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On $[m, C]$-Isometric Operators

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Abstract. In this paper we introduce an $[m, C]$-isometric operator T on a complex Hilbert space $\mathcal{H}$ and study its spectral properties. We show that if T is an $[m, C]$-isometric operator and N is an n-nilpotent operator, respectively, then $T + N$ is an $[m + 2n - 2, C]$-isometric operator. Finally we give a short proof of Duggal's result for tensor product of m-isometries and give a similar result for $[m, C]$-isometric operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. For an integer $m \in \mathbb{N}$ and an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-isometric* operator if

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^{m-j} = 0.$$

In 1995, J. Agler and M. Stankus [1] introduced an *m-isometric* operator and showed nice results. An antilinear operator C on $\mathcal{H}$ is said to be *conjugation* if C satisfies $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if $CTC = T^*$. In [11], S. Jung, E. Ko, M. Lee and J. Lee studied spectral properties of complex symmetric operators. In [4], M. Chō, E. Ko and J. Lee introduced (m, C)-isometric operators with conjugation C as follows; For an operator $T \in \mathcal{L}(\mathcal{H})$ and an integer $m \geq 1$, T is said to be an (m, C)-isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} \cdot CT^{m-j}C = 0.$$

According to definitions of m-isometry, (m, C)-isometry and complex symmetric, we define an $[m, C]$-isometry T as follows; An operator T is said to be an $[m, C]$-isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j} = 0.$$

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It is easy to see that if T is complex symmetric and an $[m, C]$-isometry, then T is an m-isometry. Throughout the paper, let I be the identity operator on $\mathcal{H}$.

2. Example

(i) Let $\mathcal{H} = \mathbb{C}^2$ and let C be a conjugation on $\mathcal{H}$ given by $C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{y} \\ \overline{x} \end{pmatrix}$.
If $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$, then $CTC = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = T^*$. Since $T^{*2} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, it follows that

$$\sum_{j=0}^{2}(-1)^j\binom{2}{j}T^{*2-j} \cdot CT^{2-j}C = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + I = 0.$$

Therefore, T is a $(2, C)$-isometric operator. On the other hand, T is not a $[2, C]$-isometric operator due to the fact that

$$\sum_{j=0}^{2}(-1)^j\binom{2}{j}CT^{2-j}C \cdot T^{2-j} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + I = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq 0.$$

(ii) Under the same space $\mathcal{H}$ and the same conjugation C to (i), let S be an operator given by $S = \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$.
Then $CSC = \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$ and $CSC = S \neq S^*$. Moreover, it holds $CSC \cdot S - I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - I = 0$ and hence S is a $[1, C]$-isometry. But $S^* \cdot CSC - I = \begin{pmatrix} 2 & -2\sqrt{2}i \\ 2\sqrt{2}i & 2 \end{pmatrix} \neq 0$ and hence S is not a $(1, C)$-isometry.

(iii) Let F and J be conjugations on a Hilbert space $\mathcal{H}$ such that $JF \neq I$. Define T and C by $T = \begin{pmatrix} 0 & FJ \\ I & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$. Then it is easy to see that C is a conjugation on $\mathcal{H} \oplus \mathcal{H}$, $CTC \cdot T = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and $T^* \cdot CTC = \begin{pmatrix} JF & 0 \\ 0 & JF \end{pmatrix} \neq \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Hence T is a $[1, C]$-isometric operator and not a $(1, C)$-isometric operator.

3. [m, C]-Isometric Operators

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation C, we define the operator $\lambda_m(T; C)$ by

$$\lambda_m(T; C) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C \cdot T^{m-j}.$$

Then T is an $[m, C]$-isometry if and only if $\lambda_m(T; C) = 0$. Moreover, it holds that

$$CTC \cdot \lambda_m(T; C) \cdot T - \lambda_m(T; C) = \lambda_{m+1}(T; C). \tag{1}$$

Hence if T is an $[m, C]$-isometry, then T is an $[n, C]$-isometry for every $n \geq m$.

Let C be a conjugation on $\mathcal{H}$. Then C satisfies $\|Cx\| = \|x\|$ and $C(\alpha x) = \overline{\alpha}Cx$ for all $x \in \mathcal{H}$ and all $\alpha \in \mathbb{C}$. Moreover, since $C^2 = I$, it follows that $(CTC)^n = CT^nC$ and $(CTC)^* = CT^*C$ for every positive integer n (see [10] for more details). For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_p(T)$ and $\sigma_a(T)$ be the point spectrum and the approximate point spectrum of T, respectively. We denote the range of T by $\mathrm{R}(T)$. Then we have

Theorem 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$-isometric operator. Then the following statements hold:*
(i) *T is bounded below.*
(ii) $0 \notin \sigma_a(T)$.
(iii) *T is injective and* $\mathrm{R}(T)$ *is closed.*

Proof. If $0 \in \sigma_a(T)$, then there exists a sequence of unit vectors $\{x_n\}$ of $\mathcal{H}$ such that $\lim_{n\to\infty} Tx_n = 0$. Since T is an $[m, C]$-isometric operator, it follows that

$$\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}CT^{m-j}C\cdot T^{m-j} = (-1)^{m+1}I. \tag{2}$$

Moreover, since $\lim_{n\to\infty}\left(\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}CT^{m-j}C\cdot T^{m-j}\right)x_n = 0$, it follows from (2) that $\lim_{n\to\infty} x_n = 0$, which is a contradiction. Hence $0 \notin \sigma_a(T)$. Since (i), (ii), and (iii) are equivalent, this completes the proof. □

Theorem 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$-isometric operator. If $\alpha \in \sigma_a(T)$, then $\overline{\alpha}^{-1} \in \sigma_a(T)$. In particular, if α is an eigenvalue of T, then $\overline{\alpha}^{-1}$ is an eigenvalue of T.*

Proof. Let $\{x_n\}$ be a sequence of unit vectors such that $\lim_{n\to\infty}(T-\alpha)x_n = 0$. Since T is an $[m, C]$-isometric operator, C is bounded, and $\lim_{n\to\infty}(T^k - \alpha^k)x_n = 0$ for all $k \in \mathbb{N}$, it holds that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C\cdot T^{m-j}x_n\right) \\ &= C\lim_{n\to\infty}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}\,\overline{\alpha}^{m-j}\right)Cx_n = C\lim_{n\to\infty}(\overline{\alpha}\,T-1)^m Cx_n. \end{aligned}$$

Moreover, since $C^2 = I$, it holds $\lim_{n\to\infty}(\overline{\alpha}\,T-1)^m Cx_n = 0$. Since $\|Cx_n\| = 1$ and $\alpha \neq 0$ by Theorem 3.1, it follows that $\lim_{n\to\infty}(T-\overline{\alpha}^{-1})^m Cx_n = 0$ and hence $\overline{\alpha}^{-1} \in \sigma_a(T)$. □

Corollary 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$-isometric operator. Then $\|T\| \geq 1$.*

Proof. If $0 < \|T\| < 1$, then there exists $\alpha \in \sigma(T)$ and a sequence $\{x_n\}$ of unit vectors such that $0 < |\alpha| < 1$ and $\|(T-\alpha)x_n\| \longrightarrow 0$. By Theorem 3.2, it holds $\overline{\alpha}^{-1} \in \sigma(T)$. Since $|\overline{\alpha}^{-1}| > 1$, it is a contradiction. □

Theorem 3.4. *Let C be a conjugation on $\mathcal{H}$ and let $T \in \mathcal{L}(\mathcal{H})$. Then the following assertions hold.*
(i) If T is an invertible, then T is an $[m, C]$-isometric operator if and only if so is T^{-1}.
(ii) If T is an $[m, C]$-isometric operator, then T^n is also an $[m, C]$-isometric operator for any $n \in \mathbb{N}$.

Proof. (i) Suppose that T is invertible and an $[m, C]$-isometry. Since $C^2 = I$, it follows that

$$\begin{aligned} 0 &= (CT^{-m}C)\Big[\sum_{j=0}^{m}(-1)^j\binom{m}{j}(CT^{m-j}C)T^{m-j}\Big]T^{-m} \\ &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}(C(T^{-1})^jC)\,(T^{-1})^j. \end{aligned}$$

Since

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}(C(T^{-1})^{m-j}C)(T^{-1})^{m-j} = 0$$

is equivalent to

$$\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}(C(T^{-1})^jC)(T^{-1})^j = 0,$$

T^{-1} is an $[m, C]$-isometry. Hence the statement (i) holds.

(ii) Since

$$\begin{aligned} a^n - 1)^m &= (a-1)^m\Big(a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a + 1\Big)^m \\ &= (a-1)^m\Big(\xi_0 a^{m(n-1)} + \xi_1 a^{m(n-1)-1} + \xi_2 a^{m(n-1)-2} + \cdots + \xi_{m(n-1)}\Big) \end{aligned}$$

where ξ_i are coefficients ($i = 0, ..., m(n-1)$), it follows that

$$\lambda_m(T^n; C) = \sum_{i=0}^{m(n-1)} \xi_i\, CT^{m(n-1)-i}C \cdot \lambda_m(T; C) \cdot T^{m(n-1)-i}. \tag{3}$$

From (3), if $\lambda_m(T; C) = 0$, then $\lambda_m(T^n; C) = 0$. Hence T^n is an $[m, C]$-isometric operator for any $n \in \mathbb{N}$. So this completes the proof. □

An operator $N \in \mathcal{L}(\mathcal{H})$ is said to be *n-nilpotent* if $N^n = 0$ ($n \in \mathbb{N}$). In [2] T. Bermúdes, A. Martinón, V. Müller and A.J. Noda proved the following.

Proposition 3.5. (Theorem 3.1, [2]) *Let T be an m-isometry on $\mathcal{H}$ and N be an n-nilpotent operator such that $TN = NT$. Then $T + N$ is an $(m + 2n - 2)$-isometry.*

We have following similar result.

Theorem 3.6. *Let T be an $[m, C]$-isometric operator on $\mathcal{H}$ and N be an n-nilpotent operator such that $TN = NT$. Then $T + N$ is an $[m + 2n - 2, C]$-isometry.*

Proof. In the proof, we denote $\lambda_m(T; C)$ by $\lambda_m(T)$ simply. First we show

$$\lambda_m(T + N) = \sum_{i+j+k=m}\begin{pmatrix} m \\ i, j, k \end{pmatrix}C(T+N)^iC \cdot CN^jC \cdot \lambda_k(T) \cdot T^j \cdot N^i, \tag{4}$$

where $\begin{pmatrix} m \\ i, j, k \end{pmatrix} = \dfrac{m!}{i! \cdot j! \cdot k!}$ and $\lambda_0(*) = I$. It is easy to see that (4) holds for $m = 1$. Assume that (4) holds for m. Then by (1) we have

$$\begin{aligned} & \lambda_{m+1}(T+N) = C(T+N)C \cdot \lambda_m(T+N) \cdot (T+N) - \lambda_m(T+N) \\ = \;& (C(T+N)C)[\sum_{i+j+k=m}\begin{pmatrix} m \\ i, j, k \end{pmatrix}C(T+N)^iC\,CN^jC\,\lambda_k(T)\,T^j\,N^i](T+N) \\ & - \sum_{i+j+k=m}\begin{pmatrix} m \\ i, j, k \end{pmatrix}C(T+N)^iC\,CN^jC\,\lambda_k(T)\,T^j\,N^i \\ = \;& \sum_{i+j+k=m}\begin{pmatrix} m \\ i, j, k \end{pmatrix}C(T+N)^iC\,CN^jC\Big(CTC\,\lambda_k(T)\,T - \lambda_k(T)\Big)T^j\,N^i \\ & + \sum_{i+j+k=m}\begin{pmatrix} m \\ i, j, k \end{pmatrix}C(T+N)^iC\,CN^{j+1}C\,\lambda_k(T)\,T^{j+1}\,N^i \\ & + \sum_{i+j+k=m}\begin{pmatrix} m \\ i, j, k \end{pmatrix}C(T+N)^{i+1}C\,CN^jC\,\lambda_k(T)\,T^j\,N^{i+1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i+j+k=m} \binom{m}{i,j,k} C(T+N)^i C\, CN^j C\, \lambda_{k+1}(T)\, T^j\, N^i \\
&\quad + \sum_{i+j+k=m} \binom{m}{i,j,k} C(T+N)^i C\, CN^{j+1} C\, \lambda_k(T)\, T^{j+1}\, N^i \\
&\quad + \sum_{i+j+k=m} \binom{m}{i,j,k} C(T+N)^{i+1} C\, CN^j C\, \lambda_k(T)\, T^j\, N^{i+1} \\
&= \sum_{i+j+k=m+1} \binom{m+1}{i,j,k} C(T+N)^i C\, CN^j C\, \lambda_k(T)\, T^j\, N^i.
\end{aligned}
$$

Hence (4) holds for $m+1$ and holds for any $m \in \mathbb{N}$. By (4) it holds

$$\lambda_{m+2n-2}(T+N) = \sum_{i+j+k=m+2n-2} \binom{m+2n-2}{i,\ j,\ k} C(T+N)^i C\, CN^j C \lambda_k(T) T^j N^i.$$

(i) If $\max\{i, j\} \geq n$, then $CN^jC = 0$ or $N^i = 0$.
(ii) If $\max\{i, j\} \leq n-1$, then $k \geq m$ and hence $\lambda_k(T) = 0$.
By (i) and (ii), we have $\lambda_{m+2n-2}(T+N) = 0$. Therefore, $T+N$ is an $[m+2n-2, C]$-isometric operator. □

Remark 3.7. *Let $T \in \mathcal{L}(\mathcal{H})$. If $\beta_m(T)$ is defined by*

$$\beta_m(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^{m-j},$$

then T is an m-isometric operator if and only if $\beta_m(T) = 0$. Since, for any commuting pair (T, S), it follows that

$$\beta_m(T+S) = \sum_{i+j+k=m} \binom{m}{i,j,k} (T+S)^{*i} \cdot S^{*j} \cdot \beta_k(T) \cdot T^j \cdot S^i.$$

So we have other proof of Proposition 3.5.

From Theorem 3.6, we get the following corollary.

Corollary 3.8. *If T is a $[1, C]$-isometric operator on $\mathcal{H}$ and N is an n-nilpotent operator such that $TN = NT$, then $T+N$ is an $[2n-1, C]$-isometry.*

Example 3.9. Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$ on $\mathbb{C}^3$. If $T = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on $\mathbb{C}^3$, then $T = I + N$ where $N = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus we have $T^2 = \begin{pmatrix} 1 & 0 & 2a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $T^3 = \begin{pmatrix} 1 & 0 & 3a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $CTC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \overline{a} & 0 & 1 \end{pmatrix}$, $CT^2C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\overline{a} & 0 & 1 \end{pmatrix}$, and $CT^3C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3\overline{a} & 0 & 1 \end{pmatrix}$. Then we have

$$\lambda_3(T; C) = CT^3CT^3 - 3CT^2CT^2 + 3CTCT - I = 0.$$

On the other hand, since $N^2 = 0$, it follows from Theorem 3.6 that T is a $[3, C]$-isometric operator.

For an operator $T \in \mathcal{L}(\mathcal{H})$, the numerical range $W(T)$ of T is $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *convexoid* if $\overline{W(T)} = \text{co}\,\sigma(T)$, that is, the closure of $W(T)$ is equal to the convex hull of $\sigma(T)$. An operator T is called *power bounded* if there exists a positive number M such that $\|T^n\| \leq M$ for all $n \in \mathbb{N}$.

Theorem 3.10. *Let T be a $[2, C]$-isometric operator. If T is power bounded and $CTC \cdot T - I$ is convexoid, then T is a $[1, C]$-isometric operator.*

Proof. For the proof, we will show that $W(CTC \cdot T - I) = \{0\}$. Assume that $W(CTC \cdot T - I) \neq \{0\}$. Since $CTC \cdot T - I$ is convexoid, it holds $\overline{W(CTC \cdot T - I)} = \text{co}\,\sigma(CTC \cdot T - I)$. Then there exist a non-zero $a \in \mathbb{C}$ and a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_{n\to\infty}(CTC \cdot T - I - a)x_n = 0$. Since T is a $[2, C]$-isometric operator, it holds $\lim_{n\to\infty}(CT^2C \cdot T^2 - (1 + 2a))x_n = 0$. Inductively, we have

$$\lim_{n\to\infty}\Big(CT^nC \cdot T^n - (1 + na)\Big)x_n = 0.$$

Therefore, it holds that $\|CT^nC \cdot T^n\| \geq |1 + na|$. Since $a \neq 0$, it follows that $\lim_{n\to\infty} |1 + na| = \infty$. Since T is power bounded, so is $CT^nC \cdot T^n$ and hence it is a contradiction. □

4. Tensor Products of [m, C]-Isometric Operators

For a complex Hilbert space $\mathcal{H}$, let $\mathcal{H} \otimes \mathcal{H}$ denote the completion of the algebraic tensor product of $\mathcal{H}$ and $\mathcal{H}$ endowed a reasonable uniform cross-norm. For operators $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$, $T \otimes S \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ denote the *tensor product* operator defined by T and S. Note that $T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$. Then B. Duggal in [9] proved the following result.

Proposition 4.1. (Theorem 2.10, [9]) *Let T and S be an m-isometry and an n-isometry on $\mathcal{H}$, respectively. Then $T \otimes S$ is an $(m + n - 1)$-isometry on $\mathcal{H} \otimes \mathcal{H}$.*

Since Duggal's proof is long and difficult, we firstly give a short proof. A pair of operators (T, S) is said to be *a doubly commuting pair* if (T, S) satisfies $TS = ST$ and $T^*S = ST^*$. Then, for a doubly commuting pair (T, S), it holds

$$\beta_m(TS) = \sum_{k=0}^{m} \binom{m}{k} T^{*k} \cdot \beta_{m-k}(T) \cdot T^k \cdot \beta_k(S). \tag{5}$$

Equation (5) is a result of Lemma 3.1 of [3]. It comes from the following equation;

$$(ab - 1)^m = \Big((a - 1) + a(b - 1)\Big)^m = \sum_{k=0}^{m} \binom{m}{k} (a - 1)^{m-k}\, a^k\, (b - 1)^k.$$

Proposition 4.2. *Let T and S be an m-isometry and an n-isometry on $\mathcal{H}$, respectively. If (T, S) is a doubly commuting pair, then TS is an $(m + n - 1)$-isometry on $\mathcal{H}$.*

Proof. By Equation (5), we have

$$\beta_{m+n-1}(TS) = \sum_{k=0}^{m+n-1} \binom{m + n - 1}{k} T^{*k} \cdot \beta_{m+n-1-k}(T) \cdot T^k \cdot \beta_k(S).$$

(i) If $0 \leq k \leq n - 1$, then $m + n - 1 - k \geq m$ and hence $\beta_{m+n-1-k}(T) = 0$.
(ii) If $k \geq n$, then $\beta_k(S) = 0$.
Therefore, $\beta_{m+n-1}(TS) = 0$ and so TS is an $(m + n - 1)$-isometry. □

Proof of Proposition 4.1. It is clear that $T \otimes I$ and $I \otimes S$ are an m-isometry and an n-isometry on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $(T \otimes I, I \otimes S)$ is a doubly commuting pair, by Proposition 4.2, $(T \otimes I)(I \otimes S) = T \otimes S$ is an $(m + n - 1)$-isometry on $\mathcal{H} \otimes \mathcal{H}$.

Next we show following similar result of Proposition 4.1. For $[m, C]$-operators, let (T, S) be a commuting pair and satisfy $S \cdot CTC = CTC \cdot S$, where C is a conjugation. Then it holds

$$\lambda_m(TS; C) = \sum_{k=0}^{m} \binom{m}{k} CT^kC \cdot \lambda_{m-k}(T; C) \cdot T^k \cdot \lambda_k(S; C). \tag{6}$$

Then, by a similar proof of Proposition 4.2, we have

Theorem 4.3. *Let T and S be an $[m, C]$-isometry and an $[n, C]$-isometry on $\mathcal{H}$, respectively. If (T, S) is a commuting pair and satisfies $S \cdot CTC = CTC \cdot S$, then TS is an $[m + n - 1, C]$-isometry on $\mathcal{H}$.*

Proof. By Equation (6), it holds

$$\lambda_{m+n-1}(TS; C) = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} CT^kC \cdot \lambda_{m+n-1-k}(T; C) \cdot T^k \cdot \lambda_k(S; C).$$

Hence TS is an $[m + n - 1, C]$-isometry on $\mathcal{H}$. □

Theorem 4.4. *Let T and S be an $[m, C]$-isometry and an $[n, D]$-isometry on $\mathcal{H}$, respectively. Then $T \otimes S$ is an $[m + n - 1, C \otimes D]$-isometry on $\mathcal{H} \otimes \mathcal{H}$.*

For conjugations C and D on $\mathcal{H}$, we define $C \otimes D$ on $\mathcal{H} \otimes \mathcal{H}$ by

$$(C \otimes D)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \overline{\alpha_j} Cx_j \otimes Dy_j.$$

First we prepare the following lemma.

Lemma 4.5. *Let C and D be conjugations on $\mathcal{H}$. Then $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$.*

Proof. Let $x = \sum_{i=1}^{n} \alpha_i x_i^1 \otimes x_i^2$ and $y = \sum_{j=1}^{m} \beta_j y_j^1 \otimes y_j^2 \in \mathcal{H} \otimes \mathcal{H}$ where $\alpha_i, \beta_j \in \mathbb{C}$. Since C and D are isometric, it follows that

$$\begin{aligned}
\langle (C \otimes D)x, (C \otimes D)y \rangle &= \langle (C \otimes D)(\sum_{i=1}^{n} \alpha_i x_i^1 \otimes x_i^2), (C \otimes D)(\sum_{j=1}^{m} \beta_i y_j^1 \otimes y_j^2) \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{\alpha_i} \langle Cx_i^1, Cy_j^1 \rangle \cdot \overline{\beta_j} \langle Dx_i^2, Dy_j^2 \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{\alpha_i} \langle y_j^1, x_i^1 \rangle \cdot \overline{\beta_j} \langle y_j^2, x_i^2 \rangle \\
&= \langle \sum_{j=1}^{m} \overline{\beta_j} y_j^1 \otimes y_j^2, \sum_{i=1}^{n} \overline{\alpha_i} x_i^1 \otimes x_i^2 \rangle = \langle y, x \rangle.
\end{aligned} \tag{7}$$

Moreover, since C and D are involutive, it follows that

$$(C \otimes D)^2 = (C^2 \otimes D^2) = I \otimes I \tag{8}$$

on the algebraic tensor product of $\mathcal{H} \otimes \mathcal{H}$. Since C and D are bounded, it follows from (7) and (8) that $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$. □

Proof of Theorem 4.4. By Lemma 4.5, $C \otimes D$ is a conjugation. It is clear that $T \otimes I$ and $I \otimes S$ are $[m, C \otimes D]$-isometry and $[n, C \otimes D]$-isometry on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $\left(T \otimes I,\ I \otimes S\right)$ is a commuting pair and satisfies

$$(I \otimes S) \cdot \Big((C \otimes D)(T \otimes I)(C \otimes D)\Big) = \Big((C \otimes D)(T \otimes I)(C \otimes D)\Big) \cdot (I \otimes S),$$

by Theorem 4.3, $(T \otimes I)(I \otimes S) = T \otimes S$ is an $[m + n - 1, C \otimes D]$-isometry.

For an (m, C)-isometric operator T, $\Lambda_m(T; C)$ is defined by

$$\Lambda_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} \cdot CT^{m-j}C.$$

Let a commuting pair (T, S) satisfy $S^* \cdot CTC = CTC \cdot S^*$, where C is a conjugation. Then it holds

$$\Lambda_m(TS; C) = \sum_{k=0}^{m} \binom{m}{k} T^{*k} \cdot \Lambda_{m-k}(T; C) \cdot CT^kC \cdot \Lambda_k(S; C). \tag{9}$$

By similar proofs of Proposition 4.2 and Theorems 4.3 and 4.4, we have following results.

Theorem 4.6. *Let T and S be an (m, C)-isometry and an (n, C)-isometry on $\mathcal{H}$, respectively. If (T, S) is a commuting pair and satisfies $S^* \cdot CTC = CTC \cdot S^*$, then TS is an $(m + n - 1, C)$-isometry on $\mathcal{H}$.*

Proof. The proof follows from Equation (9). □

Theorem 4.7. *Let T and S be an (m, C)-isometry and an (n, D)-isometry on $\mathcal{H}$, respectively. Then $T \otimes S$ is an $(m + n - 1, C \otimes D)$-isometry on $\mathcal{H} \otimes \mathcal{H}$.*

Proof. Operators $T \otimes I$ and $I \otimes S$ are $(m, C \otimes D)$-isometry and $(n, C \otimes D)$-isometry on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $\left(T \otimes I,\ I \otimes S\right)$ is a commuting pair and satisfies $(I \otimes S)^* \cdot \Big((C \otimes D)(T \otimes I)(C \otimes D)\Big) = \Big((C \otimes D)(T \otimes I)(C \otimes D)\Big) \cdot (I \otimes S)^*$, by Theorem 4.6 $(T \otimes I)(I \otimes S) = T \otimes S$ is an $(m + n - 1, C \otimes D)$-isometry on $\mathcal{H} \otimes \mathcal{H}$. □

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