Filomat 31:7 (2017), 1981–1984
DOI 10.2298/FIL1707981C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$U(X)$ as a Ring for Metric Spaces X

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Abstract. In this short paper, we will show that the space of real valued uniformly continuous functions defined on a metric space (X, d) is a ring if and only if every subset $A \subset X$ has one of the following properties:

- A is Bourbaki-bounded, i.e., every uniformly continuous function on X is bounded on A.
- A contains an infinite uniformly isolated subset, i.e., there exist $\delta > 0$ and an infinite subset $F \subset A$ such that $d(a, x) \geq \delta$ for every $a \in F, x \in X \setminus \{a\}$.

1. Introduction

Even when it is quite surprising, it seems that there is no characterization of metric spaces whose real valued uniformly continuous functions have ring structure. Everybody knows that $C(X)$ is a ring whenever X is a topological space, as well as $\mathrm{Lip}(X)$ is a ring if and only if X has finite diametre. The main result in this paper solves this lack of information about $U(X)$, but with a somehow disgusting statement. Recall the Atsuji characterization of UC spaces, where X' is the set of accumulation points in X ([1]):

Theorem 1.1. ([1]) *Let X be a metric space. Then $U(X) = C(X)$ if and only if X' is compact and every closed $I \subset X \setminus X'$ is uniformly isolated.*

An alternative way to state this result is:

Theorem 1.2. *Let X be a metric space. Then $U(X) = C(X)$ if and only if X' is compact and, for every $\delta > 0$, the set $I = \{x \in X : d(x, X') \geq \delta\}$ is uniformly isolated.*

Our ideal statement would be entirely analogous to the second one, writing "X' is Bourbaki-bounded" instead of "X' is compact", but we cannot ensure that $X \setminus X'$ will have this property. Namely, we have found two main problems. The first one is that there may be some sequences near X' formed by isolated points which do not affect to its Bourbaki-boundedness – we will put an example (see 2.8). This could be solved by changing X' by another Bourbaki-bounded A, and stating it as follows:

2010 *Mathematics Subject Classification*. Primary 54C10; Secondary 54C30, 54E40, 54D20
Keywords. Rings of uniformly continuous functions, Bourbaki-boundedness, uniform isolation
Received: 11 September 2015; Revised: 14 December 2015; Accepted: 16 December 2015
Communicated by Ljubiša D.R. Kočinac
The author was partially supported by MINECO (Ministerio de Economía y Competitividad) projects MTM2010-20190-C02-01 and MTM2013-45643-C02-01-P and by the Junta de Extremadura project GR10113
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Conjecture 1.3. *Let X be a metric space. Then $U(X)$ is a ring if and only if there exists a Bourbaki-bounded subset A such that for every $\delta > 0$, the set $\{x \in X : d(x, A) \geq \delta\}$ is uniformly isolated.*

The second problem is that we have not been able to show that this is true, we have just shown that if there exists such an A, then $U(X)$ is a ring – even this seems to be new since, in the recent paper [7], it is shown that $U(X)$ is a ring whenever $X = F \cup I$, where F is Bourbaki-bounded and I uniformly isolated.

The main problem in this paper has been widely studied (see, e.g., [1, 7, 8]) along with problems about characterizing the couples of uniformly continuous functions whom product remains uniformly continuous (again [1], but also [2, 3]).

2. Bourbaki-Bounded (Sub)spaces

We are about to explain some properties of Bourbaki-bounded spaces before coming up to the main result.

2.1. Given a metric space X, we will denote by $U(X)$ the set of real valued, uniformly continuous functions defined on X. The subset of bounded functions in $U(X)$ will be denoted by $U^*(X)$. Recall that $U^*(X)$ is always a ring.

Definition 2.2. A metric space X is said to be *Bourbaki-bounded* if $U(X) = U^*(X)$. $A \subset X$ is a Bourbaki-bounded subset if the map $f_{|A} : A \to \mathbb{R}$, $f_{|A}(x) = f(x)$ is bounded for every $f \in U(X)$.

2.3. Given $A \subset X$ and $\gamma > 0$, we will denote by A^γ the set of points whom distance to A is not greater than γ: $A^{1,\gamma} = A^\gamma = \{x \in X : d(x, A) \leq \gamma\}$. Inductively, $A^{k,\gamma} = \{x \in X : d(x, A^{k-1,\gamma}) \leq \gamma\}$

2.4. $\{U_j : j \in J\}$ is a *uniform* covering of X whenever there exists $\varepsilon > 0$ such that, for every $x \in X$ there is $j \in J$ such that $B(x, \varepsilon) \subset U_j$.

2.5. A covering $\{U_j : j \in J\}$ is said to be *star-finite* if $\{i \in J : U_i \cap U_j \neq \emptyset\}$ is finite for every $j \in J$.

Let us recall some characterizations of Bourbaki-bounded spaces:

Theorem 2.6. *Let (X, d) be a metric space. Then the following statements are equivalent:*

1. *X is a Bourbaki-bounded metric space;*
2. *For every metric space Y and every uniformly continuous function $f : X \to Y$, $f(X)$ is bounded in Y ([1]);*
3. *X is d'-bounded for every metric d' uniformly equivalent to d ([5]);*
4. *Every star-finite uniform cover of X is finite ([9]);*
5. *Every countable $B \subset X$ is a Bourbaki-bounded subset in X ([4]);*
6. *For every $\gamma > 0$, there exist $M_1, M_2 \in \mathbb{N}, x_1, x_2, \ldots, x_{M_1} \in X$ such that $X = \{x_1, x_2, \ldots, x_{M_1}\}^{M_2,\gamma}$ ([1]).*

When X fulfills any of the previous conditions, it is also said that X is finitely chainable, because of the last condition.

The following translations to Bourbaki-bounded subsets are easy to check:

Theorem 2.7. *Let $A \subset X$. Then the following statements are equivalent:*

1. *A is a Bourbaki-bounded subset;*
2. *For every metric space Y and every uniformly continuous function $f : X \to Y$, $f(A)$ is bounded in Y;*
3. *A is d'-bounded for every metric d' uniformly equivalent to d on X;*
4. *Every countable $B \subset A$ is a Bourbaki-bounded subset in X;*
5. *For every $\gamma > 0$, there exist $M_1, M_2 \in \mathbb{N}, x_1, x_2, \ldots, x_{M_1} \in X$ such that $A \subset \{x_1, x_2, \ldots, x_{M_1}\}^{M_2,\gamma}$.*

Example 2.8. *Consider B, the closed unit ball of l_2 and $\{e_n : n \in \mathbb{N}\}$ its usual basis. Let $X \subset l_2$ be given by $X = B \bigcup \{x_n^m : n \in \mathbb{N}, m = 1, \ldots, n\}$, where $x_n^m = \left(1 + \frac{m}{n}\right) e_n$. Then, X is Bourbaki-bounded.*

Proof. We will show that X is finitely chainable. Let $\gamma > 0$ and $k \in \mathbb{N}$ such that $\frac{1}{k} \leq \gamma < \frac{1}{k-1}$. Then, there are finitely many points in X such that $I(x) > \frac{1}{k}$ –namely, $\{x_n^m : n < k, m = 1, \ldots, n\}$. So we just need to *chain* $X_k = \{x \in X : I(x) \leq \frac{1}{k}\}$. Now, beginning at the origin 0, we have:

$A_0 = 0, A^{1,1/k} = B\left[0, \frac{1}{k}\right], A^{2,1/k} = B\left[0, \frac{2}{k}\right], \ldots, A^{k,1/k} = B\left[0, 1\right], A^{k+1,1/k} = B \cup \left\{x_n^m : d(x_n^m, B) \leq \frac{1}{k}\right\},$

$A^{k+2,1/k} = \left\{x_n^m : d(x_n^m, A^{k+1,1/k}) \leq \frac{1}{k}\right\}, \ldots, A^{3k} = X_k.$

Here, the worst subset to chain is $\{x_{2k-1}^m : m = 1, \ldots, 2k-1\}$, since every enlargement gives us just one more point than what we had. So, after A_k, we still need $2k-1$ enlargements to cover the whole space. In any case, we do not need more than $3k$ steps to get to any point in X_k from the origin. As $X \setminus X_k$ contains finitely many points, X is Bourbaki-bounded. $\square$

Remark 2.9. Please note that whenever X is a Bourbaki-bounded space, it is Bourbaki-bounded when considered as a subset of another metric space, but not every Bourbaki-bounded subset $A \subset X$ is a Bourbaki-bounded space.

Example 2.10. Let, again, $\{e_n : n \in \mathbb{N}\}$ be the usual basis of l_2. As shown in the previous example, it is a Bourbaki-bounded subset of X (every subset is), but it is not a Bourbaki-bounded space.

3. The Main Result

It is time to state everything properly.

3.1. For the sake of clarity, we must explicitly recall the notion of Atsuji isolation index: $I(x) = d(x, X \setminus \{x\}) = \inf\{d(x, y) : y \in X, y \neq x\}$ for every $x \in X$.

Definition 3.2. $A \subset X$ is uniformly isolated if $\inf\{I(a) : a \in A\} > 0$. This is equivalent to the existence of $\varepsilon > 0$ such that $d(a, x) \geq \varepsilon$ for every $a \in A, x \in X \setminus \{a\}$.

Lemma 3.3. ([6]) *For any $A \subset X$ and any $f_0 \in U^*(A)$, there exists $f \in U^*(X)$ such that $f_{|A} = f_0$.*

Remark 3.4. For any couple of sequences $(x_n), (y_n) \subset X$ such that $d(x_n, x_m) \geq \varepsilon > 0$, $0 < d(y_n, x_n) \leq \min\{\varepsilon/3, 1/n\}$ for every $n \neq m \in \mathbb{N}$ and any $\alpha_n \to 0$, the function

$g_0 : \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\} \to \mathbb{R},\ g_0(x_n) = \alpha_n, g_0(y_n) = 0$

is uniformly continuous. As it is bounded, too, the previous lemma shows that we can extend g_0 to $g \in U^*(X)$. This extension will be useful in the proof of the following result.

Theorem 3.5. *Let (X, d) be a metric space. Then, $U(X)$ is a ring if and only if every non Bourbaki-bounded $A \subset X$ contains an infinite uniformly isolated subset.*

Proof. **The "only if" implication:** Suppose, on the contrary, that there exists a non Bourbaki-bounded $A \subset X$ such that for every $\delta > 0$, $\{x \in A : I(x) \geq \delta\}$ is finite. Now, take $f \in U(X)$ unbounded on A and $(x_n) \subset X$ such that $f(x_n) \geq f(x_{n-1}) + 1 \geq n$, for every $n \in \mathbb{N}$. As $|f(x_n) - f(x_m)| \geq 1$ for every $m \neq n \in \mathbb{N}$ and f is uniformly continuous, there exists ε such that $d(x_n, x_m) \geq \varepsilon$ when $m \neq n$. As $I(x_n)$ tends to 0, we may take another sequence $(y_n) \subset X$ such that $d(x_n, y_n) \to 0$. The function $h_0 : \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\} \to [0, 1]$, defined by $h_0(x_n) = \frac{1}{n}, h_0(y_n) = 0$ is uniformly continuous and bounded. So, we may extend h_0 to $h \in U^(X)$ and it is pretty clear that $f \cdot h$ is not uniformly continuous, since

$$(f \cdot h)(y_n) = 0,\ (f \cdot h)(x_n) \geq 1,$$

so $U(X)$ is not a ring.

The "if" implication: Suppose $U(X)$ is not a ring. Then, there exist $f, g \in U(X)$ such that $f \cdot g \notin U(X)$, so there are $\varepsilon > 0$ and sequences $(x_n), (y_n) \subset X$ such that $d(x_n, y_n) \leq \frac{1}{n}$ and $|(f \cdot g)(x_n) - (f \cdot g)(y_n)| \geq \varepsilon$. As $f \cdot g$ is uniformly continuous whenever $f, g \in U^*(X)$, this implies that either f or g is unbounded on $A = \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$. As both $I(x_n), I(y_n)$ are not greater than $\frac{1}{n}$ because $d(x_n, y_n) \leq \frac{1}{n}$, A is a non Bourbaki-bounded subset such that $A \cap \{x : I(x) > \delta\}$ is finite for every $\delta > 0$. $\square$

Corollary 3.6. *Suppose there exists a Bourbaki-bounded subset $F \subset X$ such that $X \setminus (F^\gamma)$ is uniformly isolated for every $\gamma > 0$. Then $U(X)$ is a ring.*

Proof. It can be easily deduced from the above theorem, but we will give an alternative proof: we will show directly that $f \cdot g$ is uniformly continuous.

Suppose F is Bourbaki-bounded and such that $\{x \in X : d(x, F) \geq \gamma\}$ is uniformly isolated fo every γ and let $f, g \in U(X)$. Then, since F is Bourbaki-bounded, there exist $M, N \in \mathbb{N}$ such that $f_{|F} \leq M$ and $g_{|F} \leq N$. As both functions are uniformly continuous, there exist γ_f, γ_g such that $d(x, y) < \gamma_f$ implies $|f(x) - f(y)| < 1$ and $d(x, y) < \gamma_g$ implies $|g(x) - g(y)| < 1$. So, if we take $\gamma = \min\{\gamma_f, \gamma_g\}$, we have $\sup\{f(x) : x \in F^\gamma\} \leq M + 1$ and $\sup\{g(x) : x \in F^\gamma\} \leq N + 1$.

Now, there exists $\alpha > 0$ such that, $x \notin F^\gamma$ implies $I(x) \geq \alpha$ so, whenever $d(x, y) < \alpha$, both x and y must belong to F^γ.

We need to show that for every $\varepsilon > 0$ exists δ such that $d(x, y) < \delta$ implies $|g(x)f(x) - g(y)f(y)| < \varepsilon$. So let $\varepsilon > 0$ and take δ_f, δ_g such that $d(x, y) < \delta_f$ implies $|f(x) - f(y)| < \frac{\varepsilon}{2(M+1)}$ and $d(x, y) < \delta_g$ implies $|g(x) - g(y)| < \frac{\varepsilon}{2(N+1)}$. Now, taking $\delta = \min\left\{\alpha, \delta_f, \delta_g\right\}$, we obtain, for x, y such that $d(x, y) < \delta$:

$$|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| =$$
$$= |g(x)(f(x) - f(y))| + |f(y)(g(x) - g(y))| \leq (M + 1)\frac{\varepsilon}{2(M + 1)} + (N + 1)\frac{\varepsilon}{2(N + 1)} = \varepsilon,$$

and so, $fg \in U(X)$. □

We will put an example showing that this class of spaces does not agree with that given in [7], where the author shows that, if $X = F \cup I$, where F is Bourbaki-bounded and I is uniformly isolated, then $U(X)$ is a ring.

Example 3.7. *Let $X \subset l_2$ be given by $X = B[0, 1] \cup \left\{\left(1 + \frac{1}{m}\right) e_n : n, m \in \mathbb{N}\right\}$, where $\{e_n : n \in \mathbb{N}\}$ is the l_2 usual basis. Then $U(X)$ is a ring, but X is not the union of a Bourbaki-bounded and a uniformly isolated subsets.*

Proof. By the previous corollary, it is clear that $U(X)$ is a ring, taking $F = B[0, 1]$.

Now, suppose I is uniformly isolated in X. Then, by the very definition, we see that there exists $n \in \mathbb{N}$ such that $d(I, B[0, 1]) \geq \frac{1}{N}$. So, if $X = F \cup I$, then F must contain $\left\{\left(1 + \frac{1}{m}\right) e_n : n \in \mathbb{N}, m > N\right\}$. This subset contains infinitely many points with isolation index $\frac{1}{N^2+3N+2}$ –namely, every $\left(1 + \frac{1}{N+1}\right) e_n$. So, F is not Bourbaki-bounded in X. □

Conjecture 3.8. *It remains unclear, but we think the statement in corollary 3.6 is actually an equivalence.*

Acknowledgements

The author feels deeply grateful to Prof. Maribel Garrido, for several useful discussions, references and advices on this topic. It is also a pleasure to thank the referees for some useful comments which led to an easier proof of the main result.

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