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Neighbourhood Systems

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Abstract. We introduce the concept of neighbourhood systems which are common generalizations of topological spaces, locales and topological systems. The category **NS** of neighbourhood systems has coproducts but does not have products. The relationship between convergence of filters on neighbourhood systems and convergence of filters on locales is investigated.

1. Introduction

Topological systems were first introduced by S. Vickers in [8] arising from topological and localic aspects of domains and finite observational logic in computer science. In [1], J. Adamek and M.-C. Pedicchio showed that the category of topological systems is dually equivalent to the category of grids introduced in [3]. One of the purposes of the introduction of topological systems is to provide a single framework in which to treat both spaces and locales(see [8]). It is not difficult to see that the underlying locale of any nontrivial topological system has at least one point. This seems that topological systems can only deal with locales which has points. So for pointless locales there are no corresponding nontrivial topological systems. But in locale theory there exist a large number of pointless locales, thus for the same purpose of providing a single framework in which to treat both spaces and locales, we introduce the concept of neighbourhood systems and show that the category of neighbourhood systems contains the category of topological spaces, the category of locales and the category of topological systems respectively as full subcategories.

2. Preliminaries

Our notation and terminology follow that of Johnstone [6]. For instance, by a *point* of locale we mean a prime element, and we write $\mathrm{pt}(L)$ for the set of points of a locale L. The frame of open sets of a topological space X is denoted by ΩX. Recall that $\mathrm{pt}(L)$ can be made into a topological space such that the assignment $L \mapsto pt(L)$ defines a functor from the category of locales to the category of topological spaces which is right adjoint to the functor Ω.

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By a *filter* in a locale L we mean a proper filter; and we write $\mathrm{Fil}(L)$ for the set of all filters on a locale L. Recall that a filter F in a locale L is prime if for any $a, b \in L, a \vee b \in F$ implies $a \in F$ or $b \in F$. On the other hand, it is *completely prime* if for any $S \subseteq L$, $\bigvee S \in F$ implies $S \cap F \neq \emptyset$. Finally, we write $\mathrm{Spec}(L)$ for the set of all prime ideals on L.

3. Neighbourhood Systems

Let L be a locale. A subset $F \subseteq L$ is called filtered if for any two elements $a, b \in F$, there exists an element $c \in F$ such that $c \leq a, c \leq b$ hold. $F \subseteq L$ is said to be a filter on L if F is filtered upper set. The set of all filters on L which don't contain the bottom element 0 of L is denoted by $Fil(L)$.

Definition 3.1. Let X be a set and L be a locale. Let $N : X \to Fil(L)$ be a map which assigns to every point $x \in X$ a filter $N(x)$ in L. We call N a *neighbourhood assignment* on X to L and elements in $N(x)$ are called neighbourhoods of x. X and L with a neighbourhood assignment N is called a neighbourhood system which is denoted by (X, L, N).

Example 3.2. Let X be the set of all real numbers. Let L be the frame of all regular open sets of X, i.e., open subsets u of X in the usual topology satisfying $int(cl(u)) = u$. Then L has no points. For every $x \in X$, take $N(x) = \{u \in L \mid x \in u\}$. Then (X, L, N) is a neighbourhood system.

Definition 3.3. Let (X, L, N) and (Y, T, M) be two neighbourhood systems. A continuous map from (X, L, N) to (Y, T, M) is a pair $(f, \bar{f})$ where $f : X \to Y$ is a function, $\bar{f} : T \to L$ is a frame homomorphism and for every $x \in X$, $u \in M(f(x))$ if and only if $\bar{f}(u) \in N(x)$.

Example 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous open function where $\mathbb{R}$ is the real line. Take the neighbourhood system (X, L, N) in Example 3.2, then the pair (f, f^{-1}) is a continuous map from (X, L, N) to itself.

Continuous maps between neighbourhood systems are clearly stable under composition and for every neighbourhood system (X, L, N), there exists obviously an identity map from (X, L, N) to itself. Thus we have a category **NS** of neighbourhood systems and continuous maps.

Definition 3.5. A continuous map $(f, \bar{f}) : (X, L, N) \to (Y, T, M)$ is a homeomorphism if there is continuous map $(g, \bar{g}) : (Y, T, M) \to (X, L, L)$ such that the compositions $gf = id_X, fg = id_Y, \bar{g}\bar{f} = id_T, \bar{f}\bar{g} = id_L$.

When there is a homeomorphism from (X, L, N) to (Y, T, M), we say (X, L, N) and (Y, T, M) are homeomorphic and write $(X, L, N) \cong (Y, T, M)$. This means that not only the sets X and Y have same cardinal and the frames L and T are isomorphic, but also the restriction map $\bar{f} : N(x) \to M(f(x))$ is a bijection for each $x \in X$.

Similar with topological systems, a neighbourhood system can be described as a finite meet preserving map from a locale to the power set of a set as following.

We consider the category **Frm** $\downarrow$**Set** its objects are those finite meet preserving maps $\varphi : L \to \mathbf{P}(X)$ from a locale L to the power set $\mathbf{P}(X)$ of a set X, for two objects $\psi : L \to \mathbf{P}(X)$ and $\varphi : T \to \mathbf{P}(Y)$ in **Frm** $\downarrow$**Set**, a morphisms from $\psi : L \to \mathbf{P}(X)$ to $\varphi : T \to \mathbf{P}(Y)$ is a pair

$$f : X \to Y, \bar{f} : T \to L$$

where $f : X \to Y$ is a function, $\bar{f} : T \to L$ is a frame homomorphism such that the following square commutes:

$$\begin{tikzcd} T \arrow[r, "\bar{f}"] \arrow[d, "\varphi"'] & L \arrow[d, "\psi"] \\ \mathbf{P}(Y) \arrow[r, "f^{-1}"] & \mathbf{P}(X) \end{tikzcd}$$

Given a neighbourhood system (X, L, N), we have a map from L to the power set $\mathbf{P}(X)$ of X which preserves finite meets

$$\psi : L \to \mathbf{P}(X), a \mapsto \{x \in X \mid a \in N(x)\}.$$

It is not difficult to see that this assignment defines a functor from the category **NS** of neighbourhood systems to the category **Frm↓Set**. This functor is in fact an isomorphism of categories such that its inverse maps each object $\psi : L \to \mathbf{P}(X)$ of **Frm↓Set** to a neighbourhood system (X, L, N) defined by

$$N(x) = \{u \in L \mid x \in \psi(u)\}.$$

Let (X, L, N) be a neighbourhood system. Then the collection $\{\psi(a) = \{x \in X \mid a \in N(x)\} \mid a \in L\}$ of subsets of X forms a base for a topology on X. We denote the topological space by $\tilde{X}$. By the arguments above the following result is clear.

Proposition 3.6. *The assignment $(X, L, N) \mapsto \tilde{X}$ defines a functor from the category* **NS** *of neighbourhood systems to the category* **Top** *of topological spaces.*

4. Spatial Neighbourhood Systems

Let X be a topological space. Write $\Omega(X)$ for the topology on X. Let $N : X \to Fil(\Omega(X))$ be the ordinary neighbourhood assignment, i.e., $N(x) = \{U \in \Omega(X) \mid x \in U\}$ for each $x \in X$. Then $(X, \Omega(X), N)$ is a neighbourhood system, we call it the neighbourhood system generated by X.

Suppose X and Y are topological spaces, $(X, \Omega(X), N)$ and $(Y, \Omega(Y), M)$ are the neighbourhood systems generated by X and Y respectively. For every continuous map $(f, \bar{f}) : (X, \Omega(X), N) \to (Y, \Omega(Y), M)$, $U \in \Omega(Y)$, we have $f(x) \in U \Leftrightarrow x \in \bar{f}(U)$, which implies that $\bar{f} = f^{-1}$, so that $\bar{f}$ is the inverse image mapping induced by f. Thus the following result is clear.

Theorem 4.1. *The category* **Top** *of topological spaces and continuous functions can be embedded as a full subcategory of the category* **NS** *of neighbourhood systems.*

We call a neighbourhood system *spatial* if it is isomorphic to a neighbourhood system induced by a topological space.

Lemma 4.2. *A neighbourhood system (X, L, N) is spatial if and only if there exists a frame mono-homomorphism $g : L \to \mathbf{P}(X)$ from L to the powerset lattice $\mathbf{P}(X)$ of X such that $u \in N(x) \Leftrightarrow x \in g(u)$ for every $x \in X$.*

Theorem 4.1 implies that the category of topological spaces is equivalent to the category of spatial neighbourhood systems.

Let (X, L, N) be a neighbourhood system. Write $pt(L)$ for the set of all prime elements of L, and $\Omega(pt(L))$ for the topology on $pt(L)$ such that every open set has the form $\phi(u) = \{p \in pt(L) \mid u \nleq p\}$ for some $u \in L$. The topological space $(pt(L), \Omega(pt(L)))$ generates a neighbourhood system $(pt(L), \Omega(pt(L)), \Phi)$. We call a neighbourhood system (X, L, N) *sober neighbourhood system* if (X, L, N) and $(pt(L), \Omega(pt(L)), \Phi)$ are homeomorphic.

Lemma 4.3. *(X, L, N) is a sober neighbourhood system if and only if L is spatial and there is a one-to-one onto mapping $\varphi : X \to pt(L)$ such that $N(x) = L \backslash \downarrow \varphi(x)$ for every $x \in X$.*

Corollary 4.4. *The category* **Sob** *of sober topological spaces is equivalent to the category of sober neighbourhood systems.*

5. Topological Neighbourhood Systems

Definition 5.1. A neighbourhood system (X, L, N) is said to be a *topological neighbourhood system* if for every $x \in X$, $N(x)$ is a completely prime filter, i.e., $N(x)$ satisfies the following condition:

$$(\star) \bigvee_{i \in I} u_i \in N(x) \Rightarrow \exists i \in I, \text{such that } u_i \in N(x).$$

Let (X, L) be a topological system. For each $x \in X$, take $N(x) = \{u \in L \mid x \models u\}$. Then (X, L, N) is a neighbourhood system, we call it the neighbourhood system generated by the topological system (X, L).

Lemma 5.2. *A neighbourhood system (X, L, N) is a topological neighbourhood system if and only if it can be generated by a topological system $(X, L, \models)$.*

Proof. Suppose (X, L, N) is a topological neighbourhood system. Define a binary relation $\models$ from X to L such that

$$x \models u \Leftrightarrow u \in N(x).$$

Then $(X, L, \models)$ is a topological system and (X, L, N) is generated by $(X, L, \models)$.

Conversely, if (X, L, N) is generated by a topological system $(X, L, \models)$, and $\bigvee_{i \in I} u_i \in N(x)$ for $x \in X$, then $x \models \bigvee u_i$ which implies that $\exists i \in I$, such that $x \models u_i$. Hence $u_i \in N(x)$. □

Let (X_1, L_1) and (X_2, L_2) be two topological systems. It is not difficult to see that $(f, \bar{f}) : (X_1, L_1) \to (X_2, L_2)$ is a continuous map of topological systems if and only if $(f, \bar{f}) : (X_1, L_1, N_1) \to (X_2, L_2, N_2)$ is a continuous map of neighbourhood systems where (X_1, L_1, N_1) and (X_2, L_2, N_2) are neighbourhood systems generated by topological systems (X_1, L_1) and (X_2, L_2) respectively. Hence the following result is clear.

Theorem 5.3. *The category of topological systems and continuous maps is isomorphic to the category of topological neighbourhood systems and continuous maps, hence can be embedded as a full subcategory of the category* **NS** *of neighbourhood systems.*

6. Localic Neighbourhood Systems

Let L be a locale. The set of all prime ideals of L is denoted by Spec(L). For each $j \in$ Spec(L), put $N^L(j) = \{x \in L \mid x \notin j\}$, then $(\text{Spec}(L), L, N^L)$ is a neighbourhood system. We call it the neighbourhood system generated by L.

Lemma 6.1. *The correspondence $L \mapsto (Spec(L), L, N^L)$ between objects defines a full embedding functor from the category of locales into the category* **NS** *of neighbourhood systems.*

Proof. Suppose $f : L \to Q$ is a localic map and $f^* : Q \to L$ is the corresponding frame homomorphism. For each $j \in$ Spec(L), put $\tilde{f}(j) = \{a \in Q \mid f^*(a) \in j\}$. Then it is clear that $\tilde{f}(j)$ is a prime ideal on Q and $(\tilde{f}, f^*) : (\text{Spec}(L), L, N^L) \to (\text{Spec}(Q), Q, N^Q)$ is a continuous map of neighbourhood systems where $(\text{Spec}(L), L, N^L)$ and $(\text{Spec}(Q), Q, N^Q)$ are the neighbourhood systems generated by L and Q respectively. This implies that the correspondence $L \mapsto (\text{Spec}(L), L, N)$ is an embedding functor. To show it is full, we note that if $(g, f^*) : (\text{Spec}(L), L, N^L) \to (\text{Spec}(Q), Q, N^Q)$ is a continuous map of neighbourhood systems then for every prime ideal j on L, $a \in g(j)$ if and only if $f^*(a) \in j$. Thus $g(j) = \{a \in Q \mid f^*(a) \in j\}$. This implies that the above functor is full. □

A neighbourhood system is called *localic* if it is homeomorphic to a neighbourhood system generated by a locale.

Corollary 6.2. *The category of locales and localic maps is equivalent to the category of localic neighbourhood systems and continuous maps.*

A neighbourhood system (X, L, N) is said a T_0 neighbourhood system if for any two points $x, y \in X, x \neq y$ implies $N(x) \neq N(y)$. (X, L, N) is said a pre-localic neighbourhood system if it is a T_0 neighbourhood system and satisfying the condition that for any point $x \in X$, $u_1 \vee u_2 \in N(x)$ implies that $u_1 \in N(x)$ or $u_2 \in N(x)$. Every localic neighbourhood system is a pre-localic neighbourhood system but the converse is not true.

Theorem 6.3. *The category of localic neighbourhood systems is reflective in the category of pre-localic neighbourhood systems.*

Proof. Let (X, L, M) be a pre-localic neighbourhood system. We can regard X as a subset of the set of all prime filters on L, hence a subset of the set $\mathrm{Spec}(L)$ of prime ideals on L. Thus we have a continuous map $(f, \bar{f}) : (X, L, M) \to (\mathrm{Spec}(L), L, N^L)$ where $f : X \to \mathrm{Spec}(L)$ is an embedding and $\bar{f} : L \to L$ is the identity. Suppose $(g, \bar{g}) : (X, L, N) \to (\mathrm{Spec}(H), H, N^H)$ is a continuous map. Then g can be regarded as a restriction of the map

$$\hat{g} : \mathrm{Spec}(L) \to \mathrm{Spec}(H), J \mapsto \bar{g}^{-1}(J)$$

Hence $(g, \bar{g})$ can be uniquely factored through $(f, \bar{f})$ by the continuous map $(\hat{g}, \bar{g}) : (\mathrm{Spec}(L), L, N^L) \to (\mathrm{Spec}(H), H, N^H)$. $\square$

7. Operations of Neighbourhood Systems

In this section we consider some operations of neighbourhood systems.

Definition 7.1. A continuous map $(f, \bar{f}) : (Y, T, M) \to (X, L, N)$ of neighbourhood systems is said an *embedding* if f is injective and $\bar{f}$ is surjective. A neighbourhood system (Y, T, M) is called a neighbourhood subsystem of (X, L, N) if there is an embedding $(f, \bar{f}) : (Y, T, M) \to (X, L, N)$ of neighbourhood systems.

Let (X, L, N) be a neighbourhood system. Given a subset $Y \subseteq X$ and a sublocale $T \subseteq L$, there will be no suitable neighbourhood assignment M on Y to T such that it becomes a neighbourhood subsystem of (X, L, N) in general.

Lemma 7.2. *Let (X, L, N) be a neighbourhood system. Let $Y \subseteq X$ and $f : L \to T$ be a surjective frame morphism. Then there is a neighbourhood assignment M on Y to T such that (Y, T, M) becomes a neighbourhood subsystem of (X, L, N) if and only if $\forall y \in Y, v \in T$ such that $\bar{f}(u) \leq \bar{f}(v)$ for some $u \in N(y)$ implies that $v \in N(y)$.*

Now we consider sums of neighbourhood systems which is in fact coproducts of neighbourhood systems in categorical sense.

Let $\{(X_j, L_j, N_j) \mid j \in J\}$ be a family of neighbourhood systems. The sum of $\{(X_j, L_j, N_j) \mid j \in J\}$ is defined as following.

Take $X = \bigcup_{i \in J} X_j$ be the disjoint union of $\{X_j \mid j \in J\}$, $L = \prod L_j$ be the frame product of $\{L_j \mid j \in J\}$. Define

$$N : X \to Fil(L), N(x) = \{(u_j) \mid u_i \in N_i(x)\},\ x \in X_i$$

Theorem 7.3. *(X, L, N) is the coproduct of $\{(X_j, L_j, N_j) \mid j \in J\}$ in the category of neighbourhood systems with injections $(i_j, \bar{i}_j) : (X_j, L_j, N_j) \to (X, L, N), j \in J$ defined by $i_j(x) = x$ and $\bar{i}_j(x_t) = x_j$ for each $j \in J$.*

Proof. We only need to check the universal property that for any neighbourhood system (Y, P, M) and continuous maps $(f_j, \bar{f}_j) : (X_j, L_j, N_j) \to (Y, P, M), j \in J$, there exists a unique continuous map $(f, \bar{f}) : (X, L, N) \to (Y, P, M)$ such that $f_j = f_j i_j, \bar{f}_j = \bar{i}_j \bar{f}_j$ for each $j \in J$. But it is clear if we take $f(x) = f_j(x)$ for $x \in X_j$ and $\bar{f}(u) = (\bar{f}_j(u))$ for $u \in P$ as the following diagram shows.

$$\begin{tikzcd}
(X_j, L_j, N_j) \arrow[rr, "(f_j\text{,}\bar{f}_j)"] \arrow[dr, "(i_j\text{,}\bar{i}_j)"'] & & (Y, P, M) \arrow[dl, "(f\text{,}\bar{f})"] \\
& (X, L, N) &
\end{tikzcd}$$

$\square$

Corollary 7.4. *(1) A coproduct of a family of spatial neighbourhood systems is still spatial.*
(2) A coproduct of a family of topological neighbourhood systems is still topological.
(3) A coproduct of a family of localic neighbourhood systems is still localic.

Proof. (1) Let $\{(X_j, L_j, N_j) \mid j \in J\}$ be a family of spatial neighbourhood systems. It is clear that the frame product $\prod L_j$ is isomorphic to the topology of the sum space $\biguplus X_j$.

(2) Suppose $\{(X_j, L_j, N_j) \mid j \in J\}$ is a family of topological neighbourhood systems, it is clear that each $N(x)$ is a completely prime filter, i.e., satisfying the condition $(\star)$. Hence the coproduct (X, L, N) is topological.

(3) Suppose $\{(X_j, L_j, N_j) \mid j \in J\}$ is a family of localic neighbourhood systems. We note that every prime ideal on $\prod L_j$ has the form $p_j^{-1}(I)$ for some projection $p_j : \prod L_j \to L_j$ and a prime ideal I on L_j. Thus the set of all prime ideals $pi(\prod L_j)$ on $\prod L_j$ is one-one to the join set $\bigcup_{j \in J} pi(L_j)$. □

Let $\{(X_j, L_j, N_j) \mid j \in J\}$ be a family of neighbourhood systems. Write $\prod_{j\in J} X_j$ for the Cartesian product of $\{X_j\}$, and $\coprod_{j\in J} L_j$ for the coproduct of frames $\{L_j\}$, equivalently the product of locales $\{L_j\}$. We define a neighbourhood assignment $N : \prod_{j\in J} X_j \to Fil(\coprod_{j\in J} L_j)$ such that

$$N(x_j) = \uparrow \{\downarrow (u_j) \mid \exists \text{a finite set } F \subseteq J, \text{such that } u_j = 1, j \in J \setminus F, u_i \in N_i(x_i), i \in F\}$$

Write $p_j : \prod X_j \to X_j$ and $\bar{p}_j : L_j \to \coprod L_j$ be the $j'th$ projection and co-projection respectively. Then $(p_j, \bar{p}_j) : (\prod X_j, \coprod L_j, N) \to (X_j, L_j, N_j)$ is a continuous map.

Lemma 7.5. *Let $\{(X_j, L_j, N_j) \mid j \in J\}$ be a family of neighbourhood systems. Suppose that $(\widetilde{\prod X_j}, \widetilde{\prod L_j}, \tilde{N})$ is the product of $\{(X_j, L_j, N_j) \mid j \in J\}$ with projections $(f_j, \bar{f}_j) : (\widetilde{\prod X_j}, \widetilde{\prod L_j}, \tilde{N}) \to (X_j, L_j, N_j), j \in J$, then $\widetilde{\prod X_j}$ is equipotent with $\prod X_j$ and $\widetilde{\prod L_j}$ is isomorphic with $\coprod L_j$.*

Proof. By the universal property there is a continuous map $(h, \bar{h}) : (\prod X_j, \coprod L_j, N) \to (\widetilde{\prod X_j}, \widetilde{\prod L_j}, \tilde{N})$ such that the following diagram commutes:

$$\begin{tikzcd}
(\prod X_j, \coprod L_j, N) \arrow[rr, "{(h,\bar{h})}"] \arrow[dr, "{(p_j,\bar{p}_j)}"'] & & (\widetilde{\prod X_j}, \widetilde{\prod L_j}, \tilde{N}) \arrow[dl, "{(f_j,\bar{f}_j)}"] \\
& (X_j, L_j, N_j) &
\end{tikzcd}$$

Let $f : \widetilde{\prod X_j} \to \prod X_j$ be a map such that $p_j f = f_j$ for $\forall j \in J$, and let $\bar{f} : \coprod L_j \to \widetilde{\prod L_j}$ be a frame homomorphism such that $\bar{f}\bar{p}_j = \bar{f}_j$ for $\forall j \in J$. Then we have $p_j f h = f_j h = p_j, j \in J$, this implies that fh be an identity on $\prod X_j$. Thus h is injective. So we can regard $\prod X_j$ as a subset of $\widetilde{\prod X_j}$. But it is clear that the induced neighbourhood system $(\prod X_j, \widetilde{\prod L_j}, \tilde{N})$ is still a product of $\{(X_j, L_j, N_j) \mid j \in J\}$. Hence $\widetilde{\prod X_j}$ is equipotent with $\prod X_j$.

To show that $\widetilde{\prod L_j}$ is isomorphic with $\coprod L_j$. We first note that $\bar{f}\bar{p}_j = \bar{f}_j$ implies that $\bar{h}\bar{f}\bar{p}_j = \bar{h}\bar{f}_j = \bar{p}_j, i \in J$. It implies that $\bar{h}\bar{f}$ be an identity on $\prod L_j$.

Secondly, we show that $\{\bar{f}_j : L_j \to \widetilde{\prod L_j} \mid j \in J\}$ is an epi-sink, i.e. $g\bar{f}_j = r\bar{f}_j$ implies $g = r$ for any two frame homomorphisms g and r. Write L be the sub-frame of $\widetilde{\prod L_j}$ generated by $\{\bar{f}_j(x) \mid x \in L_j, j \in J\}$. Then it is not difficult to see that the induced neighbourhood system $(\prod X_j, L, \tilde{N})$ is still a product of $\{(X_j, L_j, N_j) \mid j \in J\}$. Thus $L = \widetilde{\prod L_j}$ and this implies that $\{\bar{f}_j : L_j \to \widetilde{\prod L_j} \mid j \in J\}$ is an epi-sink.

Now $\bar{h}\bar{f}_j = \bar{p}_j$ implies that $\bar{f}\bar{h}\bar{f}_j = \bar{f}\bar{p}_j = \bar{f}_j, j \in J$. This implies that $\bar{f}\bar{h}$ is an identity on $\widetilde{\prod L_j}$ since $\{\bar{f}_j : L_j \to \widetilde{\prod L_j} \mid j \in J\}$ is an epi-sink. Hence $\widetilde{\prod L_j}$ is isomorphic with $\coprod L_j$. □

By the above Lemma, it is clear that the continuous map $(h, \bar{h}) : (\prod X_j, \coprod L_j, N) \to (\widetilde{\prod X_j}, \widetilde{\prod L_j}, \tilde{N})$ is a homeomorphism. This shows that if the category **NS** of neighbourhood systems has products, they must be the form of $(\prod X_j, \coprod L_j, N)$ defined above. The following example shows that there are no products in the category of neighbourhood systems.

Example 7.6. Let $L = \Omega(\mathbb{R})$ be the frame of all open sets of the real line $\mathbb{R}$ and let $X = \mathbb{R}$. Let

$$M : X \to Fil(L), u \mapsto \uparrow u$$

Consider the diagonal map $(\triangle, \bar{\triangle}) : (X, L, M) \to (X \times X, L \times_l L, N)$ where $L \times_l L$ be the localic product of L and itself and the neighbourhood assignment N defined as in Lemma 6.2. Write $S = \{(x, y) \mid x^2 < y < \sqrt{x}, 0 < x < 1\}$. For $u = (0, 1)$ be the unit interval, take $J = \{(v, w) \mid v \times w \subseteq S\} \in L \times_l L$. Then $\bar{\triangle}(J) = \bigvee\{v \cap w \mid v \times w \subseteq S\} = (0, 1) = u \in M(u)$ but $J \notin N(u, u)$. Thus the diagonal map is not continuous.

Regarding the category **NS** of neighbourhood systems and continuous maps as a concrete category over **SET** $\times$ **Loc**, by the above result we know that **NS** is not topological over **SET** $\times$ **Loc** since there are no initial structures in **NS** .

8. Convergence in Neighbourhood Systems

In this section we show that neighbourhood systems provide us a unified framework to deal with convergence of filters.

The concept of convergence of filters on locales (or frames) was first considered by Banaschewski and Pultr in [2], where they define a filter in a frame to be convergent if it contains a completely prime filter. Subsequently, Hong in [5] defined a filter on a frame L to be convergent if for every cover $\bigvee S = 1$, $S \subseteq L$, there exists $s \in S$ such that $s \in F$; A filter F on a frame L is a *clustered filter* if $\bigvee\{\neg a \mid a \in F\} \neq 1$. Hong's definition is more general since it does not require the existence of completely prime filters. We will adopt Hong's definition in the following.

Recall that for a topological space X, if F is a filter on X consists of open sets, then F has a cluster point if and only if $\bigcap\{cl(u) \mid u \in F\} \neq \emptyset$, and F has a limit point if and only if it includes a neighbourhood filter of a point. So it is clear that the concept of clustered filters is a pointless extension of the concept of filters having cluster point, and the concept of convergence of filters is a pointless extension of the concept of filters which having limit point. Similar with spacial case, convergence of filters in a frame has many interesting properties and we list two basic properties which can be found in [5].

Lemma 8.1. *(1) Every convergence filter is a clustered filter.*
(2) Every clustered ultrafilter is convergence.

Let L be a locale and F a filter on L. F is said a *finite convergence filter* if every finite cover $s_1 \vee \cdots \vee s_n = 1$, $s_i \in L, i = 1, \cdots, n$, there exists $s_j, 1 \leq j \leq n$ such that $s_j \in F$.

Lemma 8.2. *Let L be a locale. The following conditions are equivalent.*
(1) L is compact.
(2) Every finite convergence filter on L is convergence.

Let (X, L, N) be a neighbourhood system and F a filter on L. A point $x \in X$ is called a *cluster point* of F if for every $u \in F$ and every $a \in N(x)$ we have $u \wedge a \neq 0$; x is called a *limit point* of F if $N(x) \subseteq F$.

A neighbourhood system (X, L, N) is said *coverable* if $\bigvee\{u_x \mid x \in X\} = 1$ for $u_x \in N(x), x \in X$.

Lemma 8.3. *Let (X, L, N) be a neighbourhood system and consider the following conditions:*
(1) (X, L, N) is coverable;
(2) Every convergence filter on L has a limit point $x \in X$;
(3) Every clustered filter on L has a cluster point $x \in X$.
Then (1)⇒ (2), (1)⇒ (3).

Proof. (1)⇒ (2) Suppose F is a convergence filter on L. If $\forall x \in X$, x is not a limit point of F then there exist $u_x \in N(x)$ such that $u_x \notin F$. Thus $\bigvee u_x = 1$ will induce a contradiction with the condition that F is convergence.

(1)⇒ (3) Suppose F is a clustered filter on L. If $\forall x \in X$, x is not a cluster point of F then there exist $u_x \in N(x), a_x \in F$ such that $u_x \wedge a_x = 0$. This implies that $1 = \bigvee\{u_x \mid x \in X\} \leq \bigvee\{\neg a_x \mid x \in X\} \leq \bigvee\{\neg a \mid a \in F\}$ it contradicts with the condition that F is clustered. □

Lemma 8.4. *Let (X, L, N) be a neighbourhood system and consider the following conditions:*
(1) (X, L, N) is topological;
(2) Every filter on L which having a limit point is convergence;
(3) Every filter on L which having a cluster point is a clustered filter.
Then (1)⇔(2), (1)⇒(3).

Proof. (1)⇔(2) Suppose (X, L, N) is topological and $F \subseteq L$ is a filter which having a limit point $x \in X$. Given a cover $\bigvee S = 1$ of L, there exists $u \in S$ such that $u \in N(x)$. Thus $u \in F$.

Conversely, if every filter on L which having a limit point is convergence, then for each $x \in X$, $N(x)$ is convergence. Hence $\bigvee S = 1$ implies that $\exists s \in S$ such that $s \in N(x)$.

(1)⇒(3) Suppose $F \subseteq L$ is a filter which having a cluster point $x \in X$. If $\bigvee\{\neg a \mid a \in F\} = 1$ then there exists $a \in F$ such that $\neg a \in N(x)$. Hence $a \wedge \neg a = 0$ contradicts with the condition that x is cluster point of F. □

Corollary 8.5. *Let (X, L, N) be a topological coverable neighbourhood system. Then*
(1) A filter F on L is convergence if and only if it has a limit point $x \in X$;
(2) A filter F on L is clustered if and only if it has a cluster point $x \in X$.

Let (X, L, N) be a neighbourhood system. We call (X, L, N) a *Hausdorff neighbourhood system* if for any two points $x, y \in X, x \neq y$, there exist $u \in N(x), v \in N(y)$ such that $u \wedge v = 0$.

Lemma 8.6. *(X, L, N) is a Hausdorff neighbourhood system if and only if every filter F on L has at most one limit.*

The Hausdorff separation (or T_2 separation) of locales has been widely studied by many authors. For example J. Paseka and B. Šmarda in [7] defined T_2-frames in the form usual in the case of regular frames by introducing a binary relation on frames. More usually one defines a locale L to be Hausdorff if the diagonal map $\triangle : L \to L \times L$ is a closed embedding.

As an application of the convergence of filters, we now give a new definition of Hausdorffness of locales which coincides with T_2-axiom defined by J. Paseka and B. Šmarda in the realm of T_0 spaces.

Let L be a locale and F a filter on L. F is said *minimal convergence* if it is convergence and there is no convergence filter $F' \subseteq F$ such that $F' \neq F$.

Definition 8.7. A locale L is said a T_2^* locale if every convergence filter on L contains at most one minimal convergence filter.

It is not difficult to see that for a T_0 space X, X is Hausdorff if and only if its open sets locale $\Omega(X)$ is a T_2^* locale.

Example 8.8. Let L be a locale. Write L^* be the locale generated by L add a new top element 1^* which does not belong to L. Then every filter on L^* converges and $\{1^*\}$ is the unique minimal convergence filter. Hence L^* is a T_2^* locale.

If we take a locale L which is not T_2 in the sense of J. Paseka and B. Šmarda then L^* is T_2^* but not T_2. This shows that T_2^* axiom does not imply T_2 axiom. We don't know whether T_2 frames are T_2^*.

Proposition 8.9. *Let (X, L, N) be a T_0 topological coverable neighbourhood system. Then (X, L, N) is a Hausdorff neighbourhood system if and only if L is a T_2^* locale.*

Note that the neighbourhood system defined in Example 3.2 is a Hausdorff neighbourhood system, so by this proposition, the locale of regular open sets of real line $\mathbb{R}$ is a T_2^* locale.

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References

[1] J. Adamek, M.-C. Pedicchio, A remark on topological spaces, grids, and topological systems, Cahiers Topologie Géom. Diff. Catégoriques 38(1997) 217–226.
[2] B. Banaschewski, A. Pultr, Samuel compactification and completion of uniform frames, Math. Proc. Camb. Philos. Soc. 108(1990) 63–78.
[3] M. Barr, M.-C. Pedicchio, Top^{op} is a quasi-variety, Cahiers Topologie Géom. Diff. Catégoriques 36(1995) 3–10.
[4] R. Engelking, General Topology, Sigma Series in Pure Math, Vol.6, Berlin: Heldermann, 1989.
[5] S.S. Hong, Convergence in frames, Kyungpook Math. J. 35(1995) 85–95.
[6] P.T.Johnstone, Stone Spaces, Cambridge University Press, Cambridge, 1982.
[7] J. Paseka, B. Šmarda, T_2 frames and almost compact frames, Czech. Math. J. 42(1992) 385–402.
[8] S. Vickers, Topology via Logic, Cambridge University Press, Cambridge, 1989.