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Existence of Fixed Point for $GP_{(\Lambda,\Theta)}$-Contractive Mappings in GP-Metric Spaces

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Abstract. We combine some classes of functions with a notion of hybrid $GP_{(\Lambda,\Theta)}$-H-F-contractive mapping for establishing some fixed point results in the setting of GP-metric spaces. An illustrative example supports the new theory.

1. Introduction

The well known Banach fixed point theorem for contraction mappings is universally recognized as the fundamental result in the metric fixed point theory; see [6]. This result is a source of continuous inspiration for researchers working in the specific topic of fixed point theory, but also for scientists working in other branches of mathematics and applied sciences. Without enlarging the discussion too much, we point out that the metric conditions of the space and the characterizations of the mappings play an essential role in establishing the existence of solution for every mathematical problem, under investigation. A promising direction of research brings authors to modify the classical metric statements for obtaining a more general setting, useful in practical problems. Following this direction, the Banach fixed point theorem [6] has been generalized and revised in various settings; see for instance [20, 21, 24, 25]. Here, we are interested in combining the peculiarities of two of these abstract settings. Precisely, we refer to the partial metric space, which is a generalized metric space introduced by Matthews [12] for application in theoretical computer science. We point out that in a partial metric space, each element of the space does not necessarily have a zero distance from itself. Subsequently, several authors studied the problem of existence and uniqueness of fixed point in this setting; they considered mappings satisfying different contractive conditions and solved various problems involving differential and functional equations; see [3, 9, 21, 23].

On the other hand, in 2006 Mustafa and Sims [13] introduced a new notion of generalized metric spaces called G-metric spaces. Based on this notion, many fixed point results for different contractive conditions have been presented and applied; for more details see [1, 5, 14–16, 19, 22]. An attempt to combine the advantages of above two settings was successfully realized by Zand and Nezhad [26]. Precisely,

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these authors introduced a new generalized metric with the name of GP-metric space and some useful properties. Following this idea, Aydi et al. [4] established some fixed point results in GP-metric spaces. Other results in the setting of GP-metric spaces are available in [18], where the authors introduce the notion of $GP_{(\Lambda,\Theta)}$-contractive mappings and give some fixed point results for $GP_{(\Lambda,\Theta)}$-contractive mappings.

In this paper we continue this line of research, combining some classes of functions in the setting of GP-metric spaces. Consequently, the presented theorems are suitable for covering a wide class of abstract problems, but without requiring rearrangements of the proofs. An illustrative example supports the new theory.

2. Preliminaries

In this section, we recall the background and some results in the setting of GP-metric spaces. Throughout this paper, let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}^+$ and $\mathbb{N}$ denote the sets of reals, nonnegative reals, nonnegative integers and positive integers, respectively.

Just to fix notation, we say that a (totally) ordered (abelian) group G is an additive group on which is defined an order relation $<$ such that if $a < b$ then $a + c < b + c$, for all $a, b, c \in G$. We write $\leq$ for $<$ or $=$, and denote by G^+ the set of nonnegative elements of G.

Definition 2.1 ([10]). *Let G be an ordered group. An ordered group metric (for short, OG-metric) on a nonempty set X is a symmetric function $d_G : X \times X \to G^+$ such that $d_G(x, y) = 0$ if and only if $x = y$ and the triangle inequality is satisfied. The pair (X, d_G) is called ordered group metric space (for short, OG-metric space).*

Definition 2.2 ([26]). *Let X be a non empty set and G be an ordered group. A function $G_p : X \times X \times X \to G^+$ is called an ordered group partial metric (for short, OGP-metric) if the following conditions are satisfied:*

(GP1) $x = y = z$ *if* $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;

(GP2) $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$;

(GP3) $G_p(x, y, z) = G_p(J\{x, y, z\})$, *where* $J\{x, y, z\}$ *is any permutation of* x, y, z *(symmetry in all three variables);*

(GP4) $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ *for all* $x, y, z, a \in X$.

The triple (X, G, G_p) is called an OGP-metric space.

As special cases, one can consider $G^+ := \mathbb{Z}^+$ or $\mathbb{R}^+$. In the case $G^+ = \mathbb{R}^+$, the triple $(X, \mathbb{R}, G_p)$ is usually denoted by (X, G_p) and called GP-metric space. In the sequel, for avoiding confusion and being more familiar with notation, we assume that $G^+ = \mathbb{R}^+$.

Example 2.3 ([26]). *Let $X = \mathbb{R}^+ = G^+$ and define $G_p(x, y, z) = \max\{x, y, z\}$, for all $x, y, z \in X$. Then (X, G_p) is a GP-metric space.*

We recall the following facts, for further use.

Proposition 2.4 ([26], Proposition 1). *Let (X, G_p) be a GP-metric space. Then, for all $x, y, z, a \in X$, the following statements hold true:*

(i) $G_p(x, y, z) \leq G_p(x, x, y) + G_p(x, x, z) - G_p(x, x, x)$;

(ii) $G_p(x, y, y) \leq 2G_p(x, x, y) - G_p(x, x, x)$;

(iii) $G_p(x, y, z) \leq G_p(x, a, a) + G_p(y, a, a) + G_p(z, a, a) - 2G_p(a, a, a)$;

(iv) $G_p(x, y, z) \leq G_p(x, a, z) + G_p(a, y, z) - G_p(a, a, a)$.

Proposition 2.5 ([26], Proposition 2). *Every GP-metric space (X, G_p) defines a metric space (X, D_{Gp}), where*

$$D_{G_p}(x, y) = G_p(x, y, y) + G_p(y, x, x) - G_p(x, x, x) - G_p(y, y, y) \text{ for all } x, y \in X.$$

Example 2.6 ([18]). *Let X, G^+ and G_p as in the Example 2.3 above. Then the metric D_{G_p}, induced by the GP-metric G_p, is defined by*

$$D_{G_p}(x, y) = \left|x - y\right| \text{ for all } x, y \in X.$$

Lemma 2.7 ([4], Lemma 1.10). *Let (X, G_p) be a GP-metric space. Then*

(i) if $G_p(x, y, z) = 0$, then $x = y = z$;

(ii) if $x \neq y$, then $G_p(x, y, y) > 0$.

Definition 2.8. *[26]. Let (X, G_p) be a GP-metric space and let $\{x_n\}$ a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$ as $n \to +\infty$, if*

$$\lim_{m,n\to+\infty} G_p(x, x_m, x_n) = G_p(x, x, x).$$

Proposition 2.9 ([26], Proposition 4). *Let (X, G_p) be a GP-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$ the following are equivalent:*

(i) $\{x_n\}$ is GP-convergent to x;
(ii) $G_p(x_n, x_n, x) \to G_p(x, x, x)$ as $n \to +\infty$;
(iii) $G_p(x_n, x, x) \to G_p(x, x, x)$ as $n \to +\infty$.

By using the definition of D_{G_p}, one can deduce an interesting proposition, as follows.

Proposition 2.10 ([18], Proposition 1.9). *Let (X, G_p) be a GP-metric space. For any sequence $\{x_n\}$ in X convergent to a point $x \in X$ such that $\lim_{n\to+\infty} G_p(x_n, x_n, x_n) = G_p(x, x, x)$, then $\lim_{n\to+\infty} D_{G_p}(x_n, x) = 0$.*

Remark 2.11. *Let (X, G_p) be a GP-metric space and let $\{x_n\} \subset X$ be a sequence convergent to a point $x \in X$ such that $G_p(x, x, x) = 0$, then $\lim_{n\to+\infty} G_p(x_n, x_n, x_n) = G_p(x, x, x)$. In fact, by property $(GP2)$ and Proposition 1.9, we have*

$$G_p(x_n, x_n, x_n) \leq G_p(x_n, x_n, x) \to 0 \quad \text{as } n \to +\infty.$$

Definition 2.12. *[26]. Let (X, G_p) be a GP-metric space.*

(i) A sequence $\{x_n\}$ is called a GP-Cauchy sequence if and only if $\lim_{m,n\to+\infty} G_p(x_n, x_m, x_m)$ exists (and is finite);

(ii) A GP-metric space (X, G_p) is said to be GP-complete if and only if every GP-Cauchy sequence in X is GP-convergent to some $x \in X$ such that $G_p(x, x, x) = \lim_{m,n\to+\infty} G_p(x_n, x_m, x_m)$.

The following lemma is a consequence of property $(GP4)$.

Lemma 2.13. *Let (X, G_p) be a GP-metric space, $x, y \in X$ and $\{x_n\}$ be a sequence in X. Assume that*

$$\lim_{n\to+\infty} G_p(x, x_n, x_n) = \lim_{n\to+\infty} G_p(x_n, y, y),$$

then $x = y$.

Lemma 2.14 ([18], lemma 1.13). *Let (X, G_p) be a GP-metric space and $\{x_n\} \subset X$ be a sequence such that*

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \lambda G_p(x_{n-1}, x_n, x_n) \text{ for all } n \in \mathbb{N},$$

for some $\lambda \in [0, 1)$. Then $\{x_n\}$ is a GP-Cauchy sequence in X such that

$$\lim_{m,n\to+\infty} G_p(x_n, x_m, x_m) = 0.$$

Definition 2.15 ([18]). *Let (X, G_p) be a GP-metric space. A mapping $f : X \to X$ is 0-GP-continuous if*

$$\lim_{n\to+\infty} G_p(x_n, x_n, x) = 0 \quad \textit{implies} \quad \lim_{n\to+\infty} G_p(fx_n, fx_n, fx) = 0.$$

Finally, we give some concepts and examples related to the classes of functions that we will use in proving our results (see [2, 11]).

Definition 2.16. *A function $H : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}$ is a subclass function if it is continuous and $H(1, y) \le H(x, y)$ for all $x \in [1, +\infty)$ and $y \in \mathbb{R}^+$.*

Example 2.17. *Let $H : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}$ be defined, for all $x \in [1, +\infty)$ and $y \in \mathbb{R}^+$, by one of the following rules:*

(1) $H(x, y) = (y + l)^x, l > 1$;

(2) $H(x, y) = (x + l)^y, l > 1$;

(3) $H(x, y) = xy^n$, $n \in \mathbb{N}$;

(4) $H(x, y) = y$;

(5) $H(x, y) = (\frac{x+1}{2})y$;

(6) $H(x, y) = \frac{2x+1}{3}y$;

(7) $H(x, y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1})y$;

(8) $H(x, y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1} + l)^y, l > 1$.

Then H is a subclass function.

Definition 2.18. *Let $F : [0, 1) \times \mathbb{R}^+ \to \mathbb{R}$ and $H : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}$ be two functions. We say that (F, H) is a pair of upclass functions if H is a subclass function and $H(1, r) \le F(s, t)$ implies $r \le st$ for all $r, t \in \mathbb{R}^+$ and $s \in [0, 1)$. We denote by $\mathcal{H}$ the family of all (F, H) pairs.*

Building on Example 2.17, we have the following example.

Example 2.19. *Let $F : [0, 1) \times \mathbb{R}^+ \to \mathbb{R}$ and $H : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}$ be two functions and let the pair (F, H) defined, for all $x \in [1, +\infty)$, $y, t \in \mathbb{R}^+$ and $s \in [0, 1)$, by one of the following rules:*

(1) $H(x, y) = (y + l)^x, l > 1$ and $F(s, t) = st + l$;

(2) $H(x, y) = (x + l)^y, l > 1$ and $F(s, t) = (1 + l)^{st}$;

(3) $H(x, y) = xy^n$ and $F(s, t) = s^n t^n$, $n \in \mathbb{N}$;

(4) $H(x, y) = y$ and $F(s, t) = st$;

(5) $H(x, y) = (\frac{x+1}{2})y$ and $F(s, t) = st$;

(6) $H(x, y) = \frac{2x+1}{3}y$ and $F(s, t) = st$;

(7) $H(x, y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1})y$ and $F(s, t) = st$;

(8) $H(x, y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1} + l)^y, l > 1$ and $F(s, t) = (1 + l)^{st}$.

Then the pair (F, H) satisfies Definition 2.18.

3. Main Results

In this section, we introduce the notion of hybrid $GP_{(\Lambda,\Theta)}$-H-F-contractive mapping and establish some results of existence of fixed point for this class of mappings.

Definition 3.1. *Let $f : X \to X$ and $\Theta, \Lambda : X \times X \times X \to \mathbb{R}^+$ be mappings. We say that f is (Λ, Θ)-admissible with respect to the real numbers $\lambda > \theta \geq 0$ if for $x, y, z \in X$ we have*

$$\Lambda(x, y, z) \geq \lambda \quad \Longrightarrow \quad \Lambda(fx, fy, fz) \geq \lambda$$

and

$$\Theta(x, y, z) \leq \theta \quad \Longrightarrow \quad \Theta(fx, fy, fz) \leq \theta.$$

Definition 3.2. *Let (X, G_p) be a GP-metric space and $f : X \to X$ be a mapping. We say that f is a hybrid $GP_{(\Lambda,\Theta)}$-H-F-contractive mapping with respect to the real numbers $\lambda > \theta \geq 0$ if there exists a pair $(F, H) \in \mathcal{H}$ such that the condition*

$$H\Big(\frac{\Lambda(x, y, z)}{\lambda}, G_p(fx, fy, fz)\Big) \leq F\Big(\frac{\Theta(x, y, z)}{\lambda}, G_p(x, y, z) + LM(x, y, z)\Big) \tag{1}$$

holds for all $x, y, z \in X$ such that $\Lambda(x, y, z) \geq \lambda$ and $\Theta(x, y, z) \leq \theta$, where $L \geq 0$ and

$$M(x, y, z) = \min\{\max\{D_{G_p}(fx, y), D_{G_p}(fx, z)\}, \max\{D_{G_p}(fy, y), D_{G_p}(fz, z)\}\}.$$

The first fixed point theorem is established for 0-GP-continuous mappings.

Theorem 3.3. *Let (X, G_p) be a GP-complete GP-metric space and let $f : X \to X$ be a mapping. Assume that there exist two real numbers $\lambda > \theta \geq 0$ such that the following conditions hold:*

(i) f is a hybrid $GP_{(\Lambda,\Theta)}$-H-F-contractive mapping with respect to λ and θ;

(ii) f is a (Λ, Θ)-admissible mapping with respect to λ and θ;

(iii) there exists $x_0 \in X$ such that $\Lambda(x_0, fx_0, fx_0) \geq \lambda$ and $\Theta(x_0, fx_0, fx_0) \leq \theta$;

(iv) f is a 0-GP-continuous mapping.

Then f has a fixed point in X.

Proof. Let $x_0 \in X$ be such that $\Lambda(x_0, fx_0, fx_0) \geq \lambda$ and $\Theta(x_0, fx_0, fx_0) \leq \theta$. Let $\{x_n\}$ be a Picard sequence starting at x_0, that is, $x_n = fx_{n-1} = f^n x_0$ for all $n \in \mathbb{N}$. Since f is a (Λ, Θ)-admissible mapping and $\Lambda(x_0, x_1, x_1) = \Lambda(x_0, fx_0, fx_0) \geq \lambda$, we deduce that $\Lambda(x_1, x_2, x_2) = \Lambda(fx_0, fx_1, fx_1) \geq \lambda$. By continuing this process, we get $\Lambda(x_n, x_{n+1}, x_{n+1}) \geq \lambda$ for all $n \in \mathbb{N} \cup \{0\}$. Similarly, $\Theta(x_n, x_{n+1}, x_{n+1}) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$. Now, if $x_{m-1} = x_m$ for some $m \in \mathbb{N}$, then x_m is a fixed point of f and we have nothing to prove. Thus, we can assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. This, by Lemma 2.7, ensures that $G_p(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, by using (1) with $x = x_{n-1}$ and $y = z = x_n$, we get

$$\begin{aligned} H(1, G_p(x_n, x_{n+1}, x_{n+1})) &\leq H\Big(\frac{\Lambda(x_{n-1}, x_n, x_n)}{\lambda}, G_p(x_n, x_{n+1}, x_{n+1})\Big) \\ &\leq F\Big(\frac{\Theta(x_{n-1}, x_n, x_n)}{\lambda}, G_p(x_{n-1}, x_n, x_n) + L\,M(x_{n-1}, x_n, x_n)\Big). \end{aligned}$$

Therefore

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \frac{\Theta(x_{n-1}, x_n, x_n)}{\lambda}[G_p(x_{n-1}, x_n, x_n) + L\,M(x_{n-1}, x_n, x_n)].$$

Since $\Theta(x_{n-1}, x_n, x_n) \leq \theta < \lambda$ for all $n \in \mathbb{N}$, we obtain

$$G_p(x_n, x_{n+1}, x_{n+1}) \quad \leq \quad \frac{\theta}{\lambda}[G_p(x_{n-1}, x_n, x_n) + L\, M(x_{n-1}, x_n, x_n)],$$

but $M(x_{n-1}, x_n, x_n) = 0$ for all $n \in \mathbb{N}$. Thus

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \frac{\theta}{\lambda} G_p(x_{n-1}, x_n, x_n) \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Since, $0 \leq \dfrac{\theta}{\lambda} < 1$, by Lemma 2.14, we deduce that $\{x_n\}$ is a GP-Cauchy sequence such that $\lim_{m,n\to+\infty} G_p(x_n, x_m, x_m) = 0$. The hypothesis that X is GP-complete ensures that there exists $z \in X$ such that the sequence $\{x_n\}$ GP-converges to z and

$$G_p(z, z, z) = \lim_{m,n\to+\infty} G_p(x_n, x_m, x_m) = 0.$$

Now, using the 0-GP-continuity of the mapping f and Proposition 2.4 (ii), we get

$$\begin{aligned} \lim_{n\to+\infty} G_p(fz, fz, x_{n+1}) \quad &\leq \quad \lim_{n\to+\infty} 2G_p(fz, x_{n+1}, x_{n+1}) - \lim_{n\to+\infty} G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\leq \quad \lim_{n\to+\infty} 2G_p(fz, fx_n, fx_n) = 0. \end{aligned}$$

Consequently,

$$\lim_{n\to+\infty} G_p(x_n, fz, fz) = 0.$$

As

$$\lim_{n\to+\infty} G_p(x_n, x_n, z) = 0,$$

by Lemma 2.13, we deduce that $z = fz$. □

The second fixed point theorem is established for hybrid $GP_{(\Lambda,\Theta)}$-H-F-contractive mappings that are not 0-GP-continuous.

Theorem 3.4. *Let (X, G_p) be a GP-complete GP-metric space and let $f : X \to X$ be a mapping. Assume that there exist two real numbers $\lambda > \theta \geq 0$ such that the following conditions hold:*

(i) f is a hybrid $GP_{(\Lambda,\Theta)}$-H-F-contractive mapping with respect to λ and θ;

(ii) f is a (Λ, Θ)-admissible mapping with respect to λ and θ;

(iii) there exists $x_0 \in X$ such that $\Lambda(x_0, fx_0, fx_0) \geq \lambda$ and $\Theta(x_0, fx_0, fx_0) \leq \theta$;

(iv) if $\{x_n\} \subset X$ is a sequence convergent to $z \in X$ such that $\Lambda(x_n, x_{n+1}, x_{n+1}) \geq \lambda$ and $\Theta(x_n, x_{n+1}, x_{n+1}) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$, then $\Lambda(x_n, z, z) \geq \lambda$ and $\Theta(x_n, z, z) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point.

Proof. Let $x_0 \in X$ be such that $\Lambda(x_0, fx_0, fx_0) \geq \lambda$ and $\Theta(x_0, fx_0, fx_0) \leq \theta$ and let $\{x_n\}$ be a Picard sequence starting at x_0. Following the proof of above Theorem 3.3, we can say that $\{x_n\}$ is a GP-Cauchy sequence such that $\Lambda(x_n, x_{n+1}, x_{n+1}) \geq \lambda$ and $\Theta(x_n, x_{n+1}, x_{n+1}) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$. Since X is GP-complete, then there is $z \in X$ such that the sequence $\{x_n\}$ GP-converges to z; again from the proof of Theorem 3.3, we have $G_p(z, z, z) = 0$. Then by (iv), we get $\Lambda(x_n, z, z) \geq \lambda$ and $\Theta(x_n, z, z) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, by using the contractive condition (1), we write

$$\begin{aligned} H(1, G_p(x_{n+1}, fz, fz)) &\leq H(\frac{\Lambda(x_n, z, z)}{\lambda}, G_p(x_n, x_{n+1}, x_{n+1})) \\ &\leq F(\frac{\Theta(x_n, z, z)}{\lambda}, G_p(x_n, z, z) + LM(x_{n-1}, z, z)). \end{aligned}$$

Since $\frac{\Theta(x_n,z,z)}{\lambda} < 1$, we have

$$G_p(x_{n+1}, fz, fz) \leq G_p(x_n, z, z) + LM(x_{n-1}, z, z).$$

Now, by (GP4), we obtain that

$$\begin{aligned} G_p(z, fz, fz) &\leq G_p(z, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, fz, fz) \\ &\leq G_p(z, x_{n+1}, x_{n+1}) + G_p(x_n, z, z) + LM(x_{n-1}, z, z) \end{aligned}$$

holds for all $n \in \mathbb{N}$.

Since the sequence $x_n \to z$ and $G_p(z, z, z) = 0$, by Proposition 2.10 and Remark 2.11, we get

$$\lim_{n\to+\infty} D_{G_p}(x_{n+1}, z) = 0.$$

Consequently, we have

$$\lim_{n\to+\infty} M(x_n, z, z) = 0.$$

It follows easily that $G_p(z, fz, fz) \leq 0$, that is, $z = fz$. Hence, f has a fixed point. □

We conclude this section with a simple illustrative example.

Example 3.5. *Let $X = \mathbb{R}^+$ and let $G_p : X \times X \times X \to \mathbb{R}^+$ be a GP-metric defined by $G_p(x, y, z) = \max\{x, y, z\}$ for all $x, y, z \in X$. Also, let $f : X \to X$ be given by*

$$f(x) = \begin{cases} \frac{1}{4}x^3 & \text{if } x \in [0, 3] \\ \frac{1}{2}\ln(x+1) & \text{if } x \in \mathbb{R}^+ \setminus [0, 3]. \end{cases}$$

Now, consider the mappings $\Theta, \Lambda : X \times X \times X \to \mathbb{R}^+$ defined by

$$\Theta(x, y, z) = 2 \text{ for all } x, y, z \in X \quad \text{and} \quad \Lambda(x, y, z) = \begin{cases} 4 & \text{if } x, y, z \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is (Λ, Θ)-admissible with respect to $\lambda = 4$ and $\theta = 2$. Now, for all $x, y, z \in X$ such that $\Lambda(x, y, z) \geq 4$ and $\Theta(x, y, z) \leq 2$, that is $x, y, z \in [0, 1]$. we have

$$\begin{aligned} G_p(fx, fy, fz) &= \frac{1}{4}\max\{x^3, y^3, z^3\} \\ &\leq \frac{1}{4}G_p(x, y, z) \\ &\leq \frac{1}{2}\frac{\Theta(x, y, z)}{\lambda}[G_p(x, y, z) + LM(x, y, z)]. \end{aligned}$$

It is immediate to conclude that f is a hybrid $GP_{(\Lambda,\Theta)}$-H-F-contractive mapping with respect to $\lambda = 4$ and $\theta = 2$, by assuming that $F : [0, 1) \times \mathbb{R}^+ \to \mathbb{R}$ and $H : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}$ are defined as in (4) of Example 2.19. Finally, we note that all the hypotheses of Theorem 3.3 hold true and hence f has a fixed point.

4. Conclusions

The abstract developments of fixed point theory in generalized metric spaces are interesting as useful exercises for investigating the possibility to enlarge applicability of the constructive techniques at the basis of the proof of Banach fixed point theorem. The proposed theorems realize this idea by combining some classes of functions in the setting of *GP*-metric spaces. Consequently, we design statements and proofs of theorems with the goal of covering a wide class of abstract problems, but without requiring specific rearrangements of the proofs.

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