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New Extensions of Cline's Formula for Generalized Inverses

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Abstract. In this paper, Cline's formula for the well-known generalized inverses such as Drazin inverse, pseudo Drazin inverse and generalized Drazin inverse is extended to the case when $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$ Also, applications are given to some interesting Banach space operator properties like algebraic, meromorphic, polaroidness and B-Fredholmness.

1. Introduction

For any associative ring R with identity 1, Jacobson's lemma states that if $1 - ab$ is invertible, then so is $1 - ba$ and

$$(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a.$$

The folklore proof of this result, which is usually ascribed to Jacobson, can be "formally" proceeded by writing

$$\begin{aligned}(1 - ba)^{-1} &= 1 + ba + baba + bababa + \cdots \\ &= 1 + b(1 + ab + abab + \cdots)a \\ &= 1 + b(1 - ab)^{-1}a,\end{aligned}$$

see [14] for details. Over the years, suitable analogues of Jacobson's lemma were found for many operator properties [2–6, 8, 20–23, 26, 30] and various kinds of generalized inverse [9, 12, 17, 24, 25, 34]. In 2013, Corach, Duggal and Harte [11] extended Jacobson's lemma and many of its relatives to the case when $aba = aca$. Take invertibility for example, in the presence of $aba = aca$, we see that if $1 - ac$ is invertible, then so is $1 - ba$ and

$$(1 - ba)^{-1} = 1 + b(1 - ac)^{-1}a.$$

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This can also be checked "formally" as follows

$$\begin{aligned}(1-ba)^{-1} &= 1+ba+baba+bababa+\cdots\\ &= 1+b(1+ac+acac+\cdots)a\\ &= 1+b(1-ac)^{-1}a.\end{aligned}$$

Straight after the authors established suitable analogues in this situation for many operator properties [31, 32]. But we also remark here that, for few operator properties, we have not yet find suitable analogues in this case (see [11, 31]). Very recently, Yan and Fang [28, 29] investigated a new extension of Jacobson's lemma and obtained many of its relatives in the case when

$$\begin{cases} acd = dbd\\ dba = aca.\end{cases}$$

It is obviously that the case $a = d$ gives $aba = aca$. Take invertibility for example again, in the presence of $acd = dbd$, we know that if $1 - ac$ is invertible, then so is $1 - bd$ and

$$(1-bd)^{-1} = 1+b(1-ac)^{-1}d.$$

This can be checked "formally" again. Combining with Jacobson's lemma and in the presence of $acd = dbd$, it is easily to see that if $1 - bd$ is invertible, then so is $1 - ac$ and

$$(1-ac)^{-1} = 1+a(1+c(1+d(1-bd)^{-1}b)a)c.$$

Corresponding to Jacobson's lemma, Cline discover in 1965 a fundamental relation between the Drazin invertibility of ab and ba. He showed that if ab is Drazin invertible, then so is ba and

$$(ba)^D = b((ab)^D)^2a.$$

Here we say that an element $a \in R$ is *Drazin invertible* [13] if there exists $s \in R$ such that

$$as = sa,\ sas = s \text{ and } a^ksa = a^k \text{ for some } k \geq 0.$$

In this case s is unique and denoted by $s = a^D$, the *Drazin inverse* of a, and the least non-negative integer k satisfying $a^ksa = a^k$ is called the *Drazin index* $i(a)$ of a. Drazin inverse is a class of important, widely-applied and uniquely-defined generalized inverse. The concept of Drazin inverse was extended until recently. Recall that an element $a \in R$ is said to be *generalized Drazin invertible* [16] if there exists $s \in R$ such that

$$s \in comm^2(a),\ sas = s \text{ and } asa - a \text{ is quasinilpotent,}$$

where $comm^2(a)$ is defined as usual by

$$comm^2(a) = \{x \in R, xy = yx \text{ for all } y \in R \text{ commuting with } a\}$$

and we say that an element $a \in R$ is *quasinilpotent* if $1 + ax$ is invertible for all $x \in R$ commuting with a. In this case s is unique if it exists and denoted by $s = a^{gD}$, the *generalized Drazin inverse* of a. An intermedium between Drazin inverse and generalized Drazin inverse was introduced in [27]: an element $a \in R$ is called *pseudo Drazin invertible* provided that there is a common solution to the equations

$$s \in comm^2(a),\ sas = s \text{ and } a^ksa - a^k \in J(R) \text{ for some } k \geq 0,$$

where $J(R)$ denotes the *Jacobson radical* of R. If such a solution exists, then it is unique and denoted by $s = a^{pD}$, the *pseudo Drazin inverse* of a, and the smallest non-negative integer k for which $a^ksa - a^k \in J(R)$ holds is called the *pseudo Drazin index* $i(a)$ of a.

Cline's formula for generalized Drazin inverse and pseudo Drazin inverse were found in [19] and [27] lately. Their proof are given through the bridges *quasipolar* and *pseudopolar*, respectively. Very recently, the authors [18, 33] established Cline's formula for Drazin inverse, pseudo Drazin inverse and generalized Drazin inverse in the case when $aba = aca$. In this paper, we extend further Cline's formula for the above three kinds of generalized inverse to the case when $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$ As corollaries, we show that operator products AC and BD share some interesting operator properties such as algebraic, meromorphic, polaroidness and B-Fredholmness in the case when $\begin{cases} ACD = DBD \\ DBA = ACA. \end{cases}$

2. Main Results

We begin with the following result, which extends Cline's formula for Drazin inverse to the case when $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$

Theorem 2.1. *Suppose that $a, b, c, d \in R$ satisfy $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$ Then*

$$ac \text{ is Drazin invertible} \Longleftrightarrow bd \text{ is Drazin invertible.}$$

In this case, we have
(1) $|i(ac) - i(bd)| \leq 2$;
(2) $(ac)^D = d((bd)^D)^3 bac$ *and* $(bd)^D = b((ac)^D)^2 d$.

Proof. Suppose that bd is Drazin invertible and let s be the Drazin inverse of bd and k its Drazin index. Then we have

$$s(bd) = (bd)s, \ s(bd)s = s \text{ and } (bd)^k s(bd) = (bd)^k.$$

Set $t = ds^3bac$. We get

$$t(ac) = ds^3bacac = ds^3bdbac = ds^2bac$$

and

$$(ac)t = (ac)ds^3bac = dbds^3bac = ds^2bac,$$

and thus

$$t(ac) = (ac)t.$$

Moreover,

$$t(ac)t = ds^3bac(ac)ds^3bac = ds^3bdbacds^3bac = ds^3bdbdbds^3bac = ds^3bac = t$$

and

$$\begin{aligned} (ac)^{k+2}t(ac) &= (ac)^{k+2}ds^3bac(ac) = d(bd)^{k+2}s^3bdb(ac) = d(bd)^k bac \\ &= (ac)^k dbac = (ac)^k acac = (ac)^{k+2}. \end{aligned}$$

Consequently, ac is Drazin invertible, $(ac)^D = d((bd)^D)^3bac$ and $i(ac) \leq i(bd) + 2$.

As the above arguments, it is easily to see that if ac is Drazin invertible, then so is bd, $(bd)^D = b((ac)^D)^2d$ and $i(bd) \leq i(ac) + 1$. □

Next we give an example to show that the difference of the Drazin index of products in Theorem 2.1 is optimal.

Example 2.2. Let A, B, C and D be operators, acting on separable Hilbert space $l_2(\mathbb{N})$, defined as follows respectively:

$$A(x_1, x_2, x_3, x_4, x_5, x_6, \cdots) = (x_2, x_3, 0, x_4, x_5, x_6, \cdots) \text{ for all } \{x_n\}_{n=1}^{\infty} \in l_2(\mathbb{N}),$$

$$B(x_1, x_2, x_3, x_4, x_5, x_6, \cdots) = (x_4, x_5, x_6, x_7, x_8, x_9, \cdots) \text{ for all } \{x_n\}_{n=1}^{\infty} \in l_2(\mathbb{N}),$$

$$C(x_1, x_2, x_3, x_4, x_5, x_6, \cdots) = (x_2, x_3, 0, x_4, x_5, x_6, \cdots) \text{ for all } \{x_n\}_{n=1}^{\infty} \in l_2(\mathbb{N}),$$

$$D(x_1, x_2, x_3, x_4, x_5, x_6, \cdots) = (0, 0, 0, x_1, x_2, x_3, \cdots) \text{ for all } \{x_n\}_{n=1}^{\infty} \in l_2(\mathbb{N}).$$

It is easy to verify that $\begin{cases} ACD = DBD \\ DBA = ACA. \end{cases}$ Noting that BD is the identity operator, we see that BD is Drazin invertible and its Drazin index $i(BD) = 0$. But the Drazin index of AC is equal to 2.

Throughout the sequel, $\mathcal{B}(X, Y)$ stands for the set of all bounded linear operators from Banach space X to Banach space Y. For $T \in \mathcal{B}(X) := \mathcal{B}(X, X)$, let $\mathcal{N}(T)$ denote its *kernel*, $\alpha(T)$ its *nullity*, $\mathcal{R}(T)$ its *range* and $\beta(T)$ its *defect*. The *ascent* and the *descent* of T are defined as

$$\text{asc}(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$$

and

$$\text{dsc}(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\},$$

respectively. It is well-known that if $\text{asc}(T)$ and $\text{dsc}(T)$ are both finite, then they are equal ([1, Theorem 3.3]). Moreover, being a Drazin invertible element in $\mathcal{B}(X)$ for T is equivalent to $\text{asc}(T) = \text{dsc}(T) < \infty$. In the following we give the operator case of Theorem 2.1.

Corollary 2.3. *Suppose that $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $\begin{cases} ACD = DBD \\ DBA = ACA. \end{cases}$ Then AC is Drazin invertible if and only if BD is Drazin invertible. In this case, we have*

$$(AC)^D = D((BD)^D)^3 BAC \text{ and } (BD)^D = B((AC)^D)^2 D.$$

Proof. Dilate A, B, C and D to be elements in the algebra $\mathcal{B}(X \oplus Y)$ as follows

$$\bar{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \in \mathcal{B}(X \oplus Y),$$

$$\bar{B} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(X \oplus Y),$$

$$\bar{C} = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(X \oplus Y)$$

and

$$\bar{D} = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \in \mathcal{B}(X \oplus Y)$$

Then by similar argument as in the proof of [19, Corollary 3.1] and using in particular Theorem 2.1, we obtain the desired result by a matrix calculation. □

Recall that an operator $T \in \mathcal{B}(X)$ is said to be *algebraic* if there exists non-zero complex polynomial p such that $p(T) = 0$; *meromorphic* if every non-zero spectral point is a pole of the resolvent of T; *polaroid* if every isolated spectral point is a pole of the resolvent of T.

Corollary 2.4. *Suppose that $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy* $\begin{cases} ACD = DBD \\ DBA = ACA. \end{cases}$ *Then*

(1) *AC is algebraic if and only if BD is algebraic;*
(2) *AC is meromorphic if and only if BD is meromorphic;*
(3) *AC is polaroid if and only if BD is polaroid.*

Proof. (1) and (2) Apply the proof of [33, Theorem 2.2], using in particular Corollary 2.3 and [28, Lemmas 2.3 and 2.4].

(3) Apply the proof of [33, Theorem 2.3], using in particular Corollary 2.3 and [28, Lemmas 2.3 and 2.4]. □

If both $\alpha(T)$ and $\beta(T)$ are finite, then $T \in \mathcal{B}(X)$ is said to be *Fredholm* and the *index* of T is then defined by $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. An operator $T \in \mathcal{B}(X)$ is called *B-Fredholm* if for some $n \in \mathbb{N}$ the range $\mathcal{R}(T^n)$ is closed and the restriction $T|_{\mathcal{R}(T^n)}$ of T to $\mathcal{R}(T^n)$ is Fredholm. In this case, [7, Proposition 2.1] enables us to define the *index* of T as the index of $T|_{\mathcal{R}(T^n)}$.

Corollary 2.5. *Suppose that $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy* $\begin{cases} ACD = DBD \\ DBA = ACA. \end{cases}$ *Then AC is B-Fredholm if and only if BD is B-Fredholm. In this case, we have*

$$\mathrm{ind}(AC) = \mathrm{ind}(BD).$$

Proof. When $X = Y$, the desired result follows by applying the proof of [33, Lemma 2.8] and using in particular Theorem 2.1. When $X \neq Y$, the desired result follows by dilating A, B, C and D as in Corollary 2.3, and then applying the proof of [33, Theorem 2.9] and using in particular [28, Lemmas 2.3 and 2.4]. □

Lemma 2.6. *Suppose that $a, b, c, d \in R$ satisfy* $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$ *Then*

$$ac \text{ is quasinilpotent} \Longleftrightarrow bd \text{ is quasinilpotent}.$$

Proof. Suppose that ac is quasinilpotent. Then for all $x \in R$ commuting with ac, $1 + xac$ is invertible. Let $y \in R$ be an element commuting with bd. Since

$$(dy^3bac)(ac) = (dy^3bdb)(ac) = (dbdy^3b)(ac) = (ac)(dy^3bac),$$

$1 + (dy^3bac)(ac)$ is invertible. Therefore by Jacobson's Lemma, we have

$$(1 + ybd)(1 - ybd + y^2bdbd) = 1 + y^3bdbdbd = 1 + y^3bacacd$$

is invertible. Thus, together with the fact $1 + ybd$ and $1 - ybd + y^2bdbd$ commute, $1 + ybd$ is invertible. Consequently, bd is quasinilpotent.

Conversely, suppose that bd is quasinilpotent. Then by [18, Lemma 2.2], we see that db is quasinilpotent. Then by similar arguments as the previous paragraph, we infer that ca is quasinilpotent. By [18, Lemma 2.2] again, we conclude that ac is quasinilpotent. □

In the following we extend Cline's formula for generalized Drazin inverse to the case when $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$

Theorem 2.7. *Suppose that $a, b, c, d \in R$ satisfy* $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$ *Then*

$$ac \text{ is generalized Drazin invertible} \Longleftrightarrow bd \text{ is generalized Drazin invertible}.$$

In this case, we have $(ac)^{gD} = d((bd)^{gD})^3bac$ and $(bd)^{gD} = b((ac)^{gD})^2d$.

Proof. Suppose that bd is generalized Drazin invertible and let $s = (bd)^{gD}$. Then

$$s \in comm^2(bd),\ s(bd)s = s \text{ and } (bd)s(bd) - bd \text{ is quasinilpotent.}$$

Put

$$t = ds^3bac.$$

In order to prove that $t = (ac)^{gD}$, it needs to show that

$$\text{(i) } t \in comm^2(ac),\ \text{(ii) } t(ac)t = t \text{ and (iii) } (ac)t(ac) - ac \text{ is quasinilpotent.}$$

(i) Let $r \in comm(ac)$. Then we have

$$rt = rds^3bac = rd(bdbds^5)bac = racacds^5bac = acacrds^5bac = d(bacrd)s^5bac. \tag{1}$$

Since

$$bd(bacrd) = bacacrd = bracacd = bracdbd = (bacrd)bd$$

and $s \in comm^2(bd)$, we have

$$(bacrd)s = s(bacrd). \tag{2}$$

Therefore, putting (2) into (1), we get

$$\begin{aligned} rt &= d(bacrd)s^5bac = ds^5(bacrd)bac \\ &= ds^5(bacra)cac = ds^5bacacacr \\ &= ds^5bdbdbacr = ds^3bacr \\ &= tr. \end{aligned}$$

(ii) We have $t(ac)t = ds^3bac(ac)ds^3bac = ds^3bdbdbds^3bac = ds^3bac = t$.

(iii) Let $a' = (1 - dsb)a$ and $b' = (1 - bds)b$. Then $b'd$ is quasinilpotent. Direct calculation shows that $a'cd = db'd$, $db'a' = a'ca'$ and $ac - (ac)^2t = a'c$. Therefore Lemma 2.6 implies that $ac - (ac)^2t$ is quasinilpotent. Consequently, $(ac)^{gD} = d((bd)^{gD})^3bac$.

As the above arguments, it is easily to see that if ac is generalized Drazin invertible, then so is bd and $(bd)^{gD} = b((ac)^{gD})^2d$. □

At last, Cline's formula for pseudo Drazin inverse is extended to the case when $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$

Theorem 2.8. *Suppose that $a, b, c, d \in R$ satisfy $\begin{cases} acd = dbd \\ dba = aca. \end{cases}$ Then*

$$ac \textit{ is pseudo Drazin invertible} \iff bd \textit{ is pseudo Drazin invertible.}$$

In this case, we have

(1) $|i(ac) - i(bd)| \le 2$;

(2) $(ac)^{pD} = d((bd)^{pD})^3bac$ and $(bd)^{pD} = b((ac)^{pD})^2d$.

Proof. Let s be the pseudo Drazin inverse of bd and let $k = i(bd)$. Then

$$s \in comm^2(bd),\ s(bd)s = s \text{ and } (bd)^ks(bd) - (bd)^k \in J(R).$$

Put

$$t = ds^3bac.$$

As in the proof of Theorem 2.7, we get $t \in comm^2(ac)$ and $t(ac)t = t$. Moreover, since $(bd)^k s(bd) - (bd)^k \in J(R)$,

$$\begin{aligned}(ac)^{k+2}t(ac) - (ac)^{k+2} &= (ac)^{k+2}ds^3bacac - (ac)^{k+2}\\ &= d(bd)^{k+2}s^3bdbac - (db)^{k+1}ac\\ &= d(bd)^{k+1}sbac - d(bd)^k bac\\ &= d((bd)^{k+1}s - (bd)^k)bac \in J(R).\end{aligned}$$

Therefore, ac is pseudo Drazin invertible, $(ac)^{pD} = d((bd)^{pD})^3bac$ and $i(ac) \leq i(bd) + 2$.

By similar arguments as above, one can show that if ac is pseudo Drazin invertible, then so is bd, $(bd)^{pD} = b((ac)^{pD})^2d$ and $i(bd) \leq i(ac) + 1$. $\square$

We conclude this paper by an example to illustrate that the results obtained in this paper are proper generalizations of the corresponding ones in [18, 33].

Example 2.9. For Banach spaces X and Y, let $S_1 \in \mathcal{B}(Y, X), T_1 \in \mathcal{B}(Y, X)$ and $T_2 \in \mathcal{B}(X, Y)$ be operators satisfying $S_1(T_2T_1 - I) \neq 0$ and let $S_2 = S_1T_2$. We consider $A, B, C, D \in \mathcal{B}(X \oplus Y)$ as follows:

$$A = \begin{pmatrix} 0 & S_1 \\ 0 & 0 \end{pmatrix}, \quad B = C = \begin{pmatrix} I & T_1 \\ T_2 & I \end{pmatrix}, \quad D = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Evidently, $CDC - CAC = \begin{pmatrix} 0 & S_1(T_2T_1 - I) \\ 0 & T_2S_1(T_2T_1 - I) \end{pmatrix} \neq 0$, but $ACD = DBD = \begin{pmatrix} S_2^2 & 0 \\ 0 & 0 \end{pmatrix}$ and $DBA = ACA = \begin{pmatrix} 0 & S_2S_1 \\ 0 & 0 \end{pmatrix}$. Hence, the common Drazin invertibility (resp. pseudo Drazin invertibility and generalized Drazin invertibility) of AC and CD can only be deduced directly from the results obtained in this paper, and not from the corresponding ones in [18, 33].

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