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Application of Fixed Point Theorem for Stability Analysis of a Nonlinear Schrodinger with Caputo-Liouville Derivative

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Abstract. Using the new Caputo-Liouville derivative with fractional order, we have modified the nonlinear Schrdinger equation. We have shown some useful in connection of the new derivative with fractional order. We used an iterative approach to derive an approximate solution of the modified equation. We have established the stability of the iteration scheme using the fixed point theorem. We have in addition presented in detail the uniqueness of the special solution.

1. Introduction

Recently, a new derivative with fractional order was proposed by Caputo and Fabrizio [1, 2]. They argued that, "the new derivative assumes two different representations for temporal and spatial variable. However, the first form of this derivative was proposed in late 1832 by Joseph Liouville [3] .The first representation works on times variable, where the real powers appearing in the solution of the usual fractional derivative will turn into integer power and the second one is related to the spatial variables, thus for the non-local fractional derivative". One of the interesting applications of this new derivative is that, it can describe material heterogeneities and structures with different scales, which obviously cannot be handling with the well-known local theories [1]. Another application is in the study of the macroscopic behaviours of some materials, connected with non-local interactions between atoms, which are established in decisive of the properties of material. On the other hand, nonlinear differential equations have been quit efficient in describing the behaviour of some interesting real world problem. For instance Schrdinger equation plays the role of the Newton's law and conservation of energy in classical mechanic; more precisely predicts the future behaviour of a dynamic system [4–8, 12]. No wonder then, why many researcher have devoted their attention in developing new adequate analytical, numerical and iterative methods which can be used to derive exact or approximate solutions of these equations. However, in the case of approximate solutions obtained via iterative methods, the major concern is to establish the stability and the convergence of the method for the concerned equation. Fewer methods are found in the literature which helps investigating the stability of iteration methods.
The aim of this paper is to promote the application of the newly proposed derivative with fractional order to the nonlinear Schrdinger equation, to derive an approximate or exact solution using iteration technique,

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and prove the stability of the technique by using the fixed point theorem technique. The rest of the paper will have the following structure:
In section 2, we present the new derivative with fractional order with some properties, in section 3 we present the derivative the solution of the modified Schrdinger equation using an iteration method, in section 4 we present the application of the fixed point theorem to establish the stability of the methods.

2. Caputo-Liouville Derivative with Fractional Order

Definition 1: *Let $f \in H^1(a,\ b)$, $b > a$, $a \in [0,\ 1]$ then, the new Caputo derivative of fractional derivative is defined as:*

$$D_t^a(f(t)) = \frac{M(a)}{1-a}\int_a^t f'(x) exp\left[-a\frac{t-x}{1-a}\right]dx \tag{1}$$

Where $M(a)$ is a normalization function such that $M(0) = M(1) = 1$ [1]. However, if the function does not belongs to $H^1(a,\ b)$ then, the derivative can be reformulated as

$$D_t^a(f(t)) = \frac{aM(a)}{1-a}\int_a^t (f(t) - f(x))\, exp\left[-a\frac{t-x}{1-a}\right]dx \tag{2}$$

Remark: The authors remarked that, if $s = \frac{1-a}{a} \in [0,\infty)$, $a = \frac{1}{1+s} \in [0,1]$, then equation (2) assumes the form

$$D_t^a(f(t)) = \frac{N(s)}{s}\int_a^t f'(x)\, exp\left[-\frac{t-x}{s}\right]dx,\ \ N(0) = N(\infty) = 1 \tag{3}$$

In addition,

$$\lim_{s\to 0}\frac{1}{s}\, exp\left[-\frac{t-x}{s}\right] = d(x-t) \tag{4}$$

Now after the introduction of a new derivative, the associate anti-derivative becomes important, the associated integral of the new Caputo derivative with fractional order was proposed by Nieto and Losada [2].
Definition 2: [2] *Let $0 < a < 1$. The fractional integral of order a of a function f is defined by*

$$I_a^t(f(t)) = \frac{2(1-a)}{(2-a)M(a)}f(t) + \frac{2a}{(2-a)M(a)}\int_0^t f(s)\,ds,\ t \geq 0 \tag{5}$$

Remark [2]. Note that, according to the above definition, the fractional integral of Caputo type of function of order $0 < a < 1$ is an average between function f and its integral of order one. This therefore imposes

$$\frac{2(1-a)}{(2-a)M(a)} + \frac{2a}{(2-a)M(a)} = 1 \tag{6}$$

The above expression yields an explicit formula for

$$M(a) = \frac{2}{2-a},\ 0 \leq a \leq 1$$

Because of the above, Nieto and Losada proposed that the Caputo-Liouville derivative of order $0 < a < 1$ can be reformulated as

$$D_t^a(f(t)) = \frac{1}{1-a}\int_a^t f'(x) exp\left[-a\frac{t-x}{1-a}\right]dx \tag{7}$$

Theorem 1: *For the new Caputo derivative with fractional order, if the function $f(t)$ is such that*

$$f^{(s)}(a) = 0,\ s = 1, 2, \ldots n$$

then, we have

$$D_t^a(D_t^n(f(t))) = D_t^n(D_t^a(f(t)))$$

For proof see [1].

3. Application of Fixed Point Theorem for Nonlinear Fractional Schrödinger Equation

The equation under consideration here is the two dimensional generalized fractional Schrdinger equation

$$iD_t^{\alpha}\left(\Psi(X,t)\right) = -\frac{1}{2}\nabla^2\Psi\left((X,t)\right) + V_d(X)\Psi\left((X,t)\right) + B_d\Psi^{m+1}\left((X,t)\right), \quad X \in \mathbb{R}^d, \quad t > 0 \tag{8}$$

the above will be subjected to the following initial condition

$$i\Psi(X,0) = f(X), X \in \mathbb{R}^d$$

preliminaries: Let $(X, \|.\|)$ be a Banach space and H a self-map of X. Let $y_{n+1} = g(H, y_n)$ be some iterative technique. Assuming that, $F(H)$ the fixed point set of H has at least one element and that y_n converges to a point $p \in F(H)$. Let $\{x_n\} \subseteq X$ and define $e_n = \left\|x_{n+1} - g(H, x_n)\right\|$. If $\lim_{n\to\infty} e^n = 0$ implies that $\lim_{n\to\infty} x_n = p$, then the iteration method $y_{n+1} = g(H, y_n)$ is said to be $H-$ Stable. Without any loss of generality, we must assume that, our sequence $\{x_n\}$ has an upper boundary; otherwise we cannot expect the possibility of convergence. If all these conditions are satisfied for $y_{n+1} = Hy_n$ which is known as Picard's iteration, consequently the iteration will is $H-$ Stable. We shall then state the following theorem.

Theorem 2: (see [7]). Let $(X, \|.\|)$ be a Banach space and H a self-map of X satisfying $\left\|Hx - Hy\right\| \leq C\|x - Hx\| + c\left\|x - y\right\|$, for all x, y in X where $0 \leq C, 0 \leq \alpha < 1$. Suppose that H has fixed point p. Then, H is Picard $H-$ Stable.
Let consider the following sequence associate to the nonlinear fractional Schrdinger equation

$$\Psi_n(X,t) + I_\alpha^t\lambda(s)\left\{iD_t^{\alpha}\left(\Psi_n(X,s)\right) + \frac{1}{2}\nabla^2\Psi_n\left((X,s)\right) + V_d(X)\Psi_n\left((X,s)\right) + B_d\tilde{\Psi}_n^{m+1}\left((X,s)\right)\right\} \tag{9}$$

where $\lambda(s)$ is the Lagrange multiplier and $\tilde{\Psi}_n^{m+1}$ is a restricted variation implying $\delta\tilde{\Psi}_n^{m+1} = 0$.
Theorem 3: *Let H be a self-map defined as*

$$\begin{aligned}&H(\Psi_n(X,t)) = \Psi_{n+1}(X,t) = \Psi_n(X,t) + \\ &I_\alpha^t\lambda(s)\left\{iD_t^{\alpha}\left(\Psi_n(X,s)\right) + \tfrac{1}{2}\nabla^2\Psi_n\left((X,s)\right) + V_d(X)\Psi_n\left((X,s)\right) + B_d\tilde{\Psi}_n^{m+1}\left((X,s)\right)\right\}\end{aligned}$$

is $H-$ Stable in $L^2(a,b)$.
Proof. The first step in this proof is to show that, H has a fixed point. Therefore for $n, m \in \mathbb{N}$ we have

$$\begin{aligned}&\|H(\Psi_n(X,t)) - H(\Psi_k(X,t))\| = \|\Psi_{n+1}(X,t) - \Psi_{k+1}(X,t)\| \\ &= \left\|\begin{array}{l}\Psi_n(X,t) + \\ I_\alpha^t\lambda(s)\left\{\begin{array}{l} iD_t^{\alpha}\left(\Psi_n(X,s)\right) + \frac{1}{2}\nabla^2\Psi_n\left((X,s)\right) \\ +V_d(X)\Psi_n\left((X,s)\right) + B_d\Psi_n^{m+1}\left((X,s)\right)\end{array}\right\} - \\ \Psi_k(X,t) - \left\{I_\alpha^t\lambda(s)\left\{\begin{array}{l} iD_t^{\alpha}\left(\Psi_k(X,s)\right) \\ +\frac{1}{2}\nabla^2\Psi_k\left((X,s)\right) + V_d(X)\Psi_k\left((X,s)\right) + B_d\Psi_k^{m+1}\left((X,s)\right)\end{array}\right\}\right\}\end{array}\right\|\end{aligned} \tag{10}$$

Now using the triangular inequality property of the norm, we obtain

$$\begin{aligned}&\|\Psi_n(X,t) - \Psi_k(X,t)\| \leq \|\Psi_n(X,t) - \Psi_k(X,t)\| + \left\|I_\alpha^t\lambda(s)\left(iD_t^{\alpha}\left(\Psi_n(X,s)\right) - iD_t^{\alpha}\left(\Psi_k(X,s)\right)\right)\right\| + \\ &\left\|I_\alpha^t\lambda(s)\left(\tfrac{1}{2}\nabla^2\Psi_n\left((X,s)\right) - \tfrac{1}{2}\nabla^2\Psi_k\left((X,s)\right)\right)\right\| + \|V_d(X)\Psi_n\left((X,s)\right) - V_d(X)\Psi_k\left((X,s)\right)\| + \\ &\left\|B_d\Psi_n^{m+1}\left((X,s)\right) - B_d\Psi_k^{m+1}\left((X,s)\right)\right\|\end{aligned} \tag{11}$$

We shall evaluate the above equation case by case starting with the fractional part. We start here with

$$\begin{aligned}&\left\|I_\alpha^t\lambda(s)\left(iD_t^{\alpha}\left(\Psi_n(X,s)\right) - iD_t^{\alpha}\left(\Psi_k(X,s)\right)\right)\right\| \leq |\lambda(s)|\left\|I_\alpha^tD_t^{\alpha}\left(\Psi_n(X,s) - \Psi_k(X,s)\right)\right\| = \\ &|\lambda(s)|\,\|(\Psi_n(X,s) - \Psi_k(X,s)) - (\Psi_n(X,0) - \Psi_k(X,0))\| = \\ &|\lambda(s)|\,\|(\Psi_n(X,s) - \Psi_k(X,s))\|, (\Psi_n(X,0) - \Psi_k(X,0)) = 0, (n > 0)\end{aligned} \tag{12}$$

Secondly

$$\begin{aligned}&\left\|I_\alpha^t\lambda\left(s\right)\left(\tfrac{1}{2}\nabla^2\Psi_n\left(\left(X,s\right)\right)-\tfrac{1}{2}\nabla^2\Psi_k\left(\left(X,s\right)\right)\right)\right\| = \left\|\tfrac{1}{2}I_\alpha^t\lambda\left(s\right)\nabla^2\left(\Psi_n\left(\left(X,s\right)\right)-\Psi_k\left(\left(X,s\right)\right)\right)\right\| \leq \\ &\tfrac{1}{2}\left|\lambda\left(s\right)\right| I_\alpha^t\left\|\left(\nabla^2\left(\Psi_n\left(\left(X,s\right)\right)-\Psi_k\left(\left(X,s\right)\right)\right)\right)\right\|\end{aligned} \tag{13}$$

Making use of the continuity properties of the derivative, together with property of the norm, it is possible for us to find two positive constants θ_1, θ_2 such that,

$$\left\|\left(\nabla^2\left(\Psi_n\left(\left(X,s\right)\right)-\Psi_k\left(\left(X,s\right)\right)\right)\right)\right\| \leq \theta_1\theta_2\left\|\Psi_n\left(\left(X,s\right)\right)-\Psi_k\left(\left(X,s\right)\right)\right\| \tag{14}$$

Thus equation (13) becomes

$$\left\|I_\alpha^t\lambda\left(s\right)\left(\frac{1}{2}\nabla^2\Psi_n\left(\left(X,s\right)\right)-\frac{1}{2}\nabla^2\Psi_k\left(\left(X,s\right)\right)\right)\right\| \leq \frac{1}{2}\left|\lambda\left(s\right)\right| I_\alpha^t\theta_1\theta_2\left\|\Psi_n\left(\left(X,s\right)\right)-\Psi_k\left(\left(X,s\right)\right)\right\|\left\|I_\alpha^t(1)\right\| \tag{15}$$

However, the Caputo-Fabrizio fractional integral of a constant can be calculated as follows

$$I_t^\alpha\left(a\right) = \frac{2\left(1-\alpha\right)}{\left(2-\alpha\right)M(\alpha)}a + \frac{2\alpha t}{\left(2-\alpha\right)M(\alpha)}a \tag{16}$$

With the above information on hand, equation (14) becomes

$$\begin{aligned}&\left\|I_\alpha^t\lambda\left(s\right)\left(\tfrac{1}{2}\nabla^2\Psi_n\left(\left(X,s\right)\right)-\tfrac{1}{2}\nabla^2\Psi_k\left(\left(X,s\right)\right)\right)\right\| \leq \\ &\left\|\Psi_n\left(\left(X,s\right)\right)-\Psi_k\left(\left(X,s\right)\right)\right\|\tfrac{1}{2}\left|\lambda\left(s\right)\right|\theta_1\theta_2\left\|\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}+\tfrac{2\alpha t}{(2-\alpha)M(\alpha)}\right\|\end{aligned} \tag{17}$$

We finally consider the following

$$\begin{aligned}&\left\|B_d\Psi_n^{m+1}\left(\left(X,s\right)\right)-B_d\Psi_k^{m+1}\left(\left(X,s\right)\right)\right\| = \left|B_d\right|\left\|\Psi_n^{m+1}\left(\left(X,s\right)\right)-\Psi_k^{m+1}\left(\left(X,s\right)\right)\right\| \\ &\quad = \left|B_d\right|\left\|\Psi_n\left(\left(X,s\right)\right)-B_d\Psi_k\left(\left(X,s\right)\right)\right\|\left\|\textstyle\sum_{j=0}^m C_m^j\Psi_n^{m-j-1}\Psi_k^j\right\|\end{aligned} \tag{18}$$

Due to the physical properties of the problem under study, for all n and k, we have that the function $\left\|\Psi_n^{m-j-1}\right\|\left\|\Psi_k^j\right\| \leq v^{m-j-1}u^j$ so that equation (18) can become

$$\left\|B_d\Psi_n^{m+1}\left(\left(X,s\right)\right)-B_d\Psi_k^{m+1}\left(\left(X,s\right)\right)\right\| = \left|B_d\right|\left\|\Psi_n\left(\left(X,s\right)\right)-B_d\Psi_k\left(\left(X,s\right)\right)\right\|\sum_{j=0}^m C_m^j v^{m-j-1}u^j \tag{19}$$

Therefore, putting together equation (19), (17) , (12) and (11) into (10), we obtain

$$\begin{aligned}&\left\|H(\Psi_n(X,t))-H(\Psi_k(X,t))\right\| \leq \left|\lambda(s)\right|\left\|\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}+\tfrac{2\alpha t}{(2-\alpha)M(\alpha)}\right\| \\ &\left\{1+\left|B_d\right|\textstyle\sum_{j=0}^m C_m^j v^{m-j-1}u^j+\tfrac{1}{2}\left|\lambda\left(s\right)\right|\theta_1\theta_2\right\} \\ &\left\|\Psi_n(X,t)-\Psi_k(X,t)\right\|\end{aligned} \tag{20}$$

Thus if we assume

$$\left|\lambda(s)\right| < \left\{\begin{array}{c}\left\|\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}+\tfrac{2\alpha t}{(2-\alpha)M(\alpha)}\right\| \\ \left\{1+\left|B_d\right|\sum_{j=0}^m C_m^j v^{m-j-1}u^j+\tfrac{1}{2}\left|\lambda\left(s\right)\right|\theta_1\theta_2+\max\left|(V(X))\right|\right\}\end{array}\right\}^{-1} \tag{21}$$

The nonlinear H has a fixed point. This completes the proof.
We next show that, H satisfies the conditions in theorem 2. Let (9) holds, thus putting

$$C = 0, c = \left|\lambda(s)\right|\left\{\begin{array}{c}\left\|\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}+\tfrac{2\alpha t}{(2-\alpha)M(\alpha)}\right\| \\ \left\{1+\left|B_d\right|\sum_{j=0}^m C_m^j v^{m-j-1}u^j+\tfrac{1}{2}\left|\lambda\left(s\right)\right|\theta_1\theta_2+\max\left|V_d\right|\right\}\end{array}\right\} \tag{22}$$

shows that conditions of theorem (2) holds for the nonlinear mapping H. Therefore since all condition in theorem (2) hold for the defined non-linear mapping H, then H is Picard's H-stable. This completes the proof of theorem (3). One can also find in the literature others ways of dealing with fixed-point theorem [9-11].

4. Unicity of the Approximate Solution

In this section, we show in detail the uniqueness of the special solution, to achieve this, we consider the following operator

$$T(\Psi(X,t)) = iD_t^{\alpha}(\Psi(X,t)) = -\frac{1}{2}\nabla^2\Psi((X,t)) + V_d(X)\Psi((X,t)) + B_d\Psi^{m+1}((X,t)) \tag{23}$$

Theorem 4: *Using the new Caputo derivative with fractional order, the time-fractional nonlinear Schrdinger equation has a unique special solution while using variational iteration method.*

Proof: Let $K = \left\{U, V/\int U^{m+1}V^{m+1} < \infty\right\}$, We assume that, Ψ is the exact solution of the time-fractional Schrdinger equation, we assume by contradiction that, we can find two different special solution U and V such that, $U \neq V$. We evaluate using the inner product the following expression. $(H(U) - H(V), U - V)$.

$$H(U) - H(V) = -\frac{1}{2}\nabla^2(U(X,t) - V(X,t)) + V_d(X)(U(X,t) - V(X,t)) + B_d\left(U^{m+1}(X,t) - V^{m+1}(X,t)\right) \tag{24}$$

Therefore,

$$\begin{aligned}(H(U) - H(V), U - V) = {} & \left(\tfrac{1}{2}\nabla^2(V(X,t) - U(X,t)), (U - V)\right) + (V_d(X)(U(X,t) - V(X,t)), U - V) \\ & + \left(B_d\left(U^{m+1}(X,t) - V^{m+1}(X,t)\right), U - V\right)\end{aligned} \tag{25}$$

Indeed using some properties of inner function which are related to the norm, we have that

$$\begin{aligned}&\left(\tfrac{1}{2}\nabla^2(V(X,t) - U(X,t)), (U - V)\right) \leq \tfrac{1}{2}\left\|\nabla^2(V - U)\right\| \|U - V\| \\ &\leq \tfrac{1}{2}\Omega_1\Omega_2\|U - V\|^2\end{aligned} \tag{26}$$

Also we have the following relationship

$$(V_d(X)(U(X,t) - V(X,t)), U - V) \leq \max|V_d(X)|\,\|U - V\|^2 \tag{27}$$

And finally we have the following result

$$\left(B_d\left(U^{m+1}(X,t) - V^{m+1}(X,t)\right), U - V\right) \leq |B_d|\,\|U - V\|^2 \sum_{j=0}^{m} C_m^j V_1^{m-j-1} U_1^j \tag{28}$$

Now putting equation (28), (27), (26) into (25) we obtain

$$(H(U) - H(V), U - V) \leq \left(|B_d| \sum_{j=0}^{m} C_m^j V_1^{m-j-1} U_1^j + \frac{1}{2}\Omega_1\Omega_2 + \max|V_d(X)| + 1\right)\|U - V\|^2 \tag{29}$$

Due to the fact that U, V are bounded in K, the above equation can be transform to

$$(H(U) - H(V), U - V) \leq V_1U_1\left(|B_d| \sum_{j=0}^{m} C_m^j V_1^{m-j-1} U_1^j + \frac{1}{2}\Omega_1\Omega_2 + \max|V_d(X)| + 1\right)\|U - V\|$$

However, since Ψ is the exact solution of the time-fractional Schrdinger equation, the above relation can further be transform to

$$(H(U) - H(V), U - V) \leq V_1U_1\left(|B_d| \sum_{j=0}^{m} C_m^j V_1^{m-j-1} U_1^j + \frac{1}{2}\Omega_1\Omega_2 + \max|V_d(X)| + 1\right)\{\|\Psi - V\| + \|U - \Psi\|\} \tag{30}$$

And we can find n and m bigger enough such that U and V converge to Ψ, with this in mind, we can therefore consider $\max(n, m)$ and then

$$\|U - \Psi\| < \frac{\iota}{2V_1U_1\left(|B_d|\sum_{j=0}^{m} C_m^j V_1^{m-j-1} U_1^j + \frac{1}{2}\Omega_1\Omega_2 + \max|V_d(X)| + 1\right)} \tag{31}$$

$$\|V - \Psi\| < \frac{\iota}{2V_1U_1\left(|B_d|\sum_{j=0}^{m} C_m^j V_1^{m-j-1}U_1^j + \frac{1}{2}\Omega_1\Omega_2 + \max|V_d(X)| + 1\right)}$$

Replacing the information of equation (31) into equation (30) yields

$$(H(U) - H(V), U - V) < \iota$$

since ι is an extremely very small parameter, according to topology law, we have that

$$(H(U) - H(V), U - V) = 0 \Rightarrow U = V \tag{32}$$

This completes the proof of theorem 4.

5. Conclusion

Recently, Caputo in collaboration with Fabrizio have proposed a new derivative with fractional order, which actually the modified version proposed by Liouville in 1832. The new derivative has more interesting properties than the old version. For instance, it can describe material heterogeneities and structures with different scales, which obviously cannot be handling with the well-known local theories. Another application is in the study of the macroscopic behaviours of some materials, connected with non-local interactions between atoms, which are established in decisive of the properties of material. To further apply this derivative, we have modified the nonlinear Schrdinger equation. An iterative procedure was used and together with the fixed point concept to show the stability of the constructed mapping. A detail analysis underpinning the uniqueness of the solution of the modified equation presented.

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