant metric satisfying

$$d(ax + by + cz, 0) \leq ad(x, 0) + bd(y, 0) + cd(z, 0)$$

then X is a generalized convex metric space.

For $x_0 \in Y$, we define the following Jungck-type implicit iterative scheme in generalized convex metric spaces:

$$\begin{aligned} Sx_{n+1} &= W(Sy_n, Tx_{n+1}, Ty_n, \alpha_n, \alpha'_n) \\ Sy_n &= W(Sz_n, Ty_n, Tz_n, \beta_n, \beta'_n) \\ Sz_n &= W(Sx_n, Tz_n, Tx_n, \gamma_n, \gamma'_n), \end{aligned} \tag{1.11}$$

where $\{\alpha_n + \alpha'_n\}, \{\beta_n + \beta'_n\}, \{\gamma_n + \gamma'_n\}$ are sequences in $[0, 1]$.

The counter part of iterative scheme (1.11) in linear space can be written as

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n - \alpha'_n)Sy_n + \alpha_nTx_{n+1} + \alpha'_nTy_n \\ Sy_n &= (1 - \beta_n - \beta'_n)Sz_n + \beta_nTy_n + \beta'_nTz_n \\ Sz_n &= (1 - \gamma_n - \gamma'_n)Sx_n + \gamma_nTz_n + \gamma'_nTx_n \end{aligned} \tag{1.12}$$

Putting $\alpha'_n = \beta'_n = \gamma'_n = 0$, in (1.11), we get Jungck implicit Noor iterative scheme

$$\begin{aligned} Sx_{n+1} &= W(Sy_n, Tx_{n+1}, \alpha_n) \\ Sy_n &= W(Sz_n, Ty_n, \beta_n) \\ Sz_n &= W(Sx_n, Tz_n, \gamma_n) \end{aligned} \tag{1.13}$$

Putting $\alpha'_n = \beta'_n = \gamma'_n = 0$ and $\gamma_n = 0$ in (1.11), we get Jungck implicit Ishikawa iterative scheme

$$\begin{aligned} Sx_{n+1} &= W(Sy_n, Tx_{n+1}, \alpha_n) \\ Sy_n &= W(Sx_n, Ty_n, \beta_n) \end{aligned} \tag{1.14}$$

Putting $\alpha'_n = \beta'_n = \gamma'_n = 0$ and $\gamma_n = \beta_n = 0$ in (1.11), we get Jungck implicit Mann iterative scheme

$$Sx_{n+1} = W(Sx_n, Tx_{n+1}, \alpha_n) = (1 - \alpha_n)Sx_n + \alpha_nTx_{n+1} \tag{1.15}$$

Putting $X = Y$ and $S = I_d$ (identity mapping), in Jungck implicit Noor (1.13), Jungck implicit Ishikawa (1.14) and the Jungck implicit Mann (1.15) iterative schemes, respectively, we get implicit Noor [8], implicit Ishikawa [8] and implicit Mann [8] iterative schemes.

Kohlenbach [17] extended convex structure as hyperbolic space as follows:

Definition 1.5 ([17]). A hyperbolic space (X, d, W) is a metric space (X, g) together with a convex structure W satisfying the following conditions:

- (W1) $d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)$
- (W2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2|d(x, y)$
- (W3) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$
- (W4) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)$ for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$.

Evidently, every hyperbolic space is a convex metric space but the converse may not true. For example, if $X = \mathbb{R}$, $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ and $d(x, y) = \frac{|x-y|}{1+|x-y|}$ for $x, y \in \mathbb{R}$, then (X, d, W) is a convex metric space but not a hyperbolic space [19].

Motivated by above facts, we introduce a generalized hyperbolic space as follows:

Definition 1.6. A generalized hyperbolic space (X, d, W) is a metric space (X, g) together with a convex structure W satisfying the following conditions:

- (W1) $d(p, W(x, y, z, \lambda_1, \lambda_2)) \leq (1 - \lambda_1 - \lambda_2)d(p, x) + \lambda_1 d(p, y) + \lambda_2 d(p, z)$
- (W2) $d(W(x, y, z, \lambda_1, \lambda_2), W(x, y, z, \lambda_3, \lambda_4)) = |\lambda_1 - \lambda_3|d(x, y) + |\lambda_2 - \lambda_4|d(y, z)$
- (W3) $W(x, y, z, \lambda_1, \lambda_2) = W(y, x, z, 1 - \lambda_1, \lambda_2) = W(x, z, y, \lambda_1, 1 - \lambda_2)$
- (W4) $d(W(x, y, z, \lambda_1, \lambda_2), W(u, v, w, \lambda_1, \lambda_2)) \leq (1 - \lambda_1 - \lambda_2)d(x, u) + \lambda_1 d(y, v) + \lambda_2 d(z, w)$ for all $x, y, z, u, v, w \in X$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, 1]$.

The stability of iterative schemes has been extensively studied by various authors [2, 3, 7–9, 11–13, 21, 39]. The concept of T-stability in convex metric space setting was given by Olatinwo and Postolache [27].

Definition 1.7 ([16]). Let f and g be two selfmaps on X . A point x in X is called (1) a fixed point of f if $f(x) = x$; (2) coincidence point of a pair (f, g) if $fx = gx$; (3) common fixed point of a pair (f, g) if $x = fx = gx$. If $w = fx = gx$ for some x in X , then w is called a point of coincidence of f and g . A pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points.

Definition 1.8 ([27]). Let $S, T : X \rightarrow X$ be operators such that $T(X) \subseteq S(X)$ and $p = Sz = Tz$ a point of coincidence of S and T . Let $\{Sx_n\}_{n=0}^\infty \subset X$, be the sequence generated by an iterative procedure

$$Sx_{n+1} = f_{T, \alpha_n}^{x_n}, \quad n = 0, 1, 2, \dots \tag{1.16}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{Sx_n\}_{n=0}^\infty$ converges to p . Let $\{Sy_n\}_{n=0}^\infty \subset X$ be an arbitrary sequence in X and set $\epsilon_n = d(Sy_{n+1}, f_{T, \alpha_n}^{x_n})$, $n = 0, 1, 2, \dots$. Then, the iterative procedure (1.16) is said to be (S, T) -stable or stable if and only if $\lim_{n \rightarrow \infty} \epsilon_n = 0$ implies $\lim_{n \rightarrow \infty} Sy_n = p$.

Definition 1.9 ([4, 32]). Suppose $\{a_n\}$ and $\{b_n\}$ are two real convergent sequences with limits a and b , respectively. Then $\{a_n\}$ is said to converge faster than $\{b_n\}$ if

$$\lim_{n \rightarrow \infty} \left| \frac{a_n - a}{b_n - b} \right| = 0.$$

Definition 1.10 ([4, 5]). Let $\{u_n\}$ and $\{v_n\}$ be two iterative schemes which converge in a normed space X to the same fixed point p such that the following error estimates

$$\|u_n - p\| \leq a_n$$

and

$$\|v_n - p\| \leq b_n$$

are available, where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers, converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then we say that $\{u_n\}$ converge faster to p than $\{v_n\}$.

Lemma 1.11 ([4]). Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are nonnegative real sequences satisfying the following condition:

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $t_n \in (0, 1]$, $\sum_{n=0}^\infty t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^\infty c_n < \infty$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

Definition 1.12. Let $(S, T), (S_1, T_1)$ be pairs of operator on X . We say (S_1, T_1) is approximate operator pair of (S, T) if for all $x \in X$ and for fixed $\epsilon > 0, \epsilon_1 > 0$, we have $d(Tx, T_1x) \leq \epsilon, d(Sx, S_1x) \leq \epsilon_1$.

Lemma 1.13 ([10]). Let $\{a_n\}_{n=0}^\infty$ be a nonnegative sequence for which there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, one has the following inequality:

$$a_{n+1} \leq (1 - r_n)a_n + r_n t_n,$$

where $r_n \in (0, 1), \sum_{n=1}^\infty r_n = \infty$ and $t_n \geq 0, \forall n \in \mathbb{N}$.

$$\text{Then } 0 \leq \lim_{n \rightarrow \infty} \sup a_n \leq \lim_{n \rightarrow \infty} \sup t_n.$$

Lemma 1.14 ([4]). If δ is a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^\infty$ is a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^\infty$ satisfying

$$u_{n+1} \leq \delta u_n + \epsilon_n, \quad n = 0, 1, 2, \dots$$

one has $\lim_{n \rightarrow \infty} u_n = 0$.

2. Convergence and Stability Results of New Jungck-Type Implicit Iterative Scheme in Generalized Convex Metric Spaces

Theorem 2.1. Let X be a generalized convex metric space and $S, T : Y \rightarrow X$ be nonself operators on an arbitrary set Y satisfying (1.10) such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of X . Let $z \in C(S, T)$ such that $Sz = Tz = p$ (say). Then for $x_0 \in Y$, the iterative scheme $\{Sx_n\}_{n=0}^\infty$ defined by (1.11) with $\sum_{n=1}^\infty (\alpha_n + \alpha'_n) = \infty$ converges strongly to p . Also, p will be the unique common fixed point of S, T provided S and T are weakly compatible.

Proof. Using (1.11) and (1.10), we have

$$\begin{aligned} d(Sx_{n+1}, p) &= d(W(Sy_n, Tx_{n+1}, Ty_n, \alpha_n, \alpha'_n), p) \\ &\leq (1 - \alpha_n - \alpha'_n)d(Sy_n, p) + \alpha_n d(Tx_{n+1}, p) + \alpha'_n d(Ty_n, p) \\ &\leq (1 - \alpha_n - \alpha'_n)d(Sy_n, p) + \alpha_n ad(Sx_{n+1}, p) + \alpha'_n ad(Sy_n, p) \end{aligned} \tag{2.1}$$

which further implies

$$\begin{aligned} (1 - \alpha_n a)d(Sx_{n+1}, p) &\leq 1 - \alpha_n - \alpha'_n(1 - a)d(Sy_n, p) \\ d(Sx_{n+1}, p) &\leq \frac{1 - \alpha_n - \alpha'_n(1 - a)}{(1 - \alpha_n a)}d(Sy_n, p). \end{aligned}$$

Again from (1.11), we have the following estimates:

$$\begin{aligned} d(Sy_n, p) &= d(W(Sz_n, Ty_n, Tz_n, \beta_n, \beta'_n), p) \\ &\leq (1 - \beta_n - \beta'_n)d(Sz_n, p) + \beta_n d(Ty_n, p) + \beta'_n d(Tz_n, p) \\ &\leq (1 - \beta_n - \beta'_n)d(Sz_n, p) + \beta_n ad(Sy_n, p) + \beta'_n ad(Sz_n, p) \end{aligned}$$

which gives

$$d(Sy_n, p) \leq \frac{1 - \beta_n - \beta'_n(1 - a)}{(1 - \beta_n a)}d(Sz_n, p). \tag{2.2}$$

Also, (1.11) gives

$$\begin{aligned} d(Sz_n, p) &= d(W(Sx_n, Tz_n, Tx_n, \gamma_n, \gamma'_n), p) \\ &\leq (1 - \gamma_n - \gamma'_n)d(Sx_n, p) + \gamma_n d(Tz_n, p) + \gamma'_n d(Tx_n, p) \\ &\leq (1 - \gamma_n - \gamma'_n)d(Sx_n, p) + \gamma_n ad(Sz_n, p) + \gamma'_n ad(Sx_n, p) \end{aligned}$$

which implies

$$d(Sz_n, p) \leq \frac{1 - \gamma_n - \gamma'_n(1 - a)}{(1 - \gamma_n a)}d(Sx_n, p). \tag{2.3}$$

Estimates (2.1)-(2.3) yield

$$\begin{aligned} d(Sx_{n+1}, p) &\leq \frac{1 - \alpha_n - \alpha'_n(1 - a)}{(1 - \alpha_n a)} \frac{1 - \beta_n - \beta'_n(1 - a)}{(1 - \beta_n a)} \frac{1 - \gamma_n - \gamma'_n(1 - a)}{(1 - \gamma_n a)} d(Sx_n, p) \\ &\leq \frac{1 - \alpha_n - \alpha'_n(1 - a)}{(1 - \alpha_n a)} \frac{1 - \beta_n}{(1 - \beta_n a)} \frac{(1 - \gamma_n)}{(1 - \gamma_n a)} d(Sx_n, p). \end{aligned} \tag{2.4}$$

But $1 - \beta_n \leq 1 - \beta_n a$, gives

$$\frac{1 - \beta_n}{1 - \beta_n a} \leq 1. \tag{2.5}$$

Similarly,

$$\frac{1 - \gamma_n}{1 - \gamma_n a} \leq 1. \tag{2.6}$$

Also

$$1 - \frac{1 - \alpha_n - \alpha'_n(1 - a)}{(1 - \alpha_n a)} = \frac{1 - [1 - \alpha_n(1 - a) - \alpha'_n(1 - a)]}{(1 - \alpha_n a)} \geq 1 - [1 - \alpha_n(1 - a) - \alpha'_n(1 - a)],$$

implies

$$\frac{1 - \alpha_n - \alpha'_n(1 - a)}{(1 - \alpha_n a)} \leq [1 - \alpha_n(1 - a) - \alpha'_n(1 - a)] \tag{2.7}$$

Using (2.5)-(2.7), (2.4) yields

$$d(Sx_{n+1}, p) \leq [1 - \alpha_n(1 - a) - \alpha'_n(1 - a)]d(Sx_n, p) \tag{2.8}$$

It can be easily seen that inequality (2.8) fulfils all the condition of Lemma 1.11 and hence by its conclusion, $\lim_{n \rightarrow \infty} d(Sx_n, p) = 0$. Therefore $\{Sx_n\}$ converges to p .

Now, we prove that p is the unique common fixed point of S and T , when $Y = X$.

Let there exist another point of coincidence say p^* . Then, there exists $z^* \in X$ such that $Sz^* = Tz^* = p^*$. But from (1.10), we have

$$0 \leq \|p - p^*\| = \|Tz - Tz^*\| \leq q\|Sz - Sz^*\| + \phi(\|Sz - Tz\|) = q\|p - p^*\|$$

which implies $p = p^*$ as $0 \leq q < 1$.

Now, as S and T are weakly compatible and $p = Tz = Sz$, so $Tp = TTz = TSz = STz$ and hence $Tp = Sp$. Therefore, Tp is a point of coincidence of S, T and as the point of coincidence is unique, so $p = Tp$. Thus $Tp = Sp = p$ and therefore p is unique common fixed point of S and T . \square

Theorem 2.2. Let X be a generalized convex metric space and $S, T : Y \rightarrow X$ are nonself operators on an arbitrary set Y satisfying (1.10) such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of X . Let $z \in C(S, T)$ such that $Sz = Tz = p$ (say) and the iterative scheme $\{Sx_n\}_{n=0}^\infty$ defined by (1.11) with $\sum_{n=1}^\infty (\alpha_n + \alpha'_n) = \infty$ converges to p . Then the iterative scheme $\{Sx_n\}_{n=0}^\infty$ is (S, T) stable provided $0 < \alpha_n + \alpha'_n < \frac{1}{1+a}, \forall n \in N$.

Proof. Suppose that $\{Sp_n\}_{n=0}^\infty \subset X$, be an arbitrary sequence, $\epsilon_n = d(Sp_{n+1}, W(Sq_n, Tp_{n+1}, Tq_n, \alpha_n, \alpha'_n))$, where $Sq_n = W(Sr_n, Tq_n, Tr_n, \beta_n, \beta'_n)$, $Sr_n = W(Sp_n, Tr_n, Tp_n, \gamma_n, \gamma'_n)$ and let $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Then, we have

$$\begin{aligned} d(Sp_{n+1}, p) &\leq d(Sp_n, W(Sq_n, Tp_{n+1}, Tq_n, \alpha_n, \alpha'_n)) + d(W(Sq_n, Tp_{n+1}, Tq_n, \alpha_n, \alpha'_n), p) \\ &\leq \epsilon_n + (1 - \alpha_n - \alpha'_n)d(Sq_n, p) + \alpha_n d(Tp_{n+1}, p) + \alpha'_n d(Tq_n, p) \\ &\leq \epsilon_n + (1 - \alpha_n - \alpha'_n)d(Sq_n, p) + \alpha_n a d(Sp_{n+1}, p) + \alpha'_n a d(Sq_n, p) \end{aligned} \tag{2.9}$$

which implies

$$[1 - \alpha_n a]d(Sp_{n+1}, p) \leq \epsilon_n + (1 - \alpha_n - \alpha'_n(1 - a))d(Sq_n, p)$$

and therefore

$$d(Sp_{n+1}, p) \leq \frac{\epsilon_n}{1 - \alpha_n a} + \frac{(1 - \alpha_n - \alpha'_n(1 - a))}{\{1 - \alpha_n a\}}d(Sq_n, p) \tag{2.10}$$

Using $\frac{[1-\alpha_n-\alpha'_n(1-a)]}{(1-\alpha_n a)} \leq [1 - \alpha_n(1 - a) - \alpha'_n(1 - a)]$, (2.10) becomes

$$d(Sp_{n+1}, p) \leq [1 - (\alpha_n + \alpha'_n)(1 - a)]d(Sq_n, p) + \frac{\epsilon_n}{1 - \alpha_n a}. \tag{2.11}$$

As in (2.2) and (2.3), respectively, we have the following estimates:

$$d(Sq_n, p) \leq \frac{1 - \beta_n}{(1 - \beta_n a)} d(Sr_n, p) \leq d(Sr_n, p), \tag{2.12}$$

and

$$d(Sr_n, p) \leq \frac{1 - \gamma_n}{1 - \gamma_n a} d(Sp_n, p) \leq d(Sp_n, p), \text{ respectively.} \tag{2.13}$$

Using estimates (2.12) and (2.13), (2.11) becomes

$$\begin{aligned} d(Sp_{n+1}, p) &\leq [1 - (\alpha_n + \alpha'_n)(1 - a)]d(Sp_n, p) + \frac{\epsilon_n}{1 - \alpha_n a} \\ &\leq [1 - (\alpha_n + \alpha'_n)(1 - a)]d(Sp_n, p) + \frac{\epsilon_n}{1 - a} \end{aligned} \tag{2.14}$$

Inequality (2.14) fulfils all conditions of Lemma 1.14, hence Lemma 1.14 gives $\lim_{n \rightarrow \infty} d(Sp_n, p) = 0$.

Conversely, if we let $\lim_{n \rightarrow \infty} Sp_n = p$, then using contractive condition (1.10), we have

$$\begin{aligned} \epsilon_n &= d(Sp_{n+1}, W(Sq_n, Tp_{n+1}, Tq_n, \alpha_n, \alpha'_n)) \\ &\leq d(Sp_{n+1}, p) + d(W(Sq_n, Tp_{n+1}, Tq_n, \alpha_n, \alpha'_n), p) \\ &\leq d(Sp_{n+1}, p) + (1 - \alpha_n - \alpha'_n)d(Sq_n, p) + \alpha_n ad(Sp_n, p) + \alpha'_n ad(Sq_n, p) \end{aligned} \tag{2.15}$$

Using estimates (2.12)-(2.13), (2.15) becomes

$$\epsilon_n \leq d(Sp_n, p) + \{1 - \alpha_n(1 - a) - \alpha'_n(1 - a)\}d(Sp_n, p),$$

from which we conclude that $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Therefore, the iterative scheme (1.11) is (S, T)-stable. \square

Remark 2.3. As contractive condition (1.10) is more general than those of (1.5)-(1.9), the convergence and stability results for new Jungck implicit iterative scheme (1.11), using contractive conditions (1.5)-(1.9), can be obtained as special cases.

Remark 2.4. As Jungck implicit Noor, Jungck implicit Ishikawa and Jungck implicit Mann iterative schemes are special case of new Jungck implicit iterative scheme (1.11), convergence and stability results for these iterative schemes can be obtained as corollaries.

Remark 2.5. As implicit Noor iterative scheme [8] is a special case of new Jungck implicit iterative scheme (1.11), convergence result of implicit Noor iterative scheme [8, Theorem 9] can be obtained as a corollary.

Remark 2.6. As implicit Noor iterative scheme [8] is a special case of new Jungck implicit iterative scheme (1.11), stability result of implicit Noor iterative scheme [8, Theorem 10] can be obtained as a corollary.

Remark 2.7. As new Jungck implicit iterative scheme (1.11) converges faster as compared to Jungck-Khan iterative scheme, our results are improvement of results of Khan et al. [21]. Moreover, our results admit weak parametric restriction $\sum_{n=1}^{\infty} (\alpha_n + \alpha'_n) = \infty$ instead of $\sum_{n=1}^{\infty} \alpha_n = \infty$.

3. Convergence Rate Comparison of Implicit Iterative Schemes

Theorem 3.1. *Let X be a generalized convex metric space and $S, T : Y \rightarrow X$ are nonself operators on an arbitrary set Y satisfying (1.10) with $a \in (0, 1)$ such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of X . Let $z \in C(S, T)$ such that $Sz = Tz = p$ (say). Then for $x_0 \in C$, the sequence $\{Sx_n\}_{n=0}^\infty$ defined by (1.11), converges faster than Jungck implicit Mann (1.15), Jungck implicit Ishikawa (1.14) and Jungck implicit Noor (1.13) iterative schemes. Also, Jungck implicit-type iterative schemes converge faster as compared to their corresponding Jungck explicit-type iterative schemes.*

Proof. For Jungck implicit Mann iteration (1.15), we have

$$\begin{aligned} d(Sx_{n+1}, p) &= d(W(Sx_n, Tx_{n+1}, \alpha_n), p) \\ &\leq (1 - \alpha_n)d(Sx_n, p) + \alpha_n d(Tx_{n+1}, p) \\ &\leq (1 - \alpha_n)d(Sx_n, p) + \alpha_n a d(Sx_{n+1}, p), \end{aligned}$$

which further yields

$$(1 - \alpha_n a)d(Sx_{n+1}, p) \leq (1 - \alpha_n)d(Sx_n, p)$$

and so

$$d(Sx_{n+1}, p) \leq \frac{1 - \alpha_n}{1 - \alpha_n a} d(Sx_n, p). \tag{3.1}$$

Similarly, for Jungck implicit Ishikawa (1.14) and Jungck implicit Noor (1.13) iterations, respectively, we have the following estimates:

$$d(Sx_{n+1}, p) \leq \left(\frac{1 - \alpha_n}{1 - \alpha_n a} \right) \left(\frac{1 - \beta_n}{1 - \beta_n a} \right) d(Sx_n, p) \tag{3.2}$$

and

$$d(Sx_{n+1}, p) \leq \left(\frac{1 - \alpha_n}{1 - \alpha_n a} \right) \left(\frac{1 - \beta_n}{1 - \beta_n a} \right) \left(\frac{1 - \gamma_n}{1 - \gamma_n a} \right) d(Sx_n, p) \tag{3.3}$$

For Jungck explicit Mann iteration (1.1), we have

$$\begin{aligned} d(Sx_{n+1}, p) &= d(W(Sx_n, Tx_n, \alpha_n), p) \\ &\leq (1 - \alpha_n)d(Sx_n, p) + \alpha_n d(Tx_n, p) \\ &\leq (1 - \alpha_n)d(Sx_n, p) + \alpha_n a d(Sx_n, p) \\ &= [1 - \alpha_n(1 - a)]d(Sx_n, p). \end{aligned} \tag{3.4}$$

For Jungck explicit Ishikawa iteration (1.2), we have

$$d(Sx_{n+1}, p) \leq [(1 - \alpha_n) + \alpha_n a(1 - \beta_n(1 - a))]d(Sx_n, p) \tag{3.5}$$

For Jungck explicit Noor iterative scheme (1.3), we have

$$d(Sx_{n+1}, p) \leq [1 - \alpha_n + \alpha_n a(1 - \beta_n + \beta_n a(1 - \gamma_n(1 - a)))]d(Sx_n, p) \tag{3.6}$$

For Jungck Khan iterative scheme (1.4), we have

$$d(Sx_{n+1}, p) \leq [1 - \alpha_n + \alpha_n a(1 - \beta'_n(1 - a) - \beta_n(1 - a)(1 + \gamma_n a))]d(Sx_n, p) \tag{3.7}$$

For new Jungck implicit iterative scheme (1.11), we have

$$d(Sx_{n+1}, p) \leq \frac{\{1 - \alpha_n - \alpha'_n(1 - a)\}}{(1 - \alpha_na)} \frac{\{1 - \beta_n - \beta'_n(1 - a)\}}{(1 - \beta_na)} \frac{\{1 - \gamma_n - \gamma'_n(1 - a)\}}{(1 - \gamma_na)} d(Sx_n, p). \tag{3.8}$$

As $1 - \frac{1 - \alpha_n}{(1 - \alpha_na)} = \frac{1 - \alpha_na - 1 + \alpha_n}{1 - \alpha_na} \geq 1 - (1 - \alpha_n(1 - a))$, and

$$1 - \frac{1 - \alpha_n - \alpha'_n(1 - a)}{(1 - \alpha_na)} = \frac{1 - (1 - \alpha_n(1 - a) - \alpha'_n(1 - a))}{1 - \alpha_na} \geq 1 - (1 - \alpha_n(1 - a) - \alpha'_n(1 - a)),$$

so we have the following estimates:

$$\frac{1 - \alpha_n}{(1 - \alpha_na)} \leq 1 - \alpha_n(1 - a), \tag{3.9}$$

$$\frac{1 - \alpha_n - \alpha'_n(1 - a)}{(1 - \alpha_na)} \leq 1 - \alpha_n(1 - a) - \alpha'_n(1 - a), \tag{3.10}$$

$$\left(\frac{\{1 - \alpha_n - \alpha'_n(1 - a)\}}{(1 - \alpha_na)}\right) \leq \frac{\{1 - \alpha_n\}}{(1 - \alpha_na)} \left(\frac{\{1 - \beta_n - \beta'_n(1 - a)\}}{(1 - \beta_na)}\right) \leq \frac{\{1 - \beta_n\}}{(1 - \beta_na)} \left(\frac{\{1 - \gamma_n - \gamma'_n(1 - a)\}}{(1 - \gamma_na)}\right) \leq \frac{(1 - \gamma_n)}{(1 - \gamma_na)} \tag{3.11}$$

$$\begin{aligned} \left(\frac{\{1 - \alpha_n - \alpha'_n(1 - a)\}}{(1 - \alpha_na)}\right) \left(\frac{\{1 - \beta_n - \beta'_n(1 - a)\}}{(1 - \beta_na)}\right) \left(\frac{\{1 - \gamma_n - \gamma'_n(1 - a)\}}{(1 - \gamma_na)}\right) &\leq \left(\frac{\{1 - \alpha_n\}}{(1 - \alpha_na)}\right) \left(\frac{\{1 - \beta_n\}}{(1 - \beta_na)}\right) \left(\frac{(1 - \gamma_n)}{(1 - \gamma_na)}\right) \\ &\leq \left(\frac{\{1 - \alpha_n\}}{(1 - \alpha_na)}\right) \left(\frac{\{1 - \beta_n\}}{(1 - \beta_na)}\right) \\ &\leq \left(\frac{\{1 - \alpha_n\}}{(1 - \alpha_na)}\right), \end{aligned} \tag{3.12}$$

$$\begin{aligned} \left(\frac{\{1 - \alpha_n - \alpha'_n(1 - a)\}}{(1 - \alpha_na)}\right) \left(\frac{\{1 - \beta_n - \beta'_n(1 - a)\}}{(1 - \beta_na)}\right) \left(\frac{\{1 - \gamma_n - \gamma'_n(1 - a)\}}{(1 - \gamma_na)}\right) \\ \leq 1 - \alpha_n(1 - a)1 - \beta_n(1 - a)1 - \gamma_n(1 - a) \end{aligned} \tag{3.13}$$

$$\begin{aligned} \left(\frac{\{1 - \alpha_n - \alpha'_n(1 - a)\}}{(1 - \alpha_na)}\right) \left(\frac{\{1 - \beta_n - \beta'_n(1 - a)\}}{(1 - \beta_na)}\right) \left(\frac{\{1 - \gamma_n - \gamma'_n(1 - a)\}}{(1 - \gamma_na)}\right) \\ \leq 1 - \alpha_n(1 - a) - \alpha'_n(1 - a)1 - \beta_n(1 - a) - \beta'_n(1 - a)1 - \gamma_n(1 - a)\gamma'_n(1 - a) \\ \leq 1 - \alpha_n + \alpha_na1 - \beta'_n(1 - a) - \beta_n(1 - a)(1 + \gamma_na) \end{aligned} \tag{3.14}$$

$$\begin{aligned} \left(\frac{\{1 - \alpha_n - \alpha'_n(1 - a)\}}{(1 - \alpha_na)}\right) \left(\frac{\{1 - \beta_n - \beta'_n(1 - a)\}}{(1 - \beta_na)}\right) \leq \{1 - \alpha_n(1 - a)\}\{1 - \beta_n(1 - a)\} \\ \leq 1 - \alpha_n + \alpha_na\{1 - \beta_n(1 - a)\} \end{aligned} \tag{3.15}$$

Keeping in mind Berinde’s Definition 1.10 and inequalities (3.9)-(3.15), from estimates (3.1)-(3.8), we conclude that

1. Ascending order of convergence rate of Jungck-type implicit iterative schemes goes as follows: Jungck implicit Mann, Jungck implicit Ishikawa, Jungck implicit Noor, new Jungck implicit iterative scheme
2. Jungck implicit type iterative schemes converge faster as compared to their corresponding Jungck explicit type iterative schemes.

Example 3.2. Let $Y = [0, 1]$, $X = [0, 0.5]$, $T(x) = \frac{x}{4}$, $S(x) = \frac{x}{2}$, $\alpha_n = \beta_n = \gamma_n = \frac{4}{\sqrt{n}}$, $n \geq 64$ and for $n = 1, 2, \dots, 63$,

$\alpha_n = \beta_n = \gamma_n = 0$. Then for iterative schemes (1.1)-(1.4), (1.11), (1.13)-(1.15), we have

$$Sx_{n+1}(1.1) = \prod_{i=64}^n \left(\frac{\sqrt{i}-2}{2\sqrt{i}} \right) x_0. \tag{3.16}$$

$$Sx_{n+1}(1.15) = \prod_{i=64}^n \frac{1}{2} \left(\frac{\sqrt{i}-4}{\sqrt{i}-2} \right) x_0. \tag{3.17}$$

$$Sx_{n+1}(1.2) = \prod_{i=64}^n \left(\frac{i^{\frac{3}{2}} - 2i - 4\sqrt{i}}{2i^{\frac{3}{2}}} \right) x_0. \tag{3.18}$$

$$Sx_{n+1}(1.14) = \prod_{i=64}^n \frac{1}{2} \left(\frac{\sqrt{i}-4}{\sqrt{i}-2} \right)^2 x_0. \tag{3.19}$$

$$Sx_{n+1}(1.3) = \prod_{i=64}^n \left(\frac{i^{\frac{3}{2}} - 2i - 4\sqrt{i} - 8}{2i^{\frac{3}{2}}} \right) x_0. \tag{3.20}$$

$$Sx_{n+1}(1.13) = \prod_{i=64}^n \frac{1}{2} \left(\frac{\sqrt{i}-4}{\sqrt{i}-2} \right)^3 x_0. \tag{3.21}$$

$$Sx_{n+1}(1.4) = \prod_{i=64}^n \left(\frac{i^{\frac{3}{2}} - 4i - 8\sqrt{i} - 8}{2i^{\frac{3}{2}}} \right) x_0. \tag{3.22}$$

$$Sx_{n+1}(1.11) = \prod_{i=64}^n \frac{1}{2} \left(\frac{\sqrt{i}-6}{\sqrt{i}-2} \right)^3 x_0. \tag{3.23}$$

Using (3.16) and (3.17), we have

$$\frac{x_n(1.15)}{x_n(1.1)} = \prod_{i=64}^n \left(\frac{i-4\sqrt{i}}{i+4-4\sqrt{i}} \right).$$

But

$$0 \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(\frac{i-4\sqrt{i}}{i+4-4\sqrt{i}} \right) \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(1 - \frac{1}{i} \right) = \lim_{n \rightarrow \infty} \frac{63}{64} \times \frac{64}{65} \dots \frac{n-1}{n} = \lim_{n \rightarrow \infty} \frac{63}{n} = 0.$$

Hence $\lim_{n \rightarrow \infty} \left| \frac{x_n(1.15)-0}{x_n(1.1)-0} \right| = 0$. Therefore, by Definition 1.9, Jungck implicit Mann iterative scheme (1.15) converges faster than the corresponding Jungck Mann iterative scheme (1.1) to $p = 0$, the point of coincidence of S and T .

Also, using (3.18) and (3.19)

$$\begin{aligned} \frac{x_n(1.14)}{x_n(1.2)} &= \prod_{i=64}^n \left(\frac{2i^{3/2}}{i^{3/2} - 2i - 4\sqrt{i}} \right) \left(\frac{\sqrt{i}-2}{\sqrt{i}-4} \right)^2 \\ &= \prod_{i=64}^n \left(\frac{2i^{5/2} + 32i^{3/2} - 16i^2}{i^{5/2} - 6i^2 + 8i^{3/2} + 8i - 16\sqrt{i}} \right) \\ &= \prod_{i=64}^n \left[1 - \frac{(-i^{5/2} + 10i^2 - 24i^{3/2} + 8i - 16\sqrt{i})}{i^{5/2} - 6i^2 + 8i^{3/2} + 8i - 16\sqrt{i}} \right] \end{aligned}$$

with

$$0 \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left[1 - \frac{(-i^{5/2} + 10i^2 - 24i^{3/2} + 8i - 16\sqrt{i})}{i^{5/2} - 6i^2 + 8i^{3/2} + 8i - 16\sqrt{i}} \right] \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(1 - \frac{1}{i} \right) = 0$$

implies

$$\left| \frac{x_n(1.14) - 0}{x_n(1.2) - 0} \right| = 0.$$

Therefore, Jungck implicit Ishikawa iterative scheme converges faster as compared to the corresponding Jungck explicit Ishikawa iterative scheme.

Also, using (3.20) and (3.21)

$$\begin{aligned} \frac{x_n(1.13)}{x_n(1.3)} &= \prod_{i=64}^n \left(\frac{i^3 - 12i^{5/2} - 64i^{3/2} + 48i^2}{i^3 - 8i^{5/2} - 16i^{3/2} + 20i^2 + 16i - 88\sqrt{i} + 64} \right) \\ &= \prod_{i=64}^n \left(1 - \frac{4i^{5/2} + 48i^{3/2} - 28i^2 + 16i - 88\sqrt{i} + 64}{i^3 - 8i^{5/2} - 16i^{3/2} + 20i^2 + 16i - 88\sqrt{i} + 64} \right) \end{aligned}$$

with

$$0 \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(1 - \frac{4i^{5/2} + 48i^{3/2} - 28i^2 + 16i - 88\sqrt{i} + 64}{i^3 - 8i^{5/2} - 16i^{3/2} + 20i^2 + 16i - 88\sqrt{i} + 64} \right) \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(1 - \frac{1}{i} \right) = 0$$

implies

$$\left| \frac{x_n(1.13) - 0}{x_n(1.3) - 0} \right| = 0.$$

Therefore, Jungck implicit Noor iterative scheme converges faster as compared to the corresponding Jungck explicit Noor iterative scheme.

Also, using (3.21) and (3.23)

$$\frac{x_n(1.11)}{x_n(1.13)} = \prod_{i=64}^n \left(\frac{\sqrt{i} - 6}{\sqrt{i} - 4} \right)^3$$

with

$$0 \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(\frac{\sqrt{i} - 6}{\sqrt{i} - 4} \right)^3 \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(1 - \frac{1}{i} \right) = 0$$

implies

$$\left| \frac{x_n(1.11) - 0}{x_n(1.13) - 0} \right| = 0.$$

Therefore, new implicit Jungck type iterative scheme converges faster as compared to the Jungck implicit Noor iterative scheme.

Moreover, using (3.22) and (3.23) we have

$$\begin{aligned} \frac{x_n(1.11)}{x_n(1.4)} &= \prod_{i=64}^n \left(\frac{i^3 - 18i^{5/2} - 216i^{3/2} + 108i^2}{i^3 - 10i^{5/2} - 16i^{3/2} + 28i^2 - 16i - 32\sqrt{i} + 64} \right) \\ &= \prod_{i=64}^n \left(1 - \frac{8i^{5/2} + 200i^{3/2} - 80i^2 - 16i - 32\sqrt{i} + 64}{i^3 - 10i^{5/2} - 16i^{3/2} + 28i^2 - 16i - 32\sqrt{i} + 64} \right) \end{aligned}$$

with

$$0 \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(1 - \frac{8i^{5/2} + 200i^{3/2} - 80i^2 - 16i - 32\sqrt{i} + 64}{i^3 - 10i^{5/2} - 16i^{3/2} + 28i^2 - 16i - 32\sqrt{i} + 64} \right) \leq \lim_{n \rightarrow \infty} \prod_{i=64}^n \left(1 - \frac{1}{i} \right) = 0$$

which implies

$$\left| \frac{x_n(1.11) - 0}{x_n(1.4) - 0} \right| = 0.$$

Therefore new implicit Jungck type iterative scheme converges faster as compared to the Jungck Khan iterative scheme.

4. Data Dependency of New Jungck Type Implicit Iterative Scheme in Generalized Hyperbolic Spaces

The study of data dependence of fixed point has great importance from numerical and theoretical point of view and hence become a new trend among researchers; (see [4, 8–10, 21, 40] and references therein).

Theorem 4.1. Let (X, d, W) be a generalized hyperbolic space and $(S_1, T_1) : Y \rightarrow X$ be an approximate operator of the pair $(S, T) : Y \rightarrow X$ on an arbitrary set Y with (S, T) satisfying contractive condition (1.10). Assume that $T(Y) \subseteq S(Y)$, $T_1(Y) \subseteq S_1(Y)$, where $S(Y)$ and $S_1(Y)$ are complete subspaces of X with $Sz = Tz = p$ and $Sz^* = T_1z^* = p^*$. Suppose that the iterative scheme $\{Sx_n\}_{n=0}^\infty$ defined by (1.11) converges to p and $\{Su_n\}_{n=0}^\infty$ be the iterative scheme generated by

$$\begin{aligned} S_1u_{n+1} &= W(S_1v_n, T_1u_{n+1}, T_1v_n, \alpha_n, \alpha'_n) \\ S_1v_n &= W(S_1w_n, T_1v_n, T_1w_n, \beta_n, \beta'_n) \\ S_1w_n &= W(S_1u_n, T_1w_n, T_1u_n, \gamma_n, \gamma'_n) \end{aligned} \tag{4.1}$$

converges to p^* . Then we have

$$d(p, p^*) \leq \frac{3(\epsilon + a\epsilon_1)}{(1-a)^2} \text{ provided } \sum_n (\alpha_n + \alpha'_n) = \infty \text{ and } \beta_n + \beta'_n, \gamma_n + \gamma'_n \leq \alpha_n + \alpha'_n.$$

Proof. Using Definition 1.6, iterative schemes (1.11) and (4.1) yield the following estimates:

$$\begin{aligned} d(Sx_{n+1}, S_1u_{n+1}) &= d(W(Sy_n, Tx_{n+1}, Ty_n, \alpha_n, \alpha'_n), W(S_1v_n, T_1u_{n+1}, T_1v_n, \alpha_n, \alpha'_n)) \\ &\leq (1 - \alpha_n - \alpha'_n)d(Sy_n, S_1v_n) + \alpha_n d(Tx_{n+1}, T_1u_{n+1}) + \alpha'_n d(Ty_n, T_1v_n) \\ &\leq (1 - \alpha_n - \alpha'_n)d(Sy_n, S_1v_n) + \alpha_n d(Tx_{n+1}, Tu_{n+1}) + d(Tu_{n+1}, T_1u_{n+1}) \\ &\quad + \alpha'_n d(Ty_n, Tv_n) + d(Tv_n, T_1v_n) \\ &\leq (1 - \alpha_n - \alpha'_n)d(Sy_n, S_1v_n) + \alpha_n \{\epsilon + \varphi d(Sx_{n+1}, Tx_{n+1}) + ad(Sx_{n+1}, Su_{n+1})\} \\ &\quad + \alpha'_n \{\epsilon + \varphi d(Sy_n, Ty_n) + ad(Sy_n, Sv_n)\} \\ &\leq (1 - \alpha_n - \alpha'_n)d(Sy_n, S_1v_n) + \alpha_n \{\epsilon + \varphi d(Sx_{n+1}, Tx_{n+1}) + ad(Sx_{n+1}, S_1u_{n+1}) + ad(S_1u_{n+1}, Su_{n+1})\} \\ &\quad + \alpha'_n \{\epsilon + \varphi d(Sy_n, Ty_n) + ad(Sy_n, S_1v_n) + ad(S_1v_n, Sv_n)\}, \\ &\leq (1 - \alpha_n - \alpha'_n)d(Sy_n, S_1v_n) + \alpha_n \{\epsilon + \varphi d(Sx_{n+1}, Tx_{n+1}) + ad(Sx_{n+1}, S_1u_{n+1}) + a\epsilon_1\} \\ &\quad + \alpha'_n \{\epsilon + \varphi d(Sy_n, Ty_n) + ad(Sy_n, S_1v_n) + a\epsilon_1\} \end{aligned}$$

which gives

$$\begin{aligned} d(Sx_{n+1}, S_1u_{n+1}) &\leq \frac{(1 - \alpha_n - \alpha'_n + a\alpha'_n)}{1 - a\alpha_n} d(Sy_n, S_1v_n) + \frac{\alpha_n}{1 - a\alpha_n} \varphi d(Sx_{n+1}, Tx_{n+1}) \\ &\quad + \frac{\alpha'_n}{1 - a\alpha_n} \varphi d(Sy_n, Ty_n) + \frac{\epsilon(\alpha_n + \alpha'_n)}{1 - a\alpha_n} + \frac{a\epsilon_1(\alpha_n + \alpha'_n)}{1 - a\alpha_n} \end{aligned} \tag{4.2}$$

Also,

$$\begin{aligned}
 d(Sy_n, S_1v_n) &= d(W(Sz_n, Ty_n, Tz_n, \beta_n, \beta'_n), W(S_1w_n, T_1v_n, T_1w_n, \beta_n, \beta'_n)) \\
 &\leq (1 - \beta_n - \beta'_n)d(Sz_n, S_1w_n) + \beta_n d(Ty_n, T_1v_n) + \beta'_n d(Tz_n, T_1w_n) \\
 &\leq (1 - \beta_n - \beta'_n)d(Sz_n, S_1w_n) + \beta_n \{d(Ty_n, T_1v_n) + d(T_1v_n, T_1w_n)\} \\
 &\quad + \beta'_n d(Tz_n, T_1w_n) + d(T_1w_n, T_1w_n) \\
 &\leq (1 - \beta_n - \beta'_n)d(Sz_n, S_1w_n) + \beta_n \{\epsilon + \varphi d(Sy_n, Ty_n) + ad(Sy_n, Sv_n)\} \\
 &\quad + \beta'_n \{\epsilon + \varphi d(Sz_n, Tz_n) + ad(Sz_n, Sw_n)\} \\
 &\leq (1 - \beta_n - \beta'_n)d(Sz_n, S_1w_n) + \beta_n \{\epsilon + \varphi d(Sy_n, Ty_n) + ad(Sy_n, S_1v_n) + ad(S_1v_n, Sv_n)\} \\
 &\quad + \beta'_n \{\epsilon + \varphi d(Sz_n, Tz_n) + ad(Sz_n, S_1w_n) + ad(S_1w_n, Sw_n)\} \\
 &\leq (1 - \beta_n - \beta'_n)d(Sz_n, S_1w_n) + \beta_n \{\epsilon + \varphi d(Sy_n, Ty_n) + ad(Sy_n, S_1v_n) + a\epsilon_1\} \\
 &\quad + \beta'_n \{\epsilon + \varphi d(Sz_n, Tz_n) + ad(Sz_n, S_1w_n) + a\epsilon_1\}
 \end{aligned}$$

which gives

$$\begin{aligned}
 d(Sy_n, S_1v_n) &\leq \frac{(1 - \beta_n - \beta'_n + a\beta'_n)}{1 - a\beta_n} d(Sz_n, S_1w_n) + \frac{\beta_n}{1 - a\beta_n} \varphi d(Sy_n, Ty_n) \\
 &\quad + \frac{\beta'_n}{1 - a\beta_n} \varphi d(Sz_n, Tz_n) + \frac{\epsilon(\beta_n + \beta'_n)}{1 - a\beta_n} + \frac{a\epsilon_1(\beta_n + \beta'_n)}{1 - a\beta_n}
 \end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned}
 d(Sz_n, S_1w_n) &\leq \frac{(1 - \gamma_n - \gamma'_n + a\gamma'_n)}{1 - a\gamma_n} d(Sx_n, S_1u_n) + \frac{\gamma_n}{1 - a\gamma_n} \varphi(d(Sz_n, Tz_n)) + \frac{\gamma'_n}{1 - a\gamma_n} \varphi(d(Sx_n, Tx_n)) \\
 &\quad + \frac{\epsilon}{1 - a\gamma_n} (\gamma_n + \gamma'_n) + \frac{a\epsilon_1}{1 - a\gamma_n} (\gamma_n + \gamma'_n)
 \end{aligned} \tag{4.4}$$

Estimates (4.2)-(4.4), together with the inequalities

$$\begin{aligned}
 \frac{(1 - \alpha_n - \alpha'_n + a\alpha'_n)}{1 - a\gamma_n} &\leq 1 - \alpha_n(1 - a) - \alpha'_n(1 - a), \\
 \frac{(1 - \beta_n - \beta'_n + a\beta_n)}{(1 - a\gamma_n)(1 - a\beta_n)} &\leq 1 - \beta_n(1 - a) - \beta'_n(1 - a) < 1 \text{ and } 1 - \gamma_n(1 - a) - \gamma'_n(1 - a) < 1,
 \end{aligned}$$

yield

$$\begin{aligned}
 d(Sx_{n+1}, S_1u_{n+1}) &\leq [1 - \alpha_n(1 - a) - \alpha'_n(1 - a)]d(Sx_n, S_1u_n) + \left[\frac{\gamma_n}{1 - a\gamma_n} + \frac{\beta'_n}{1 - a\beta_n} \right] \varphi(d(Sz_n, Tz_n)) \\
 &\quad + \left[\frac{\beta_n}{1 - a\beta_n} + \frac{\alpha'_n}{1 - a\alpha_n} \right] \varphi(d(Sy_n, Ty_n)) + \frac{\alpha_n}{1 - a\alpha_n} \varphi(d(Sx_{n+1}, Tx_{n+1})) \\
 &\quad + \frac{\gamma'_n}{(1 - a\gamma_n)} \varphi(d(Sx_n, Tx_n)) \\
 &\quad + \epsilon \left[\frac{(\gamma_n + \gamma'_n)}{1 - a\gamma_n} + \frac{(\beta_n + \beta'_n)}{1 - a\beta_n} + \frac{\alpha_n + \alpha'_n}{1 - a\alpha_n} \right] + a\epsilon_1 \left[\frac{(\gamma_n + \gamma'_n)}{1 - a\gamma_n} + \frac{(\beta_n + \beta'_n)}{1 - a\beta_n} + \frac{\alpha_n + \alpha'_n}{1 - a\alpha_n} \right] \\
 &\leq [1 - \alpha_n(1 - a) - \alpha'_n(1 - a)]d(Sx_n, S_1u_n) \\
 &\quad + \frac{(\alpha_n + \alpha'_n)}{1 - a} [2\varphi(d(Sz_n, Tz_n)) + 2\varphi(d(Sy_n, Ty_n)) + \varphi(d(Sx_{n+1}, Tx_{n+1})) \\
 &\quad + \varphi(d(Sx_n, Tx_n)) + 3(\epsilon + a\epsilon_1)]
 \end{aligned} \tag{4.5}$$

Let $r_n = \alpha_n(1 - a) - \alpha'_n(1 - a)$,

$$t_n = \frac{[2\varphi(d(Sz_n, Tz_n)) + 2\varphi(d(Sy_n, Ty_n)) + \varphi(d(Sx_{n+1}, Tx_{n+1})) + \varphi(d(Sx_n, Tx_n)) + 3(\epsilon + a\epsilon_1)]}{(1 - a)^2}.$$

Then using continuity of φ together with $\lim_{n \rightarrow \infty} \|Sx_{n+1} - Tx_{n+1}\| = 0$, $\lim_{n \rightarrow \infty} \|Sy_n - Ty_n\| = 0$, $\lim_{n \rightarrow \infty} \|Sx_n - Tx_n\| = 0$, $\lim_{n \rightarrow \infty} \|Sz_n - Tz_n\| = 0$ and applying Lemma 1.13 to (4.5) yields

$$d(p, p^*) \leq \frac{3(\epsilon + a\epsilon_1)}{(1 - a)^2}.$$

□

Remark 4.2. Since Jungck implicit Mann (1.15), Jungck implicit Ishikawa (1.14) and Jungck implicit Noor (1.13) iterative schemes are special cases of the new Jungck-type implicit iterative scheme (1.11), the data dependence results similar to Theorem 4.1 also hold for Jungck implicit Mann (1.15), Jungck implicit Ishikawa (1.14) and Jungck implicit Noor (1.13) iterative schemes.

Remark 4.3. As implicit Noor iterative scheme [8] is a special case of new Jungck implicit iterative scheme (1.11), data dependence result of implicit Noor iterative scheme [8, Theorem 15] can be obtained as a corollary.

5. Applications

In this section, we apply the newly introduced Jungck-type implicit iterative scheme to find the solution of a quadratic equation and show that the new iterative scheme converges faster to the solution of a given quadratic equation as compared to other iterative schemes mentioned in this paper.

Given an equation of one variable, $f(x) = 0$, we use fixed point iterative scheme as follows:

1. Convert the equation to the form $S(x) = T(x)$, with $T(Y) \subseteq S(Y)$ where $S, T : Y \rightarrow X$ are differentiable functions such that derivative $(T) \leq a$ (derivative (S)), $a < 1$.
2. Start with an initial guess $x_0 \approx r$, where r is the actual solution (root) of the equation.
3. Iterate, using $Sx_{n+1} = Tx_n$ for $n = 0, 1, 2, \dots$
4. The iterative scheme $\{Sx_{n+1}\}$ will converge to the common fixed point of S and T (the solution of equation) when $Y = X$, otherwise $\{Sx_{n+1}\}$ will converge to the point of coincidence of S and T . In the second case, using Sx or Tx , we find coincidence point which will be the desired solution.

By applying above procedure, we find the solution of the quadratic equation $x^2 - 2x - 3 = 0$.

Let $S, T : [0, 3] \rightarrow [0, 9]$ be defined by $Sx = x^2$, $T(x) = 2x + 3$. Obviously, $T[0, 3] \subseteq S[0, 3]$. Then for the new Jungck-type implicit iterative scheme (1.11), we have

$$\begin{aligned} x_{n+1} &= (\alpha_n \pm \sqrt{\alpha_n^2 + (1 - \alpha_n)x_n^2 + 3\alpha_n}) \\ &\times (\beta_n \pm \sqrt{\beta_n^2 + (1 - \beta_n - \beta'_n)y_n^2(3\beta_n + 6\beta'_ny_n + 3\beta'_n)}) \\ &\times (\gamma_n \pm \sqrt{\gamma_n^2 + (1 - \gamma_n - \gamma'_n)y_n^2(3\gamma_n + 6\gamma'_ny_n + 3\gamma'_n)}) \end{aligned}$$

Making same calculations as above for all other iterative schemes and taking $\alpha_n = \alpha'_n = \beta_n = \beta'_n = \gamma_n = \gamma'_n = \frac{1}{\sqrt{n+0.5}}$ and initial approximation $x_0 = 2.9$, with the help computer programs in C++, we obtain the following table:

No. of iterations	New Jungck implicit iterative scheme	Jungck Khan iterative scheme	Jungck implicit Noor iterative scheme	Jungck Noor iterative scheme	Jungck implicit Ishikawa iterative scheme	Jungck Ishikawa iterative scheme	Jungck implicit Mann iterative scheme	Jungck Mann iterative scheme
n	Sx_{n+1}	Sx_{n+1}	Sx_{n+1}	Sx_{n+1}	Sx_{n+1}	Sx_{n+1}	Sx_{n+1}	Sx_{n+1}
0	18.324043	8.662796	9.281449	9.350374	8.638365	9.226238	9.457278	8.961543
1	8.817944	9.007301	9.007318	9.093414	8.976985	9.06939	9.115153	8.982465
2	9.000222	8.998022	9.011625	9.043955	8.995007	9.033949	9.053609	8.989857
3	9.000111	8.999213	9.009737	9.025	8.998398	9.019693	9.030358	8.993471
4	9.000102	8.99967	9.007487	9.015713	8.999371	9.01253	9.019037	8.995523
5	9.00006	8.999849	9.00569	9.010521	8.999719	9.008462	9.012729	8.996795
6	9.000034	8.999925	9.004351	9.007363	8.999863	9.00596	9.0089	8.997633
7	9.000019	8.999961	9.003361	9.005326	8.999928	9.004332	9.006433	8.998209
8	9.00001	8.999979	9.002625	9.003953	8.999962	9.003228	9.004771	8.998619
9	9.000006	8.999988	9.002073	9.002995	8.999979	9.002454	9.003613	8.998918
10	9.000005	8.999993	9.001656	9.002309	8.999988	9.001898	9.002786	8.99914
11	9.000003	8.999996	9.001332	9.001806	8.999993	9.001488	9.002179	8.999309
12	9.000002	8.999997	9.001081	9.00143	8.999996	9.001181	9.001725	8.99944
13	9	8.999998	9.000019	9.00141	8.999996	9.000151	9.000725	8.999448
-	-	-	-	-	-	-	-	-
17	9	9	9.000018	9.000020	8.999997	9.0000110	9.000220	8.999240
18	9	9	9.000017	9.000019	8.999997	9.000097	9.0000218	8.9999201
-	-	-	-	-	-	-	-	-
41	9	9	9.000016	9.000017	8.999997	9.000014	9.00002	8.999991
42	9	9	9.000014	9.000015	8.999997	9.000013	9.000017	8.999992
43	9	9	9.000013	9.000013	8.999997	9.000011	9.000016	8.999993
44	9	9	9.000011	9.000012	8.999997	9.00001	9.000014	8.999993
45	9	9	9.00001	9.000011	8.999997	9.000009	9.000013	8.999994
46	9	9	9.000009	9.00001	8.999997	9.000008	9.000011	8.999995
47	9	9	9.000008	9.000009	8.999997	9.000007	9.00001	8.999995
48	9	9	9.000008	9.000008	8.999997	9.000007	9.000009	8.999996
49	9	9	9.000008	9.000007	8.999997	9.000006	9.000008	8.999996
50	9	9	9.000008	9.000006	8.999997	9.000005	9.000008	8.999996
-	-	-	-	-	-	-	-	-
70	9	9	9	9.000001	8.999997	9.000001	9.000006	8.999999
71	9	9	9	9.000001	8.999997	9.000001	9.000006	8.999999
72	9	9	9	9.000001	8.999999	9.000001	9.000006	8.999999
73	9	9	9	9.000001	9	9.000001	9.000006	9
74	9	9	9	9.000001	9	9.000001	9	9
75	9	9	9	9.000001	9	9.000001	9	9
76	9	9	9	9	9	9	9	9
77	9	9	9	9	9	9	9	9

From the above table, we conclude that newly introduced Jungck implicit type iterative scheme converges to the point of coincidence 9 of S and T , then to find solution of the given equation use $Sx = x^2$ to get $x = 3$ (coincidence point of S and T). Also new iterative scheme converges faster as compared to other implicit as well as explicit iterative schemes and hence has good potential for further applications.

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