A Contribution to the Study of Soft Proximity Spaces

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Abstract. In this paper we study the soft proximity spaces. First, we investigate the relation between proximity spaces and soft proximity spaces. Also, we define the notion of a soft $\delta-$neighborhood in the soft proximity spaces which offer an alternative approach to the study of soft proximity spaces. Later, we show how a soft proximity space is derived from a soft uniform space. Finally, we obtain the initial soft proximity space determined by a family of soft proximity spaces.

1. Introduction

In 1999, Molodtsov [20] initiated the concept of soft set theory as a new approach for coping with uncertainties and also presented the basic results of the new theory. This new theory does not require the specification of a parameter. We can utilize any parametrization with the aid of words, sentences, real numbers and so on. This implies that the problem of setting the membership function does not arise. Hence, soft set theory has compelling applications in several diverse fields, most of these applications was shown by Molodtsov [20].

Maji et al. [19] gave the first practical application of soft sets in decision making problems. Chen et al. [4] presented a novel concept of parameterization reduction in soft sets. Kong et al. [15] introduced the notion of a normal parameter reduction and presented an algorithm for normal parameter reduction. Then, Ma et al. [17] proposed a simpler and more easily comprehensible algorithm. Pei and Miao [24] showed that soft sets are a class of special information systems. Maji et al. [18] studied on soft set theory in detail. Ali et al. [2] presented some new algebraic operations on soft sets. Aktaş and Çağman [1] introduced the soft group and also compared soft sets to fuzzy set and rough set. Feng et al. [8] worked on soft semirings, soft ideals and idealistic soft semirings. Shabir and Naz [25] initiated the study of soft topological spaces. Studies on the soft topological spaces have been accelerated [3, 5, 9, 16, 22, 23, 26].

Proximity structure was introduced by Efremovic in 1951 [6, 7]. It can be considered either as axiomatizations of geometric notions or as suitable tools for an investigation of topology. Moreover, this structure has a very significant role in many problems of topological spaces such as compactification and extension problems etc. The most comprehensive work on the theory of proximity spaces was done by Naimpally and Warrack [21]. Then, many authors have obtained the concept of a proximity in both the fuzzy setting and soft setting.

Extension of proximity structures to the soft sets has been studied by some authors. Hazra et al. [10] defined the notion of a proximity in soft setting for the first time, which is termed as soft proximity. Also, by using soft sets, Hazra et al. [11] introduced the different notion of a proximity on the lines of basic proximity and called it proximity of soft sets. Then, Kandil et al. [12] defined soft proximity spaces on the base of the axioms suggested by Efremovic. Moreover, Kandil et al. [13] studied on soft *I*-proximity spaces, where *I* is an ideal. All these works have generalized versions of many of the well known results on proximity spaces.

In this work, we continue investigating the properties of soft proximity spaces in Kandil et al.'s sense. Also, we give the notion of a soft δ−neighborhood in soft proximity spaces and obtain a few results analogous to the ones that hold for δ-neighborhood in proximity spaces. Moreover, we show that each soft uniform space on *X* induces a soft proximity space on the same set. Finally, we prove the existences of initial soft proximity spaces.

2. Preliminaries

In this section, we recollect some basic notions regarding soft sets. Throughout this work, let *X* be an initial universe, *P(X)* be the power set of *X* and *E* be a set of parameters for *X*.

Definition 2.1. ([20]) A soft set *F* on the universe *X* with the set *E* of parameters is defined by the set of ordered pairs

$$F = \{(e, F(e)) : e \in E, F(e) \in P(X)\}$$

where F is a mapping given by $F : E \to P(X)$.

Throughout this paper, the family of all soft sets over *X* is denoted by $S(X, E)$ [3].

Definition 2.2. ([2, 18, 24]) Let $F, G \in S(X, E)$. Then:
(i) The soft set *F* is called null soft set, denoted by Φ, if $F(e) = \emptyset$ for every $e \in E$.
(ii) If $F(e) = X$ for all $e \in E$, then *F* is called absolute soft set, denoted by $\widetilde{X}$.
(iii) *F* is a soft subset of *G* if $F(e) \subseteq G(e)$ for every $e \in E$. It is denoted by $F \sqsubseteq G$.
(iv) *F* and *G* are equal if $F \sqsubseteq G$ and $G \sqsubseteq F$. It is denoted by $F = G$.
(v) The complement of *F* is denoted by F^c, where $F^c : E \to P(X)$ is a mapping defined by $F^c(e) = X - F(e)$ for all $e \in E$. Clearly, $(F^c)^c = F$.
(vi) The union of *F* and *G* is a soft set *H* defined by $H(e) = F(e) \cup G(e)$ for all $e \in E$. *H* is denoted by $F \sqcup G$.
(vii) The intersection of *F* and *G* is a soft set *H* defined by $H(e) = F(e) \cap G(e)$ for all $e \in E$. *H* is denoted by $F \sqcap G$.

Definition 2.3. ([5, 16, 22]) A soft set *P* over X is said to be a soft point if there exists $e \in E$ such that $P(e) = \{x\}$ for some $x \in X$ and $P(e') = \emptyset$ for all $e' \in E \setminus \{e\}$. The soft point denoted as x^e.

From now on, let $SP(X)$ be the family of all soft points over *X*.

Definition 2.4. ([5, 22]) A soft point x^e is said to belongs to a soft set *F*, denoted by $x^e \widetilde{\in} F$, if $x \in F(e)$.

Definition 2.5. ([5]) Two soft points $x_1^{e_1}, x_2^{e_2}$ are said to be equal if $e_1 = e_2$ and $x_1 = x_2$. Thus, $x_1^{e_1} \neq x_2^{e_2} \Leftrightarrow x_1 \neq x_2$ or $e_1 \neq e_2$.

Definition 2.6. ([14]) Let $S(X, E)$ and $S(Y, K)$ be the families of all soft sets over *X* and *Y*, respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two mappings. Then, the mapping φ_ψ is called a soft mapping from *X* to *Y*, denoted by $\varphi_\psi : S(X, E) \to S(Y, K)$.
(i) Let $F \in S(X, E)$. Then $\varphi_\psi(F)$ is the soft set over *Y* defined as follows:

$$\varphi_\psi(F)(k) = \begin{cases} \bigcup_{e \in \psi^{-1}(k)} \varphi(F(e)), & \text{if } \psi^{-1}(k) \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

for all $k \in K$.

$\varphi_\psi(F)$ is called a soft image of a soft set F.

(ii) Let $G \in S(Y, K)$. Then $\varphi_\psi^{-1}(G)$ is the soft set over X defined as follows:

$$\varphi_\psi^{-1}(G)(e) = \varphi^{-1}(G(\psi(e)))$$

for all $e \in E$.

$\varphi_\psi^{-1}(G)$ is called a soft inverse image of a soft set G.

The soft mapping φ_ψ is called injective, if φ and ψ are injective. The soft mapping φ_ψ is called surjective, if φ and ψ are surjective [3, 26].

Theorem 2.7. ([14]) *Let $F_i \in S(X, E)$ and $G_i \in S(Y, K)$ for all $i \in J$ where J is an index set. Then, for a soft mapping $\varphi_\psi : S(X, E) \to S(Y, K)$, the following conditions are satisfied.*

(i) If $F_1 \sqsubseteq F_2$, then $\varphi_\psi(F_1) \sqsubseteq \varphi_\psi(F_2)$.

(ii) If $G_1 \sqsubseteq G_2$, then $\varphi_\psi^{-1}(G_1) \sqsubseteq \varphi_\psi^{-1}(G_2)$.

(iii) $\varphi_\psi\left(\bigsqcup_{i \in J} F_i\right) = \bigsqcup_{i \in J} \varphi_\psi(F_i)$.

(iv) $\varphi_\psi^{-1}\left(\bigsqcup_{i \in J} G_i\right) = \bigsqcup_{i \in J} \varphi_\psi^{-1}(G_i)$.

(v) $\varphi_\psi^{-1}\left(\bigsqcap_{i \in J} G_i\right) = \bigsqcap_{i \in J} \varphi_\psi^{-1}(G_i)$.

(vi) $\varphi_\psi^{-1}(\widetilde{Y}) = \widetilde{X}$, $\varphi_\psi^{-1}(\Phi) = \Phi$ and $\varphi_\psi(\Phi) = \Phi$.

Theorem 2.8. ([3, 26]) *Let $F, F_i \in S(X, E)$ for all $i \in J$ where J is an index set and let $G \in S(Y, K)$. Then, for a soft mapping $\varphi_\psi : S(X, E) \to S(Y, K)$, the following conditions are satisfied.*

(i) $F \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(F))$, the equality holds if φ_ψ is injective.

(ii) $\varphi_\psi(\varphi_\psi^{-1}(G)) \sqsubseteq G$, the equality holds if φ_ψ is surjective.

Definition 2.9. ([3]) Let $F \in S(X, E), G \in S(Y, K)$ and let $p_X : X \times Y \to X$, $q_E : E \times K \to E$ and $p_Y : X \times Y \to Y$, $q_K : E \times K \to K$ be the projection mappings in classical meaning. The soft mappings $(p_X)_{q_E}$ and $(p_Y)_{q_K}$ are called soft projection mappings from $X \times Y$ to X and from $X \times Y$ to Y, respectively, where $(p_X)_{q_E}(F \times G) = F$ and $(p_Y)_{q_K}(F \times G) = G$.

Definition 2.10. ([25]) Let τ be a collection of soft sets over X, then τ is said to be a soft topology on X if:

(st_1) $\Phi, \widetilde{X}$ belong to τ.

(st_2) the union of any number of soft sets in τ belongs to τ.

(st_3) the intersection of any two soft sets in τ belongs to τ.

(X, τ, E) is called a soft topological space. The members of τ are called soft open sets in X. A soft set F over X is called a soft closed in X if $F^c \in \tau$.

Definition 2.11. ([26]) Let (X, τ, E) be a soft topological space and $F \in S(X, E)$. The soft interior of F is the soft set $F^o = \bigsqcup \{G : G \text{ is soft open set and } G \sqsubseteq F\}$.

Definition 2.12. ([25]) Let (X, τ, E) be a soft topological space and $F \in S(X, E)$. The soft closure of F is the soft set $\overline{F} = \bigsqcap \{G : G \text{ is soft closed set and } F \sqsubseteq G\}$.

Theorem 2.13. ([22]) *Let us consider an operator associating with each soft set F on X another soft set $\overline{F}$ such that the following properties hold:*

(so_1) $F \sqsubseteq \overline{F}$,

(so_2) $\overline{\overline{F}} = \overline{F}$,

(so_3) $\overline{F \sqcap G} = \overline{F} \sqcap \overline{G}$,

(so_4) $\overline{\Phi} = \Phi$.

Then the family $\tau = \{F \in S(X, E) : \overline{F^c} = F^c\}$ defines a soft topology on X and for every $F \in S(X, E)$, the soft set $\overline{F}$ is the soft closure of F in the soft topological space (X, τ, E).

This operator is called the soft closure operator.

Definition 2.14. ([23]) Let (X, τ_1, E) and (Y, τ_2, K) be two soft topological spaces and $\varphi_\psi : (X, \tau_1, E) \to (Y, \tau_2, K)$ be a soft mapping. Then φ_ψ is called soft continuous at $x^e \widetilde{\in} X$ if for every soft neighborhood G of $\varphi_\psi(x^e)$ in Y, there exists a soft neighborhood F of x^e in X such that $\varphi_\psi(F) \sqsubseteq G$.

A soft mapping φ_ψ is called soft continuous on X if it is soft continuous at each $x^e \widetilde{\in} \widetilde{X}$.

Theorem 2.15. ([23]) *Let (X, τ_1, E) and (Y, τ_2, K) be two soft topological spaces and $\varphi_\psi : (X, \tau_1, E) \to (Y, \tau_2, K)$ be a soft mapping. Then the following conditions are equivalent:*

(i) φ_ψ is soft continuous.

(ii) For every soft open set G in (Y, τ_2, K), $\varphi_\psi^{-1}(G)$ is soft open in (X, τ_1, E).

(iii) For every soft closed set F in (Y, τ_2, K), $\varphi_\psi^{-1}(F)$ is soft closed in (X, τ_1, E).

(iv) For every $F \in S(X, E)$, $\varphi_\psi(\overline{F}) \sqsubseteq \overline{\varphi_\psi(F)}$.

Definition 2.16. ([23]) The non-empty family $\mathcal{U} \subseteq S(SP(X) \times SP(X), E)$ is called a soft uniformity for X if the following axioms are satisfied:

(su_1) If $U \in \mathcal{U}$, then $\Delta \sqsubseteq U$.

(su_2) If $U \in \mathcal{U}$, then there exists a $V \in \mathcal{U}$ such that $V \circ V \sqsubseteq U$.

(su_3) If $U \in \mathcal{U}$, then there exists a $V \in \mathcal{U}$ such that $V^{-1} \sqsubseteq U$.

(su_4) If $U, V \in \mathcal{U}$, then $U \sqcap V \in \mathcal{U}$.

(su_5) If $U \in \mathcal{U}$ and $U \sqsubseteq V$, then $V \in \mathcal{U}$.

The triplet $(X, \mathcal{U}, E)$ is called a soft uniform space on X.

Definition 2.17. ([23]) (i) The soft set $\Delta \in S(SP(X) \times SP(X), E)$ is said to be diagonal soft set which is defined by $\Delta(e) = \{(x^\alpha, x^\alpha) : x^\alpha \in SP(X)\}$, for every $e \in E$.

(ii) Let $U \in S(SP(X) \times SP(X), E)$. Then,

$$U^{-1}(e) = \{(x_1^{e_1}, x_2^{e_2}) : (x_2^{e_2}, x_1^{e_1}) \in U(e)\}$$

for every $e \in E$. If $U = U^{-1}$, then U is said to be symmetric.

(iii) Let $U, V \in S(SP(X) \times SP(X), E)$. Then,

$$U \circ V(e) = \{(x_1^{e_1}, x_2^{e_2}) : for\ some\ z^\alpha \in SP(X), (x_1^{e_1}, z^\alpha) \in V(e)\ and\ (z^\alpha, x_2^{e_2}) \in U(e)\}$$

for every $e \in E$.

Theorem 2.18. ([23]) *Let $(X, \mathcal{U}, E)$ be a soft uniform space.*

(i) If $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$.

(ii) The conjunction of axioms (su_2) and (su_3) is equivalent to the following axiom:

For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V^{-1} \sqsubseteq U$

(iii) The family $\mathcal{B}$ consisting of the symmetric soft sets in $\mathcal{U}$ form a soft base for $\mathcal{U}$.

Definition 2.19. ([23]) Let $(X, \mathcal{U}, E)$ be a soft uniform space and $x^e \widetilde{\in} \widetilde{X}$. Then, for every $U \in \mathcal{U}$

$$U[x^e] = \bigsqcup \left\{ z^\beta \widetilde{\in} \widetilde{X} : (x^e, z^\beta) \in U(\alpha), \forall \alpha \in E \right\}$$

is a soft set on X. This is extended to the soft set F on X, denoted by

$$U[F] = \bigsqcup_{x^e \widetilde{\in} F} U[x^e] = \bigsqcup \left\{ z^\beta \widetilde{\in} \widetilde{X} : for\ some\ x^e \widetilde{\in} F,\ (x^e, z^\beta) \in U(\alpha), \forall \alpha \in E \right\}.$$

Recall that a binary relation δ on the power set of a set X is called a proximity on X if the following axioms are satisfied (see, [21])

(p_1) $\emptyset \overline{\delta} A$,

(p_2) If $A \cap B \neq \emptyset$, then $A \delta B$,

(p_3) If $A \delta B$, then $B \delta A$,

(p_4) $A \delta (B \cup C)$ if and only if $A \delta B$ or $A \delta C$,

(p_5) If $A \overline{\delta} B$, then there exists a subset C of X such that $A \overline{\delta} C$ and $B \overline{\delta}(X - C)$,

where $\overline{\delta}$ means negation of δ.

The pair (X, δ) is called a proximity space; two subsets A and B of the set X are close with respect to δ if $A \delta B$, otherwise they are remote with respect to δ.

3. Soft Proximity Spaces

In this section, we study some basic properties of soft proximity spaces. Also, we give an alternative description of the concept of soft proximity spaces, which is called soft δ-neighborhood.

Definition 3.1. ([12]) A binary relation δ on $S(X, E)$ is called a proximity of soft sets on X if for any $F, G, H \in S(X, E)$, the following conditions are satisfied:

(sp_1) $\Phi \overline{\delta} F$,

(sp_2) If $F \sqcap G \neq \Phi$, then $F \delta G$,

(sp_3) If $F \delta G$, then $G \delta F$,

(sp_4) $F \delta (G \sqcup H)$ if and only if $F \delta G$ or $F \delta H$,

(sp_5) If $F \overline{\delta} G$, then there exists an $H \in S(X, E)$ such that $F \overline{\delta} H$ and $G \overline{\delta}(\widetilde{X} - H)$.

A soft proximity space is a triple (X, δ, E) consisting of a set X, a set of parameters E and a proximity relation on $S(X, E)$. We shall write $F \delta G$ if the soft sets $F, G \in S(X, E)$ are δ-related, otherwise we shall write $F \overline{\delta} G$.

Example 3.2. (i) On any set X, let us define $F \delta G$ iff $F \neq \Phi$ and $G \neq \Phi$. This defines a proximity relation on $S(X, E)$.

(ii) On any set X, let us define $F \delta G$ iff $F \sqcap G \neq \Phi$. This defines a proximity relation on $S(X, E)$.

We obtain the connection between proximity spaces and soft proximity spaces as shown in the following theorem.

Theorem 3.3. *Let (X, δ) be a proximity space. By letting for $F, G \in S(X, E)$*

$F \overline{\delta^i} G$ *iff there exist subsets A, B of X such that $F \sqsubseteq \widetilde{A}, G \sqsubseteq \widetilde{B}$ and $A \overline{\delta} B$*

we define a proximity relation on $S(X, E)$.

(Here, for every $A \subseteq X$, $\widetilde{A}$ is the soft set over X defined by $\widetilde{A}(e) = A$ for all $e \in E$).

Proof. We shall show that δ^i satisfies axioms (sp_1) − (sp_5).

(sp_1) From $\Phi \sqsubseteq \widetilde{\emptyset}$, $F \sqsubseteq \widetilde{X}$ and $\emptyset \overline{\delta} X$ it follows that $\Phi \overline{\delta^i} F$.

(sp_2) Let $F \overline{\delta^i} G$. Then, there are subsets A and B of X such that $F \sqsubseteq \widetilde{A}, G \sqsubseteq \widetilde{B}$ and $A \overline{\delta} B$. By $A \overline{\delta} B$, we have $A \cap B = \emptyset$, so that $\widetilde{A} \sqcap \widetilde{B} = \Phi$. Thus, we get $F \sqcap G = \Phi$.

(sp_3) It is clear because $A \overline{\delta} B$ implies $B \overline{\delta} A$.

(sp_4) It is easy to see that if $F \overline{\delta^i}(G \sqcup H)$ then $F \overline{\delta^i} G$ and $F \overline{\delta^i} H$. Conversely, suppose that $F \overline{\delta^i} G$ and $F \overline{\delta^i} H$. Then, there exist subsets A and B of X such that $F \sqsubseteq \widetilde{A}, G \sqsubseteq \widetilde{B}$ and $A \overline{\delta} B$. Likewise, there exist subsets C and D of X such that $F \sqsubseteq \widetilde{C}, H \sqsubseteq \widetilde{D}$ and $C \overline{\delta} D$. Since $F \sqsubseteq \widetilde{A} \sqcap \widetilde{C} = \widetilde{A \cap C}$, $G \sqcup H \sqsubseteq \widetilde{B} \sqcup \widetilde{D} = \widetilde{B \cup D}$ and $(A \cap C) \overline{\delta}(B \cup D)$, we conclude that $F \overline{\delta^i}(G \sqcup H)$.

(sp_5) If $F \overline{\delta^i} G$, then there are subsets A and B of X such that $F \sqsubseteq \widetilde{A}, G \sqsubseteq \widetilde{B}$ and $A \overline{\delta} B$. Since $A \overline{\delta} B$, by (p_5) there is a $C \subseteq X$ such that $A \overline{\delta} C$ and $B \overline{\delta}(X - C)$. Therefore, for a soft set $\widetilde{C}$, we obtain $F \overline{\delta^i} \widetilde{C}$ and $G \overline{\delta^i}(\widetilde{X} - \widetilde{C})$, which completes the proof. $\square$

Lemma 3.4. ([12]) *Let (X, δ, E) is a soft proximity space. If $F \delta G$ and $F \sqsubseteq H_1$, $G \sqsubseteq H_2$, then $H_1 \delta H_2$.*

Theorem 3.5. ([12]) *Let (X, δ, E) be a soft proximity space. Then, the mapping $F \to \overline{F}$, where*

$$\overline{F} = \bigsqcup \{ x^e \widetilde{\in} \widetilde{X} : x^e \delta F \}$$

satisfies the conditions $(so_1) - (so_4)$. Therefore, the collection

$$\tau(\delta) = \{ F \in S(X, E) : \overline{F^c} = F^c \}$$

is a soft topology on X.

Trivially, the soft proximity spaces defined in Example 3.2 (i) and (ii) induce the soft topological spaces $\tau(\delta) = \{\Phi, \widetilde{X}\}$ and $\tau(\delta) = S(X, E)$, respectively.

Corollary 3.6. *Let (X, δ, E) is a soft proximity space and $F \in S(X, E)$. Then,*

$$F \in \tau(\delta) \quad iff \quad x^e \overline{\delta}(\widetilde{X} - F) \text{ for every } x^e \widetilde{\in} F.$$

Proof. Let $x^e \widetilde{\in} F$. Then, $x^e \widetilde{\notin} (\widetilde{X} - F)$. Since $\overline{(\widetilde{X} - F)} = (\widetilde{X} - F)$, by Theorem 3.5, we have $x^e \overline{\delta}(\widetilde{X} - F)$.

Conversely, for every $x^e \widetilde{\in} F$, let $x^e \overline{\delta}(\widetilde{X} - F)$. Therefore, $x^e \widetilde{\notin} \overline{(\widetilde{X} - F)}$. From the fact that $\overline{(\widetilde{X} - F)} = (\widetilde{X} - F^o)$ it follows that $x^e \widetilde{\in} F^o$. Thus, we have $F \sqsubseteq F^o$, that is, $F \in \tau(\delta)$. $\square$

Definition 3.7. If δ_1 and δ_2 are two proximities of soft sets on X, we define

$$\delta_1 < \delta_2 \quad iff \quad F \delta_2 G \text{ implies } F \delta_1 G.$$

The above is expressed by saying that δ_2 is finer than δ_1, or δ_1 is coarser than δ_2.

The following theorem shows that a finer soft proximity structure induces a finer soft topology:

Theorem 3.8. *Let δ_1 and δ_2 be two proximities of soft sets on X. Then,*

$$\delta_1 < \delta_2 \text{ implies } \tau(\delta_1) \subseteq \tau(\delta_2).$$

Proof. Let $F \in \tau(\delta_1)$. It follows from Corollary 3.6 that $x^e \overline{\delta_1}(\widetilde{X} - F)$ for every $x^e \widetilde{\in} F$. Since $\delta_1 < \delta_2$, we get $x^e \overline{\delta_2}(\widetilde{X} - F)$ for every $x^e \widetilde{\in} F$. Hence, $F \in \tau(\delta_2)$. $\square$

Definition 3.9. Let (X, δ, E) be a soft proximity space. For $F, G \in S(X, E)$, the soft set G is said to be a soft δ-neighborhood of F if $F \overline{\delta}(\widetilde{X} - G)$; we write this in symbols as $F \Subset G$.

Theorem 3.10. *Let (X, δ, E) be a soft proximity space. Then the relation $\Subset$ satisfies the following properties:*
   (spn_1) $\Phi \Subset F$.
   (spn_2) $F \Subset G$ implies $(\widetilde{X} - G) \Subset (\widetilde{X} - F)$.
   (spn_3) $F \Subset G$ implies $F \sqsubseteq G$.
   (spn_4) $F \Subset (G \sqcap H)$ if and only if $F \Subset G$ and $F \Subset H$.
   (spn_5) $F_1 \sqsubseteq F \Subset G \sqsubseteq G_1$ implies $F_1 \Subset G_1$.
   (spn_6) $F \Subset G$ implies there is an $H \in S(X, E)$ such that $F \Subset H \Subset G$.

Proof. (spn_1) is obvious.
   (spn_2) If $F \Subset G$, then $F \overline{\delta}(\widetilde{X} - G)$. By (sp_3), $(\widetilde{X} - G)\overline{\delta} F$, that is, $(\widetilde{X} - G) \Subset (\widetilde{X} - F)$.

   (spn_3) Let $F \Subset G$. Then from (sp_2) it follows that $F \sqcap G^c = \Phi$. Thus, we have $F \sqsubseteq G$.

(spn_4) $F \Subset (G \sqcap H) \iff F\bar{\delta}\big(\widetilde{X} - (G \sqcap H)\big)$

$\qquad\qquad\qquad \iff F\bar{\delta}(\widetilde{X} - G)$ and $F\bar{\delta}(\widetilde{X} - H)$

$\qquad\qquad\qquad \iff F \Subset G$ and $F \Subset H$.

(spn_5) If $F_1 \overline{\Subset} G_1$, where $\overline{\Subset}$ means negation of $\Subset$, then $F_1\delta(\widetilde{X} - G_1)$. Since $F_1 \sqsubseteq F$ and $(\widetilde{X} - G_1) \sqsubseteq (\widetilde{X} - G)$, we have $F\delta(\widetilde{X} - G)$. Therefore, $F \overline{\Subset} G$, which is a contradiction.

(spn_6) $F \Subset G$ implies $F\bar{\delta}(\widetilde{X} - G)$. Then by (sp_5), there exists an $H \in S(X, E)$ such that $F\bar{\delta}(\widetilde{X} - H)$ and $H\bar{\delta}(\widetilde{X} - G)$. Hence, $F \Subset H \Subset G$. $\quad\square$

Theorem 3.11. *Let $\Subset$ be a relation on $S(X, E)$ satisfying $(spn_1) - (spn_6)$. Then, δ is a proximity relation on $S(X, E)$ defined as follows:*

$$F\bar{\delta}G \quad iff \quad F \Subset (\widetilde{X} - G).$$

Also, according to this proximity relation, G is a soft δ-neighbourhood of F if and only if $F \Subset G$.

Proof. We first need to verify axioms $(sp_1) - (sp_5)$.

(sp_1) Let $F \in S(X, E)$. By (spn_1), we have $\Phi \Subset (\widetilde{X} - F)$ and thus $\Phi \bar{\delta} F$.

(sp_2) Let $F\bar{\delta}G$. Then, $F \Subset (\widetilde{X} - G)$ and from (spn_3) it follows that $F \sqcap G = \Phi$.

(sp_3) If $F\bar{\delta}G$, then $F \Subset (\widetilde{X} - G)$. By (spn_2), $G \Subset (\widetilde{X} - F)$ and hence $G\bar{\delta}F$.

(sp_4) $F\bar{\delta}(G \sqcup H) \iff F \Subset \big(\widetilde{X} - (G \sqcup H)\big)$

$\qquad\qquad\qquad \iff F \Subset (\widetilde{X} - G)$ and $F \Subset (\widetilde{X} - H)$

$\qquad\qquad\qquad \iff F\bar{\delta}G$ and $F\bar{\delta}H$.

(sp_5) Let $F\bar{\delta}G$. Then $F \Subset (\widetilde{X} - G)$. Therefore, by (spn_6), there is a soft set H such that $F \Subset H \Subset (\widetilde{X} - G)$. Thus, $F\bar{\delta}(\widetilde{X} - H)$ and $H\bar{\delta}G$.

Hence δ is a proximity of soft sets on X. From the definitions of the terms involved it follows easily that G is a soft δ-neighbourhood of F if and only if $F \Subset G$. $\quad\square$

Lemma 3.12. *Let (X, δ, E) be a soft proximity space. For $F, G \in S(X, E)$,*

$$F\delta G \quad iff \quad \overline{F}\delta\overline{G},$$

where the soft closure is taken with respect to $\tau(\delta)$.

Proof. Necessity follows immediately from Lemma 3.4. For sufficiency, suppose that $F\bar{\delta}G$. Then, by (sp_5), there is an $H \in S(X, E)$ such that $F\bar{\delta}H$ and $G\bar{\delta}(\widetilde{X} - H)$. We claim that $\overline{G} \sqsubseteq H$. Indeed, let $x^e \widetilde{\notin} H$. Then, we have $x^e \sqsubseteq (\widetilde{X} - H)$. From Lemma 3.4, we see that $x^e\bar{\delta}G$. Therefore, $x^e \widetilde{\notin} \overline{G}$ and our claim is proved. From this and $F\bar{\delta}H$ we obtain $F\bar{\delta}\overline{G}$. Repeating the argument shows that $\overline{F}\bar{\delta}\overline{G}$. $\quad\square$

Theorem 3.13. *Let (X, δ, E) be a soft proximity space and $F, G \in S(X, E)$. Then, the following statements are satisfied:*
(i) $F \Subset G$ if and only if $\overline{F} \Subset G$.
(ii) If $F\bar{\delta}G$, then there exist soft sets F_1, G_1 such that $F \Subset F_1$, $G \Subset G_1$ and $F_1\bar{\delta}G_1$.
(iii) If $F \Subset G$, then there is an $H \in \tau(\delta)$ such that $F \Subset H \sqsubseteq \overline{H} \Subset G$.

Proof. (i) It is clear from Lemma 3.12.

(ii) If $F\bar{\delta}G$, then from (sp_5), there is a soft set G_1 such that $F\bar{\delta}G_1$ and $G\bar{\delta}(\widetilde{X} - G_1)$. Because $G_1\bar{\delta}F$, there is a soft set F_1 such that $G_1\bar{\delta}F_1$ and $F\bar{\delta}(\widetilde{X} - F_1)$. Thus, there exist soft sets F_1 and G_1 such that $F \Subset F_1$, $G \Subset G_1$ and $F_1\bar{\delta}G_1$.

(iii) Let $F \Subset G$. Then, by (ii) there exist soft sets F_1, G_1 such that $F \Subset F_1$, $(\widetilde{X} - G) \Subset G_1$ and $F_1 \sqcap G_1 = \Phi$. By virtue of (spn_2), we have $F_1 \sqsubseteq (\widetilde{X} - G_1) \Subset G$. Therefore, $F_1\bar{\delta}(\widetilde{X} - G)$. Since $F\bar{\delta}(\widetilde{X} - F_1)$, it follows from Lemma 3.12 that $F\bar{\delta}(\widetilde{X} - F_1)$, i.e., $F\bar{\delta}(\widetilde{X} - (F_1)^o)$. Letting $H = (F_1)^o$, we get $F \Subset H \sqsubseteq F_1$. Because $F_1\bar{\delta}(\widetilde{X} - G)$, we have $H\bar{\delta}(\widetilde{X} - G)$. This means that $\overline{H}\bar{\delta}(\widetilde{X} - G)$, i.e., $\overline{H} \Subset G$. $\quad\square$

Theorem 3.14. *If (X, δ, E) is a soft proximity space and $F \in S(X, E)$, then*

$$\overline{F} = \bigsqcap \{G : F \Subset G\}.$$

Proof. Let us take a soft set G such that $F \Subset G$. Therefore, $\overline{F} \Subset G$ and by (spn_3) we obtain $\overline{F} \sqsubseteq G$. Hence,

$$\overline{F} \sqsubseteq \bigsqcap \{G : F \Subset G\}.$$

On the other hand, suppose that $x^e \widetilde{\notin} \overline{F}$. Then, $x^e \overline{\delta} F$. Therefore, by Theorem 3.13 (ii), there exist soft sets F_1, G_1 such that $\overline{F} \Subset F_1$, $x^e \widetilde{\in} G_1$ and $F_1 \sqcap G_1 = \Phi$. Thus, F has a soft δ-neighbourhood F_1 not containing x^e. This implies that $x^e \widetilde{\notin} \bigsqcap \{G : F \Subset G\}$. $\square$

Definition 3.15. Let (X, δ_1, E) and (Y, δ_2, K) be two soft proximity spaces. A soft mapping $\varphi_\psi : (X, \delta_1, E) \to (Y, \delta_2, K)$ is a soft proximity mapping if it satisfies

$$F \delta_1 G \Rightarrow \varphi_\psi(F) \, \delta_2 \, \varphi_\psi(G)$$

for every $F, G \in S(X, E)$.

Using the above definition, we can easily prove the following propositions.

Proposition 3.16. *Let (X, δ_1, E) and (Y, δ_2, K) be two soft proximity spaces. A soft mapping $\varphi_\psi : (X, \delta_1, E) \to (Y, \delta_2, K)$ is a soft proximity mapping if and only if*

$$F_1 \overline{\delta_2} G_1 \Rightarrow \varphi_\psi^{-1}(F_1) \overline{\delta_1} \varphi_\psi^{-1}(G_1),$$

or in other form

$$F_1 \Subset_2 G_1 \Rightarrow \varphi_\psi^{-1}(F_1) \Subset_1 \varphi_\psi^{-1}(G_1),$$

for every $F_1, G_1 \in S(Y, K)$.

Proposition 3.17. *The composition of two soft proximity mappings is a soft proximity mapping.*

Theorem 3.18. *A soft proximity mapping $\varphi_\psi : (X, \delta_1, E) \to (Y, \delta_2, K)$ is soft continuous with respect to $\tau(\delta_1)$ and $\tau(\delta_2)$.*

Proof. Let $\varphi_\psi : (X, \delta_1, E) \to (Y, \delta_2, K)$ be a soft proximity mapping. To show φ_ψ is soft continuous, it is enough to show that $\varphi_\psi(\overline{F}) \sqsubseteq \overline{\varphi_\psi(F)}$ for every $F \in S(X, E)$. Let $y^k \widetilde{\in} \varphi_\psi(\overline{F})$. Then, there is a soft point $x^e \widetilde{\in} \overline{X}$ such that $x^e \widetilde{\in} \overline{F}$ and $\varphi_\psi(x^e) = y^k$. Therefore, $x^e \delta_1 F$ and it follows from our hypothesis that $y^k \delta_2 \varphi_\psi(F)$. Thus, $y^k \widetilde{\in} \overline{\varphi_\psi(F)}$. $\square$

Lemma 3.19. *Let $(X, \mathcal{U}, E)$ be a soft uniform space and $F, G \in S(X, E)$. Then, there exists a $U \in \mathcal{U}$ such that $U[F] \sqcap U[G] = \Phi$ if and only if there exists a $U \in \mathcal{U}$ such that $U[F] \sqcap G = \Phi$.*

Proof. The necessity is clear since $F \sqsubseteq U[F]$ for every $F \in S(X, E)$ and every $U \in \mathcal{U}$. For the sufficiency, take a $U \in \mathcal{U}$ such that $U[F] \sqcap G = \Phi$. By Theorem 2.18, there is a symmetric soft set $V \in \mathcal{U}$ such that $V \circ V \sqsubseteq U$. We shall show that $V[F] \sqcap V[G] = \Phi$, which will complete the proof. Suppose $x^e \widetilde{\in} V[F]$ and $x^e \widetilde{\in} V[G]$ for some $x^e \in SP(X)$. From the definition of symmetric soft set and Definition 2.19, it follows that there exist an $x_1^{e_1} \widetilde{\in} F$ and an $x_2^{e_2} \widetilde{\in} G$ with $(x_1^{e_1}, x^e) \in V(\alpha)$ and $(x^e, x_2^{e_2}) \in V^{-1}(\alpha) = V(\alpha)$ for each $\alpha \in E$. Since

$$(x_1^{e_1}, x_2^{e_2}) \in (V \circ V)(\alpha) \subseteq U(\alpha) \text{ for each } \alpha \in E$$

we obtain $x_2^{e_2} \widetilde{\in} U[F]$. Therefore, $U[F] \sqcap G \neq \Phi$, which is a contradiction. $\square$

Lemma 3.20. *Let $(X, \mathcal{U}, E)$ be a soft uniform space and $F, G \in S(X, E)$. Then, the following results hold:*
 (i) $(U \sqcap V)[F] \sqsubseteq U[F]$ for every $U, V \in \mathcal{U}$.
 (ii) $U[F \sqcup G] = U[F] \sqcup U[G]$ for every $U \in \mathcal{U}$.

Proof. (i) It is clear from Definition 2.19.

(ii) Let $x^e \,\widetilde{\in}\, U[F \sqcup G]$. Then, there exists an $x_1^{e_1} \,\widetilde{\in}\, (F \sqcup G)$ such that $(x_1^{e_1}, x^e) \in U(\alpha)$ for each $\alpha \in E$. Suppose $x_1^{e_1} \,\widetilde{\in}\, F$. Therefore, we get $x^e \,\widetilde{\in}\, U[F]$, and so $x^e \,\widetilde{\in}\, U[F] \sqcup U[G]$. Thus, we have $U[F \sqcup G] \sqsubseteq U[F] \sqcup U[G]$. By a similar argument, we can show that $U[F] \sqcup U[G] \sqsubseteq U[F \sqcup G]$, completing the proof. $\square$

Theorem 3.21. *Let $(X, \mathcal{U}, E)$ be a soft uniform space. Then, we define a proximity of soft sets on X by*

$$F \overline{\delta} G \quad iff \quad there\ exists\ a\ U \in \mathcal{U}\ such\ that\ U[F] \sqcap U[G] = \Phi.$$

Proof. To show that δ is a proximity of soft sets on X it suffices to prove that (sp_4) and (sp_5) is satisfied, since the other proximity axioms are easily verified.

(sp_4) It follows from Lemma 3.20 (ii) that if $F \overline{\delta} (G \sqcup H)$, then $F \overline{\delta} G$ and $F \overline{\delta} H$. Conversely, let $F \overline{\delta} G$ and $F \overline{\delta} H$. Then, there exist $U_1, U_2 \in \mathcal{U}$ such that $U_1[F] \sqcap U_1[G] = \Phi$ and $U_2[F] \sqcap U_2[H] = \Phi$. By (su_4), we have $U_3 = U_1 \sqcap U_2 \in \mathcal{U}$. Using Lemma 3.20 (i), we obtain

$$U_3[F] \sqcap U_3[G] = \Phi \quad and \quad U_3[F] \sqcap U_3[H] = \Phi.$$

Since

$$U_3[F] \sqcap U_3[G \sqcup H] = (U_3[F] \sqcap U_3[G]) \sqcup (U_3[F] \sqcap U_3[H]) = \Phi$$

it follows that $F \overline{\delta} (G \sqcup H)$, as required.

(sp_5) If $F \overline{\delta} G$, then there is a $U \in \mathcal{U}$ such that $U[F] \sqcap U[G] = \Phi$. By Theorem 2.18, there is a symmetric soft set $V \in \mathcal{U}$ such that $V \circ V \sqsubseteq U$. We first verify that $V[U[G]] \sqcap F = \Phi$. Suppose instead that there exists an $x^e \,\widetilde{\in}\, V[U[G]] \sqcap F$. Then, there exists an $x_1^{e_1} \,\widetilde{\in}\, U[G]$ such that $(x_1^{e_1}, x^e) \in V(\alpha)$ for each $\alpha \in E$. Because V is symmetric soft set and $V \sqsubseteq U$, we obtain $x_1^{e_1} \,\widetilde{\in}\, U[F]$. But this contradicts the fact that $U[F] \sqcap U[G] = \Phi$. Hence, from Lemma 3.19 it follows that $F \overline{\delta} U[G]$. Now, we verify that $V[\widetilde{X} - U[G]] \sqcap G = \Phi$. Supposing the contrary, we find an $x^e \,\widetilde{\in}\, V[\widetilde{X} - U[G]] \sqcap G$. Then, there exists an $x_1^{e_1} \,\widetilde{\in}\, (\widetilde{X} - U[G])$ such that $(x_1^{e_1}, x^e) \in V(\alpha)$ for each $\alpha \in E$. Therefore, we have $(x^e, x_1^{e_1}) \notin U(\alpha)$ for some $\alpha \in E$, because otherwise we would have $x_1^{e_1} \,\widetilde{\in}\, U[G]$. Since V is symmetric soft set and $V \sqsubseteq U$, we get $(x_1^{e_1}, x^e) \notin V(\alpha)$ for some $\alpha \in E$, a contradiction. Thus, applying Lemma 3.19, we infer that $G \overline{\delta} (\widetilde{X} - U[G])$. $\square$

4. Initial Soft Proximity Spaces

We prove the existences of initial soft proximity space. Based on this fact, we define the product of soft proximity spaces.

Definition 4.1. Let X be a set, $\{(X_\alpha, \delta_\alpha, E_\alpha) : \alpha \in \Delta\}$ be a family of soft proximity spaces, and for each $\alpha \in \Delta$ let $(\varphi_\psi)_\alpha : S(X, E) \to (X_\alpha, \delta_\alpha, E_\alpha)$ be a soft mapping. The initial structure δ is the coarsest proximity of soft sets on X for which all mappings $(\varphi_\psi)_\alpha : (X, \delta, E) \to (X_\alpha, \delta_\alpha, E_\alpha)$ $(\alpha \in \Delta)$ are soft proximity mapping.

Theorem 4.2. (Existence of initial structures) *Let X be a set, $\{(X_\alpha, \delta_\alpha, E_\alpha) : \alpha \in \Delta\}$ be a family of soft proximity spaces, and for each $\alpha \in \Delta$ let $(\varphi_\psi)_\alpha : S(X, E) \to (X_\alpha, \delta_\alpha, E_\alpha)$ be a soft mapping. For any $F, G \in S(X, E)$, define $F \delta G$ iff for every finite families $\{F_i : i = 1, ..., n\}$ and $\{G_j : j = 1, ..., m\}$ where $F = \bigsqcup_{i=1}^{n} F_i$ and $G = \bigsqcup_{j=1}^{m} G_j$, there exist an F_i and a G_j such that*

$$(\varphi_\psi)_\alpha(F_i)\, \delta_\alpha\, (\varphi_\psi)_\alpha(G_j) \quad for\ each\ \alpha \in \Delta.$$

Then δ is the coarsest proximity of soft sets on X for which all mappings $(\varphi_\psi)_\alpha : (X, \delta, E) \to (X_\alpha, \delta_\alpha, E_\alpha)$ $(\alpha \in \Delta)$ are soft proximity mapping.

Proof. We first prove that δ is a proximity of soft sets on X.

(sp_1) is obvious.

(sp_2) We will show that if $F\overline{\delta}G$, then $F \sqcap G = \Phi$. Let $F\overline{\delta}G$. Then, there exist finite covers $F = \bigsqcup_{i=1}^{n} F_i$ and $G = \bigsqcup_{j=1}^{m} G_j$ of F and G respectively such that $(\varphi_\psi)_\alpha(F_i)\overline{\delta_\alpha}(\varphi_\psi)_\alpha(G_j)$ for some $\alpha = s_{ij} \in \Delta$, where $i = 1, ..., n$ and $j = 1, ..., m$. Since each δ_α is a proximity of soft sets on X_α, $(\varphi_\psi)_\alpha(F_i) \sqcap (\varphi_\psi)_\alpha(G_j) = \Phi$. From this it follows that

$$(\varphi_\psi)_\alpha\Big(\bigsqcup_{i=1}^{n} F_i\Big) \sqcap (\varphi_\psi)_\alpha\Big(\bigsqcup_{j=1}^{m} G_j\Big) = (\varphi_\psi)_\alpha(F) \sqcap (\varphi_\psi)_\alpha(G) = \Phi.$$

Thus, we have $F \sqcap G = \Phi$.

(sp_3) Since each δ_α is a proximity of soft sets on X_α, it is clear that $F\delta G$ implies $G\delta F$.

(sp_4) It is easy to verify that if $F\delta G$, then $F\delta H$ for each $H \sqsupseteq G$. Therefore, $F\delta G$ or $F\delta H$ implies $F\delta(G \sqcup H)$. Conversely, assume that $F\overline{\delta}G$ and $F\overline{\delta}H$. Then, there exist finite covers $F = \bigsqcup_{i=1}^{n} F_i$ and $G = \bigsqcup_{j=1}^{m} G_j$ of F and G respectively such that $(\varphi_\psi)_\alpha(F_i)\overline{\delta_\alpha}(\varphi_\psi)_\alpha(G_j)$ for some $\alpha = s_{ij} \in \Delta$, where $i = 1, ..., n$ and $j = 1, ..., m$. In the same way, there are finite covers $F = \bigsqcup_{p=1}^{q} F_p'$ and $H = \bigsqcup_{j=m+1}^{m+l} G_j$ of F and H respectively such that $(\varphi_\psi)_\alpha(F_p')\overline{\delta_\alpha}(\varphi_\psi)_\alpha(G_j)$ for some $\alpha = t_{pj} \in \Delta$, where $p = 1, ..., q$ and $j = m + 1, ..., m + l$. Now, $F = \bigsqcup\{F_i \sqcap F_p' : i = 1, ..., n; p = 1, ..., q\}$ and $G \sqcup H = \bigsqcup\{G_j : j = 1, ..., m + l\}$ are finite covers of F and $G \sqcup H$, respectively. Hence, from the fact that $(\varphi_\psi)_\alpha(F_i \sqcap F_p')\overline{\delta_\alpha}(\varphi_\psi)_\alpha(G_j)$ for $\alpha = s_{ij}$ or $\alpha = t_{pj}$ it follows that $F\overline{\delta}(G \sqcup H)$.

(sp_5) Let us define the set Ω of all pairs (F, G) such that $F\overline{\delta}G$ and we have either $F\delta H$ or $G\delta(\widetilde{X} - H)$ for each $H \in S(X, E)$. The validity of (sp_5) will follow from the fact that Ω is empty. Suppose, on the contrary, that $(F, G) \in \Omega$. Then, $(\varphi_\psi)_\alpha(F)\delta_\alpha(\varphi_\psi)_\alpha(G)$ for each $\alpha \in \Delta$. Indeed, let $H' \in S(X_\alpha, E_\alpha)$ and $H = (\varphi_\psi)_\alpha^{-1}(H')$. If $F\delta H$, then $(\varphi_\psi)_\alpha(F)\delta_\alpha(\varphi_\psi)_\alpha(H)$. Because $(\varphi_\psi)_\alpha(H) \sqsubseteq H'$, we have $(\varphi_\psi)_\alpha(F)\delta_\alpha H'$. Similarly, if $G\delta(\widetilde{X} - H)$, then $(\varphi_\psi)_\alpha(G)\delta_\alpha(\widetilde{X_\alpha} - H')$. Hence, since δ_α is a proximity of soft sets on X_α, we obtain $(\varphi_\psi)_\alpha(F)\delta_\alpha(\varphi_\psi)_\alpha(G)$. Also, we observe that for each $(F, G) \in \Omega$ there are positive integers n, m and covers $F = \bigsqcup_{i=1}^{n} F_i$ and $G = \bigsqcup_{j=1}^{m} G_j$ such that for every pair $(i, j) \in \{1, ..., n\} \times \{1, ..., m\}$, there exists an $\alpha \in \Delta$ satisfying $(\varphi_\psi)_\alpha(F_i)\overline{\delta_\alpha}(\varphi_\psi)_\alpha(G_j)$. Let $l = n + m$. It easy to see that $l > 2$. Then, for each $(F, G) \in \Omega$ let us choose such an integer l. But l is not uniquely determined by (F, G). Let κ be the set of all integers corresponding to members of Ω and let l be the smallest member of κ. Take a $(F, G) \in \Omega$ such that l is the integer corresponding to it. Then, there are covers $F = \bigsqcup_{i=1}^{n} F_i$ and $G = \bigsqcup_{j=1}^{m} G_j$ such that $l = n + m$ and for every pair $(i, j) \in \{1, ..., n\} \times \{1, ..., m\}$ there exists an $\alpha \in \Delta$ satisfying $(\varphi_\psi)_\alpha(F_i)\overline{\delta_\alpha}(\varphi_\psi)_\alpha(G_j)$. One of the n, m is greater than 1. Consider $n > 1$ and let $F' = F_1 \sqcup ... \sqcup F_{n-1}$. In this case, one of the following conditions should be true:

(i) For every $H \in S(X, E)$, either $F'\delta H$ or $G\delta(\widetilde{X} - H)$,

(ii) For every $H \in S(X, E)$, either $F_n\delta H$ or $G\delta(\widetilde{X} - H)$. In fact, suppose that neither (i) nor (ii) holds. Then, there are $H_1, H_2 \in S(X, E)$ such that $F'\overline{\delta}H_1, G\overline{\delta}(\widetilde{X} - H_1)$ and $F_n\overline{\delta}H_2, G\overline{\delta}(\widetilde{X} - H_2)$. Letting $H = H_1 \sqcap H_2$, we obtain $F\overline{\delta}H$ and $G\overline{\delta}(\widetilde{X} - H)$, contradicting the fact that $(F, G) \in \Omega$. Suppose that (i) holds. Because $F' \sqsubseteq F$ and $F\overline{\delta}G$, this means that $F'\overline{\delta}G$. Hence by (i), we have $(F', G) \in \Omega$. But this is now a contradiction since $(n - 1) + m = l - 1 \in \kappa$, contrary to the choice of l. If (ii) holds, we get a contradiction in a similar way. Therefore, the set Ω is empty. Thus, δ is a proximity of soft sets on X.

It is easy to see that all mappings $(\varphi_\psi)_\alpha : (X, \delta, E) \to (X_\alpha, \delta_\alpha, E_\alpha)$ are soft proximity mapping. Let δ^* be another proximity of soft sets on X making each of the mappings $(\varphi_\psi)_\alpha : (X, \delta^*, E) \to (X_\alpha, \delta_\alpha, E_\alpha)$ soft proximity mapping. We shall show that $\delta < \delta^*$, which will complete the proof. Let $F\delta^*G$ and consider any covers $F = \bigsqcup_{i=1}^{n} F_i$ and $G = \bigsqcup_{j=1}^{m} G_j$ of F and G respectively. Since $F = (F_1 \sqcup ... \sqcup F_n)\delta^*G$, by ($sp_4$), there is an $i \in \{1, .., n\}$ such that $F_i\delta^*G$. In the same way, since $F_i\delta^*G = (G_1 \sqcup ... \sqcup G_m)$, by ($sp_4$), there is a $j \in \{1, .., m\}$ such that $F_i\delta^*G_j$. From the fact that all mappings $(\varphi_\psi)_\alpha : (X, \delta^*, E) \to (X_\alpha, \delta_\alpha, E_\alpha)$ are soft proximity mapping it follows that $(\varphi_\psi)_\alpha(F_i) \delta_\alpha (\varphi_\psi)_\alpha(G_j)$ for each $\alpha \in \Delta$. Thus, we get $F\delta G$. $\square$

Theorem 4.3. *A soft mapping* $\varphi_\psi : (Y, \delta^*, K) \to (X, \delta, E)$ *is a soft proximity mapping if and only if* $(\varphi_\psi)_\alpha \circ \varphi_\psi : (Y, \delta^*, K) \to (X_\alpha, \delta_\alpha, E_\alpha)$ *is a soft proximity mapping for every* $\alpha \in \Delta$.

Proof. The necessity is easy. We prove the sufficiency. Suppose that $(\varphi_\psi)_\alpha \circ \varphi_\psi$ is a soft proximity mapping for every $\alpha \in \Delta$. Let $F\delta^*G$ and let $\varphi_\psi(F) = \bigsqcup_{i=1}^{n} F_i$, $\varphi_\psi(G) = \bigsqcup_{j=1}^{m} G_j$. Then, we have

$$F \sqsubseteq \bigsqcup_{i=1}^{n} \varphi_\psi^{-1}(F_i) \quad and \quad G \sqsubseteq \bigsqcup_{j=1}^{m} (\varphi_\psi)^{-1} G_j.$$

Since $F\delta^*G$, by (sp_4), there exist i, j such that $\varphi_\psi^{-1}(F_i)\delta^*\varphi_\psi^{-1}(G_j)$. Because

$$(\varphi_\psi)_\alpha \circ \varphi_\psi \circ \varphi_\psi^{-1}(F_i) \sqsubseteq (\varphi_\psi)_\alpha(F_i) \quad and \quad (\varphi_\psi)_\alpha \circ \varphi_\psi \circ \varphi_\psi^{-1}(G_j) \sqsubseteq (\varphi_\psi)_\alpha(G_j)$$

it follows from our hypothesis that $(\varphi_\psi)_\alpha(F_i)\delta_\alpha(\varphi_\psi)_\alpha(G_j)$ for every $\alpha \in \Delta$. This shows that $\varphi_\psi(F)\delta\varphi_\psi(G)$. $\quad\square$

Definition 4.4. Let $\{(X_\alpha, \delta_\alpha, E_\alpha) : \alpha \in \Delta\}$ be a family of soft proximity spaces and let $X = \prod_{\alpha \in \Delta} X_\alpha$, $E = \prod_{\alpha \in \Delta} E_\alpha$ be product sets. An initial proximity structure $\delta = \prod_{\alpha \in \Delta} \delta_\alpha$ of soft sets on X with respect to all the soft projection mappings $(p_{X_\alpha})_{q_{E_\alpha}}$, where $p_{X_\alpha} : X \to X_\alpha$ and $q_{E_\alpha} : E \to E_\alpha$, is called the product proximity structure.

The triplet (X, δ, E) is said to be a product soft proximity space.

From Theorem 4.2 and Theorem 4.3, we obtain the following corollary.

Corollary 4.5. $\{(X_\alpha, \delta_\alpha, E_\alpha) : \alpha \in \Delta\}$ *be a family of soft proximity spaces. Let* $X = \prod_{\alpha \in \Delta} X_\alpha$ *and* $E = \prod_{\alpha \in \Delta} E_\alpha$ *be sets and for each* $\alpha \in \Delta$ *let* $(p_{X_\alpha})_{q_{E_\alpha}}$ *be a soft mapping. For any* $F, G \in S(X, E)$, *define* $F\delta G$ *iff for every finite families* $\{F_i : i = 1, ..., n\}$ *and* $\{G_j : j = 1, ..., m\}$ *where* $F = \bigsqcup_{i=1}^{n} F_i$ *and* $G = \bigsqcup_{j=1}^{m} G_j$, *there exist an* F_i *and a* G_j *such that* $(p_{X_\alpha})_{q_{E_\alpha}}(F_i)\delta_\alpha(p_{X_\alpha})_{q_{E_\alpha}}(G_j)$ *for each* $\alpha \in \Delta$. *Then:*

(i) $\delta = \prod_{\alpha \in \Delta} \delta_\alpha$ *is the coarsest proximity of soft sets on* X *for which all mappings* $(p_{X_\alpha})_{q_{E_\alpha}}$ ($\alpha \in \Delta$) *are soft proximity mapping.*

(ii) A soft mapping $\varphi_\psi : (Y, \delta^*, K) \to (X, \delta, E)$ *is a soft proximity mapping if and only if* $(p_{X_\alpha})_{q_{E_\alpha}} \circ \varphi_\psi : (Y, \delta^*, K) \to (X_\alpha, \delta_\alpha, E_\alpha)$ *is a soft proximity mapping for every* $\alpha \in \Delta$.

5. Conclusion

Each proximity space determines in a natural way a topological space with beneficial properties. Also, this theory possesses deep results, rich machinery and tools. With the development of topology, the theory of proximity makes a great progress. Hence, the concept of proximity have been studied by many authors in both the fuzzy setting and the soft setting. In the present work, we give some properties of soft proximity spaces. We present an alternative description of the concept of soft proximity spaces, which is called soft δ-neighborhood. Also, we have shown that each soft uniform space determines a soft proximity space. We believe that these notions will help the researchers to advance and promote the further study on soft proximity spaces.

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