Filomat 31:8 (2017), 2281–2292
DOI 10.2298/FIL1708281A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Two Point Trapezoidal Like Inequalities Involving Hypergeometric Functions

Muhammad Uzair Awan[a], Muhammad Aslam Noor[b,c], Marcela V. Mihai[d], Khalida Inayat Noor[b]

[a]*Government College University, Faisalabad, Pakistan.*
[b]*Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.*
[c]*Mathematics Department, King Saud University, Riyadh, Saudi Arabia.*
[d]*Department scientific-methodical sessions, Romanian Mathematical Society-branch Bucharest, Academy Street no. 14, RO-010014, Bucharest, Romania.*

Abstract. In this paper, we derive a new integral identity for differentiable function. Using this new integral identity as an auxiliary result, we derive some new two point trapezoidal like inequalities for differentiable harmonic h-convex functions. These inequalities can also be viewed as Hermite-Hadamard type inequalities. We also discuss some new special cases which can be deduced from our main results.

1. Introduction and Preliminaries

In past few years, several new extensions of classical convexity have been proposed, see [1–9, 12, 16, 18, 19]. Iscan [8] introduced and studied the class of harmonic convex functions. Noor et al. [16] extended the class of harmonic convex functions and introduced the notion of harmonic h-convex functions, which unifies some new classes of harmonic convex functions. For some recent studies on harmonic convexity, see [8, 9, 11, 12, 16, 17].

Definition 1.1 ([18]). *A set $K_h \subset \mathbb{R} \setminus \{0\}$ is said to be harmonic convex, if*

$$\frac{xy}{tx+(1-t)y} \in K \quad \forall x, y \in K, t \in [0,1].$$

Harmonic convex functions are defined as:

Definition 1.2 ([8]). *Let K_h be a harmonic convex set. A function $f : K_h \to \mathbb{R}$ is said to be harmonic convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in K_h, t \in [0,1].$$

2010 *Mathematics Subject Classification*. 26A51; 26D15
Keywords. Convex functions, harmonic, functions, integral inequalities.
Received: 21 October 2015; Accepted: 23 November 2015
Communicated by Dragan S. Djordjević
Corresponding Author: Muhammad Uzair Awan
Email addresses: awan.uzair@gmail.com (Muhammad Uzair Awan), noormaslam@hotmail.com (Muhammad Aslam Noor), marcelamihai58@yahoo.com (Marcela V. Mihai), khalidanoor@hotmail.com (Khalida Inayat Noor)

Noor et al. [16] defined the class of harmonic h-convex functions as:

Definition 1.3 ([16]). *Let K_h be a harmonic convex set and $h : [0,1] \to \mathbb{R}$ be a real function. A function $f : K_h \to \mathbb{R}$ is said to be harmonic h-convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(1-t)f(x) + h(t)f(y), \forall x, y \in K_h \text{ and } t \in [0,1]. \tag{1.1}$$

Remark 1.4. *Note that, if $h(t) = t, t^s, t^{-1}, t^{-s}$ and $h(t) = 1$ in Definition 1.3, then, we have definitions of harmonic convex, harmonic s-convex, harmonic Godunova-Levin, harmonic s-Godunova-Levin convex and harmonic p-functions respectively. This shows that the class of harmonic h-convex functions is quite unifying one.*

The main objective of this paper is to derive some new two point trapezoidal like inequalities via harmonic h-convex functions. For this, we derive a new integral identity which will serve as an auxiliary result for the development of main results. We also discuss some special cases which are included in our main results. The results obtained in this paper can be useful in the field of numerical analysis where error analysis is required.

Before proceeding further, we recall some concepts from special functions, which are widely used in the development of our main results.

Gamma and Beta functions are defined respectively as:

$$\Gamma(x) = \int_0^\infty e^{-x} t^{x-1} \mathrm{d}t,$$

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2F_1(x,y;c;z) = \frac{1}{\mathrm{B}(y,c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\mathrm{d}t$$

for $|z| < 1, c > y > 0$. For more information, see [10].

From now onward, we take the notations $\mathcal{I}_h \subset \mathbb{R} \setminus \{0\}$ be the real interval and $\mathcal{I}_h^0$ be the interior of $\mathcal{I}_h$ unless otherwise specified.

2. A New Integral Identity

In this section, we derive our auxiliary result which plays an important role in the development of our main results.

Lemma 2.1. *Let $f : \mathcal{I}_h \to \mathbb{R}$ be a differentiable function on I_h^0 where $a, b \in I_h$ with $a < b$. If $f' \in L[a,b]$, then*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\
&= ab(b-a)\left[\int_0^1 \left(\frac{1}{2}-t\right)\left(\frac{1}{(1-t)a+(1+t)b}\right)^2 f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\mathrm{d}t\right. \\
&\quad \left. + \int_0^1 \left(t-\frac{1}{2}\right)\left(\frac{1}{(1+t)a+(1-t)b}\right)^2 f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\mathrm{d}t\right].
\end{aligned}$$

Proof. Let

$$V_1 = \int_0^1 \left(\frac{1}{2} - t\right)\left(\frac{1}{(1-t)a + (1+t)b}\right)^2 f'\left(\frac{2ab}{(1-t)a + (1+t)b}\right)\mathrm{d}t$$

$$= \frac{1}{4ab(b-a)} f(a) + \frac{1}{4ab(b-a)} f\left(\frac{2ab}{a+b}\right) - \frac{ab}{ab(b-a)^2}\int_a^{\frac{2ab}{a+b}} \frac{f(x)}{x^2}\mathrm{d}x. \tag{2.1}$$

Similarly

$$V_2 = \int_0^1 \left(t - \frac{1}{2}\right)\left(\frac{1}{(1+t)a + (1-t)b}\right)^2 f'\left(\frac{2ab}{(1+t)a + (1-t)b}\right)\mathrm{d}t$$

$$= \frac{1}{4ab(b-a)} f(b) + \frac{1}{4ab(b-a)} f\left(\frac{2ab}{a+b}\right) - \frac{ab}{ab(b-a)^2}\int_{\frac{2ab}{a+b}}^{b} \frac{f(x)}{x^2}\mathrm{d}x. \tag{2.2}$$

Combining (2.1), (2.2) and then multiplying by $ab(b-a)$ completes the proof. □

3. Some New Integral Inequalities

In this section, using Lemma 2.1 we derive some new inequalities of Hermite-Hadamard type for the class of harmonic h-convex functions.

Theorem 3.1. *Let $f : \mathcal{I}_h \to \mathbb{R}$ be a differentiable function on I_h^0 where $a, b \in I_h$ with $a < b$ and $f' \in L[a, b]$. If $|f'|$ is harmonic h-convex function, then*

$$\left|\frac{1}{2}\left[\frac{f(a) + f(b)}{2} + f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq ab(b-a)\left[\{|f'(a)| + |f'(b)|\}\left\{\int_0^1 \left|t - \frac{1}{2}\right|(c_1 + c_2)(c_3 + c_4)\,\mathrm{d}t\right\}\right],$$

where

$$c_1 = h\left(\frac{1+t}{2}\right), \tag{3.1}$$

$$c_2 = h\left(\frac{1-t}{2}\right), \tag{3.2}$$

$$c_3 = \frac{1}{((1-t)a + (1+t)b)^2}, \tag{3.3}$$

and

$$c_4 = \frac{1}{((1+t)a + (1-t)b)^2}, \tag{3.4}$$

respectively.

Proof. Using Lemma 2.1, property of the modulus and the fact that $|f'|$ is harmonic h-convex function, we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right| \\
&=\left|ab(b-a)\left[\int_0^1\left(\frac{1}{2}-t\right)\left(\frac{1}{(1-t)a+(1+t)b}\right)^2 f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\mathrm{d}t\right.\right. \\
&\qquad\left.\left.+\int_0^1\left(t-\frac{1}{2}\right)\left(\frac{1}{(1+t)a+(1-t)b}\right)^2 f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\mathrm{d}t\right]\right| \\
&\le ab(b-a)\left[\int_0^1\left|t-\frac{1}{2}\right|\left(\frac{1}{(1-t)a+(1+t)b}\right)^2\left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|\mathrm{d}t\right. \\
&\qquad\left.+\int_0^1\left|t-\frac{1}{2}\right|\left(\frac{1}{(1+t)a+(1-t)b}\right)^2\left|f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|\mathrm{d}t\right] \\
&\le ab(b-a)\left[\int_0^1\left|t-\frac{1}{2}\right|\left(\frac{1}{(1-t)a+(1+t)b}\right)^2\left\{h\left(\frac{1+t}{2}\right)|f'(a)|+h\left(\frac{1-t}{2}\right)|f'(b)|\right\}\mathrm{d}t\right. \\
&\qquad\left.+\int_0^1\left|t-\frac{1}{2}\right|\left(\frac{1}{(1-t)a+(1+t)b}\right)^2\left\{h\left(\frac{1-t}{2}\right)|f'(a)|+h\left(\frac{1+t}{2}\right)|f'(b)|\right\}\mathrm{d}t\right] \\
&=ab(b-a)\left[\{|f'(a)|+|f'(b)|\}\left\{\int_0^1\left|t-\frac{1}{2}\right|\left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right)\right.\right. \\
&\qquad\left.\left.\times\left(\frac{1}{((1-t)a+(1+t)b)^2}+\frac{1}{((1+t)a+(1-t)b)^2}\right)\mathrm{d}t\right\}.\right.
\end{aligned}$$

This completes the proof. □

Corollary 3.2. *Under the assumptions of Theorem 3.1 and $h(t)=t$, we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right| \\
&\le\frac{ab(b-a)}{2(a+b)^2}\left[|f'(a)|+|f'(b)|\right](\varphi_1+\varphi_2),
\end{aligned}$$

where

$$\varphi_1={}_2F_1\left(2,2;3;\frac{a-b}{a+b}\right)-{}_2F_1\left(2,1;2;\frac{a-b}{a+b}\right)+\frac{1}{4}\cdot{}_2F_1\left(2,1;3;\frac{a-b}{a+b}\right),$$

and

$$\varphi_2={}_2F_1\left(2,2;3;\frac{b-a}{a+b}\right)-{}_2F_1\left(2,1;2;\frac{b-a}{a+b}\right)+\frac{1}{4}\cdot{}_2F_1\left(2,1;3;\frac{b-a}{a+b}\right).$$

Corollary 3.3. *Under the assumptions of Theorem 3.1 and* $h(t) = t^s$*, we have*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq ab(b-a)\left[|f'(a)|+|f'(b)|\right](I_1+I_2),$$

where

$$I_1=\int_0^{1/2}\left(\frac{1}{2}-t\right)(c_1+c_2)(c_3+c_4)\,\mathrm{d}t=J_1+J_2+J_3+J_4,$$

with:

$$J_1=\int_0^{1/2}\left(\frac{1}{2}-t\right)c_1c_3\mathrm{d}t\leq\frac{3^s}{2^{2s+3}(a+b)^2}\cdot{}_2F_1\left(2,1;3;\frac{a-b}{2(a+b)}\right),$$

$$J_2=\int_0^{1/2}\left(\frac{1}{2}-t\right)c_1c_4\mathrm{d}t\leq\frac{3^s}{2^{2s+3}(a+b)^2}\cdot{}_2F_1\left(2,1;3;\frac{b-a}{2(a+b)}\right),$$

$$\begin{aligned}J_3&=\int_0^{1/2}\left(\frac{1}{2}-t\right)c_2c_3\mathrm{d}t\\&=\frac{1}{2^{s+3}b^2}\left[\frac{2}{s+2}\cdot{}_2F_1\left(2,s+2;s+3;\frac{b-a}{2b}\right)\right.\\&\left.-\frac{1}{s+1}\cdot{}_2F_1\left(2,s+1;s+2;\frac{b-a}{2b}\right)+\frac{1}{2^{s+1}(s+1)(s+2)}\cdot{}_2F_1\left(2,s+1;s+3;\frac{b-a}{4b}\right)\right],\end{aligned}$$

$$\begin{aligned}J_4&=\int_0^{1/2}\left(\frac{1}{2}-t\right)c_2c_4\mathrm{d}t\\&=\frac{1}{2^{s+3}a^2}\left[\frac{2}{s+2}\cdot{}_2F_1\left(2,s+2;s+3;\frac{a-b}{2a}\right)\right.\\&\left.-\frac{1}{s+1}\cdot{}_2F_1\left(2,s+1;s+2;\frac{a-b}{2a}\right)+\frac{1}{2^{s+1}(s+1)(s+2)}\cdot{}_2F_1\left(2,s+1;s+3;\frac{a-b}{4a}\right)\right],\end{aligned}$$

and

$$I_2=\int_{1/2}^1\left(t-\frac{1}{2}\right)(c_1+c_2)(c_3+c_4)\,\mathrm{d}t=K_1+K_2+K_3+K_4,$$

with:

$$K_1=\int_{1/2}^1\left(t-\frac{1}{2}\right)c_1c_3\mathrm{d}t\leq\frac{1}{8(a+b)^2}\cdot{}_2F_1\left(2,1;3;\frac{a-b}{2(a+b)}\right)+\frac{1}{4b(a+b)},$$

$$K_2=\int_{1/2}^1\left(t-\frac{1}{2}\right)c_1c_4\mathrm{d}t\leq\frac{1}{8(a+b)^2}\cdot{}_2F_1\left(2,1;3;\frac{b-a}{2(a+b)}\right)+\frac{1}{4a(a+b)},$$

$$K_3 = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)c_2c_3\mathrm{d}t = \frac{1}{2^{2s+4}b^2(s+1)(s+2)}\cdot {}_2F_1\left(2,2;s+3;\frac{b-a}{4b}\right),$$

$$K_4 = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)c_2c_4\mathrm{d}t = \frac{1}{2^{2s+4}a^2(s+1)(s+2)}\cdot {}_2F_1\left(2,2;s+3;\frac{a-b}{4a}\right).$$

Corollary 3.4. *Under the assumptions of Theorem 3.1 and $h(t)=t^{-s}$ we have*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq ab(b-a)\left[|f'(a)|+|f'(b)|\right]\left(I_1^*+I_2^*\right),$$

where

$$I_1^* = \int_0^{1/2}\left(\frac{1}{2}-t\right)(c_1+c_2)(c_3+c_4)\,\mathrm{d}t = J_1^*+J_2^*+J_3^*+J_4^*,$$

with:

$$J_1^* = \int_0^{1/2}\left(\frac{1}{2}-t\right)c_1c_3\mathrm{d}t \leq \frac{1}{2^{-s+3}(a+b)^2}\cdot {}_2F_1\left(2,1;3;\frac{a-b}{2(a+b)}\right),$$

$$J_2^* = \int_0^{1/2}\left(\frac{1}{2}-t\right)c_1c_4\mathrm{d}t \leq \frac{1}{2^{-s+3}(a+b)^2}\cdot {}_2F_1\left(2,1;3;\frac{b-a}{2(a+b)}\right),$$

$$\begin{aligned}J_3^* &= \int_0^{1/2}\left(\frac{1}{2}-t\right)c_2c_3\mathrm{d}t\\ &= \frac{1}{2^{-s+3}b^2}\left[\frac{2}{-s+2}\cdot {}_2F_1\left(2,-s+2;-s+3;\frac{b-a}{2b}\right)-\frac{1}{-s+1}\cdot {}_2F_1\left(2,-s+1;-s+2;\frac{b-a}{2b}\right)\right.\\ &\quad\left.+\frac{1}{2^{-s+1}(-s+1)(-s+2)}\cdot {}_2F_1\left(2,-s+1;-s+3;\frac{b-a}{4b}\right)\right],\end{aligned}$$

$$\begin{aligned}J_4^* &= \int_0^{1/2}\left(\frac{1}{2}-t\right)c_2c_4\mathrm{d}t\\ &= \frac{1}{2^{-s+3}a^2}\left[\frac{2}{-s+2}\cdot {}_2F_1\left(2,-s+2;-s+3;\frac{a-b}{2a}\right)\right.\\ &\quad\left.-\frac{1}{-s+1}\cdot {}_2F_1\left(2,-s+1;-s+2;\frac{a-b}{2a}\right)+\frac{1}{2^{-s+1}(-s+1)(-s+2)}\cdot {}_2F_1\left(2,-s+1;-s+3;\frac{a-b}{4a}\right)\right],\end{aligned}$$

and

$$I_2^* = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)(c_1+c_2)(c_3+c_4)\,\mathrm{d}t = K_1^*+K_2^*+K_3^*+K_4^*,$$

with:

$$K_1^* = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)c_1c_3\mathrm{d}t \le \frac{1}{2^{-2s+3}\cdot 3^s(a+b)^2}\cdot{}_2F_1\left(2,1;3;\frac{a-b}{2(a+b)}\right)+\frac{1}{2^{-2s+2}\cdot 3^s b(a+b)},$$

$$K_2^* = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)c_1c_4\mathrm{d}t \le \frac{1}{2^{-2s+3}\cdot 3^s(a+b)^2}\cdot{}_2F_1\left(2,1;3;\frac{b-a}{2(a+b)}\right)+\frac{1}{2^{-2s+2}\cdot 3^s a(a+b)},$$

$$K_3^* = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)c_2c_3\mathrm{d}t = \frac{1}{2^{-2s+4}b^2(-s+1)(-s+2)}\cdot{}_2F_1\left(2,2;-s+3;\frac{b-a}{4b}\right),$$

$$K_4^* = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)c_2c_4\mathrm{d}t = \frac{1}{2^{-2s+4}a^2(-s+1)(-s+2)}\cdot{}_2F_1\left(2,2;-s+3;\frac{a-b}{4a}\right).$$

Corollary 3.5. *Under the assumptions of Theorem 3.1 and $h(t) = 1$, we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\le \frac{1}{(a+b)^2}\left[\frac{1}{4}\cdot{}_2F_1\left(2,1;3;\frac{a-b}{2(a+b)}\right)+\frac{1}{4}\cdot{}_2F_1\left(2,1;3;\frac{b-a}{2(a+b)}\right)\right.\\
&\quad+{}_2F_1\left(2,2;3;\frac{b-a}{a+b}\right)-{}_2F_1\left(2,1;2;\frac{b-a}{a+b}\right)+\frac{1}{2}\cdot{}_2F_1\left(2,1;3;\frac{b-a}{2(a+b)}\right)\\
&\quad\left.+{}_2F_1\left(2,2;3;\frac{a-b}{a+b}\right)-{}_2F_1\left(2,1;2;\frac{a-b}{a+b}\right)+\frac{1}{2}\cdot{}_2F_1\left(2,1;3;\frac{a-b}{2(a+b)}\right)\right].
\end{aligned}$$

Corollary 3.6. *Under the assumptions of Theorem 3.1, if*

$$\frac{f(a)+f(b)}{2} = f\left(\frac{2ab}{a+b}\right),$$

we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|\\
&=\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\le ab(b-a)\left[\left[\{|f'(a)|+|f'(b)|\}\left\{\int_0^1\left|t-\frac{1}{2}\right|(c_1+c_2)(c_3+c_4)\,\mathrm{d}t\right\}\right]\right],
\end{aligned}$$

where c_1, c_2, c_3 and c_4 are given by (3.1), (3.2), (3.3) and (3.4) respectively.

Theorem 3.7. *Let $f : \mathcal{I}_h \to \mathbb{R}$ be a differentiable function on I_h^0 where $a, b \in I_h$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonic h-convex function where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq ab(b-a)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}\left[\left(\int_0^1 c_3^q\{c_1|f'(a)|^q + c_2|f'(b)|^q\}\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_0^1 c_4^q\{c_2|f'(a)|^q + c_1|f'(b)|^q\}\,\mathrm{d}t\right)^{\frac{1}{q}}\right],$$

where c_1, c_2 are given by (3.1), (3.2) and

$$c_3^q = \frac{1}{((1-t)a + (1+t)b)^{2q}}, \tag{3.5}$$

and

$$c_4^q = \frac{1}{((1+t)a + (1-t)b)^{2q}}, \tag{3.6}$$

respectively.

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|f'|^q$ is harmonic h-convex function, we have

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$= \left|ab(b-a)\left[\int_0^1\left(\frac{1}{2}-t\right)\left(\frac{1}{(1-t)a+(1+t)b}\right)^2 f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\mathrm{d}t\right.\right.$$

$$\left.\left. + \int_0^1\left(t-\frac{1}{2}\right)\left(\frac{1}{(1+t)a+(1-t)b}\right)^2 f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\mathrm{d}t\right]\right|$$

$$\leq ab(b-a)\left[\left(\int_0^1\left|t-\frac{1}{2}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{((1-t)a+(1+t)b)^{2q}}\left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1\left|t-\frac{1}{2}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{((1-t)a+(1+t)b)^{2q}}\left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$\leq ab(b-a)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(\frac{1}{((1-t)a+(1+t)b)^{2q}}\right)\left[h\left(\frac{1+t}{2}\right)|f'(a)|^q + h\left(\frac{1-t}{2}\right)|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1\left(\frac{1}{((1-t)a+(1+t)b)^{2q}}\right)\left[h\left(\frac{1-t}{2}\right)|f'(a)|^q + h\left(\frac{1+t}{2}\right)|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$= ab(b-a)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}\left[\left(\int_0^1 c_3^q\{c_1|f'(a)|^q + c_2|f'(b)|^q\}\,\mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_0^1 c_4^q\{c_2|f'(a)|^q + c_1|f'(b)|^q\}\,\mathrm{d}t\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Corollary 3.8. *Under the assumptions of Theorem 3.7 and $h(t) = t$, we have*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq ab(b-a)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}\left(\psi_1^{1/q}+\psi_2^{1/q}\right),$$

where

$$\psi_1=\int\limits_0^1 c_3^q\left\{c_1|f'(a)|^q+c_2|f'(b)|^q\right\}\mathrm{d}t$$
$$=\frac{1}{2(a+b)^{2q}}\left\{|f'(a)|^q\left[{}_2F_1\left(2q,1;2;\frac{a-b}{a+b}\right)+\frac{1}{2}\cdot{}_2F_1\left(2q,2;3;\frac{a-b}{a+b}\right)\right]+\frac{1}{2}|f'(b)|^q\cdot{}_2F_1\left(2q,1;3;\frac{a-b}{a+b}\right)\right\},$$

and

$$\psi_2=\int\limits_0^1 c_4^q\left\{c_2|f'(a)|^q+c_1|f'(b)|^q\right\}\mathrm{d}t$$
$$=\frac{1}{2(a+b)^{2q}}\left\{\frac{1}{2}|f'(a)|^q\cdot{}_2F_1\left(2q,1;3;\frac{b-a}{a+b}\right)+|f'(b)|^q\left[{}_2F_1\left(2q,1;2;\frac{b-a}{a+b}\right)+\frac{1}{2}\cdot{}_2F_1\left(2q,2;3;\frac{b-a}{a+b}\right)\right]\right\}.$$

Corollary 3.9. *Under the assumptions of Theorem 3.7 and $h(t) = t^s$, we have*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq ab(b-a)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}\left(\psi_1^{1/q}+\psi_2^{1/q}\right),$$

where

$$\psi_1=\int\limits_0^1 c_3^q\left\{c_1|f'(a)|^q+c_2|f'(b)|^q\right\}\mathrm{d}t=|f'(a)|^qM_1+|f'(b)|^qM_2,$$

with

$$M_1=\int_0^1 c_1c_3^q\mathrm{d}t=\frac{1}{2^s(a+b)^{2q}}\int_0^1(1+t)^s\left(1-\frac{a-b}{a+b}t\right)^{-2q}\mathrm{d}t$$
$$\leq\frac{1}{(2q-1)(a-b)}\left[\frac{1}{(2b)^{2q-1}}-\frac{1}{(a+b)^{2q-1}}\right],$$

$$M_2=\int_0^1 c_2c_3^q\mathrm{d}t$$
$$=\frac{1}{2^s(a+b)^{2q}}\int_0^1(1-t)^s\left(1-\frac{a-b}{a+b}t\right)^{-2q}\mathrm{d}t$$
$$=\frac{1}{2^s(s+1)(a+b)^{2q}}\cdot{}_2F_1\left(2q,1;s+2;\frac{a-b}{a+b}\right),$$

and

$$\psi_2 = \int_0^1 c_4^q \{c_2|f'(a)|^q + c_1|f'(b)|^q\}\,\mathrm{d}t = |f'(a)|^q N_1 + |f'(b)|^q N_2,$$

with

$$\begin{aligned} N_1 &= \int_0^1 c_2 c_4^q \mathrm{d}t \\ &= \frac{1}{2^s(a+b)^{2q}} \int_0^1 (1-t)^s \left(1 - \frac{b-a}{a+b}t\right)^{-2q} \mathrm{d}t \\ &= \frac{1}{2^s(s+1)(a+b)^{2q}} \cdot {}_2F_1\left(2q, 1; s+2; \frac{b-a}{a+b}\right), \end{aligned}$$

$$\begin{aligned} N_2 = \int_0^1 c_1 c_4^q \mathrm{d}t &= \frac{1}{2^s(a+b)^{2q}} \int_0^1 (1+t)^s \left(1 - \frac{b-a}{a+b}t\right)^{-2q} \mathrm{d}t \\ &\le \frac{1}{(2q-1)(b-a)} \left[\frac{1}{(2a)^{2q-1}} - \frac{1}{(a+b)^{2q-1}}\right]. \end{aligned}$$

Corollary 3.10. *Under the assumptions of Theorem 3.7 and $h(t) = t^{-s}$, we have*

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\ &\le ab(b-a)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}} \left(\psi_1^{1/q} + \psi_2^{1/q}\right), \end{aligned}$$

where

$$\psi_1 = \int_0^1 c_3^q \{c_1|f'(a)|^q + c_2|f'(b)|^q\}\,\mathrm{d}t = |f'(a)|^q M_1^* + |f'(b)|^q M_2^*,$$

with

$$\begin{aligned} M_1^* = \int_0^1 c_1 c_3^q \mathrm{d}t &= \frac{1}{2^{-s}(a+b)^{2q}} \int_0^1 (1+t)^{-s} \left(1 - \frac{a-b}{a+b}t\right)^{-2q} \mathrm{d}t \\ &\le \frac{1}{2^{-s}(2q-1)(a-b)} \left[\frac{1}{(2b)^{2q-1}} - \frac{1}{(a+b)^{2q-1}}\right], \end{aligned}$$

$$\begin{aligned} M_2^* &= \int_0^1 c_2 c_3^q \mathrm{d}t = \frac{1}{2^{-s}(a+b)^{2q}} \int_0^1 (1-t)^{-s} \left(1 - \frac{a-b}{a+b}t\right)^{-2q} \mathrm{d}t \\ &= \frac{1}{2^{-s}(-s+1)(a+b)^{2q}} \cdot {}_2F_1\left(2q, 1; -s+2; \frac{a-b}{a+b}\right), \end{aligned}$$

and

$$\psi_2 = \int_0^1 c_4^q \{c_2|f'(a)|^q + c_1|f'(b)|^q\}\,\mathrm{d}t = |f'(a)|^q N_1^* + |f'(b)|^q N_2^*,$$

with

$$N_1^* = \int_0^1 c_2 c_4^q \mathrm{d}t = \frac{1}{2^{-s}(a+b)^{2q}} \int_0^1 (1-t)^{-s} \left(1 - \frac{b-a}{a+b}t\right)^{-2q} \mathrm{d}t$$
$$= \frac{1}{2^{-s}(-s+1)(a+b)^{2q}} \cdot {}_2F_1\left(2q, 1; -s+2; \frac{b-a}{a+b}\right),$$

$$N_2^* = \int_0^1 c_1 c_4^q \mathrm{d}t = \frac{1}{2^{-s}(a+b)^{2q}} \int_0^1 (1+t)^{-s} \left(1 - \frac{b-a}{a+b}t\right)^{-2q} \mathrm{d}t$$
$$\leq \frac{1}{(2q-1)(b-a)} \left[\frac{1}{(2a)^{2q-1}} - \frac{1}{(a+b)^{2q-1}}\right].$$

Corollary 3.11. *Under the assumptions of Theorem 3.7 and $h(t) = 1$, we have*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq \frac{ab(b-a)}{(a+b)^2}\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}} \left(|f'(a)|^q + |f'(b)|^q\right)^{1/q} \left[{}_2F_1\left(2q, 1; 2; \frac{a-b}{a+b}\right) + {}_2F_1\left(2q, 1; 2; \frac{b-a}{a+b}\right)\right]^{1/q}.$$

Corollary 3.12. *Under the assumptions of Theorem 3.7, if*

$$\frac{f(a)+f(b)}{2} = f\left(\frac{2ab}{a+b}\right),$$

we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$= \left|f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq ab(b-a)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}} \left[\left(\int_0^1 c_3^q \left\{c_1|f'(a)|^q + c_2|f'(b)|^q\right\} \mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_0^1 c_4^q \left\{c_2|f'(a)|^q + c_1|f'(b)|^q\right\} \mathrm{d}t\right)^{\frac{1}{q}}\right],$$

where c_1, c_2, c_3^q and c_4^q are given by (3.1), (3.2), (3.5) and (3.6) respectively.

Conclusion

In this paper, we have derived several new trapezoidal like inequalities via harmonic h-convex functions. We have derived a new integral identity for differentiable functions. In order to obtain our main results, we have used this new integral identity as an auxiliary result. We have also discussed several new special cases which can be deduced from our main results. It is expected that the ideas and techniques of this paper may inspire interested readers to further explore this area of research. It is an interesting problem for future research to obtain these results using the concepts of quantum calculus. For some useful details, see [13–15].

Acknowledgements

Authors are thankful to anonymous referee and to the editor for their valuable comments and suggestions. Authors also would like to thank Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research and academic environment. Authors are pleased to acknowledge the "support of Distinguished Scientist Fellowship Program(DSFP), King Saud University," Riyadh, Saudi Arabia.

References

[1] W. W. Breckner, Stetigkeitsaussagen fiir eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen, Pupl. Inst. Math., 23, 13-20, (1978).
[2] G. Cristescu and L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, 2002.
[3] G. Cristescu, M. A. Noor, M. U. Awan, Bounds of the second degree cumulative frontier gaps of functions with generalized convexity, Carpathian J. Math. 31(2), 173-180, (2015).
[4] S. S. Dragomir, Inequalities of HermiteHadamard type for h-convex functions on linear spaces, RGMIA Res. Rep. Coll., Article 72 (2013).
[5] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, Australia (2000).
[6] S. S. Dragomir, J. Pečarić, L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21, 335-341, (1995).
[7] E. K. Godunova, V. I. Levin, Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii. Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva. (1985) 138-142, (in Russian).
[8] I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe J. Math. Stat., 43(6), 935-942, (2014).
[9] I. Iscan, Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions, J. Math. Article ID 346305, 10 pages, (2014).
[10] A. Kilbas , H. M. Srivastava, J. J. Trujillo: Theory and applications of fractional differential equations, Elsevier B.V., Amsterdam, Netherlands, (2006).
[11] M. V. Mihai, M. A. Noor, K. I. Noor, M. U. Awan, Some integral inequalities for harmonic h-convex functions involving hypergeometric functions, Appl. Math. Comput. 252, 257-262, (2015).
[12] M. A. Noor, K. I. Noor, M. U. Awan, Integral inequalities for coordinated harmonically convex functions, Complex Var. Elliptic Equat. 60(6), 776-786 (2015).
[13] M. A. Noor, K. I. Noor, M. U. Awan, Some Quantum estimates for Hermite-Hadamard inequalities, Appl. Math. Comput. 251, 675-679 (2015).
[14] M. A. Noor, K. I. Noor, M. U. Awan, Some quantum integral inequalities via preinvex functions, Appl. Math. Comput., 269, 242-251, (2015).
[15] M. A. Noor, K. I. Noor, M. U. Awan, Quantum Ostrowski inequalities for q-differentiable convex functions, J. Math. Inequal. 10(4), 1013-1018, (2016).
[16] M. A. Noor, K. I. Noor, M. U. Awan, S. Costache, Some integral inequalities for harmonically h-convex functions U. P. B. Sci. Bull., Series A. 77(1), 5-16, (2015).
[17] J. Park, Hermite-Hadamard-like and Simpson-like type inequalities for harmonically convex functions. Int. J. Math. Anal. 8(27), 1321-1337, (2014).
[18] H.-N. Shi and J. Zhang, Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions, J. Inequal. Appl. 2013(2013), 527.
[19] S. Varošanec, On h-convexity, J. Math. Anal. Appl., 326, 303-311, (2007).