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A Common Fixed Point Theorem for Semigroups of Nonlinear Uniformly Continuous Mappings with an Application to Asymptotic Stability of Nonlinear Systems

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Abstract. In this paper, we study the existence of a common fixed point for uniformly continuous one parameter semigroups of nonlinear self-mappings on a closed convex subset C of a real Banach space X with uniformly normal structure such that the semigroup has a bounded orbit. This result applies, in particular, to the study of an asymptotic stability criterion for a class of semigroup of nonlinear uniformly continuous infinite-dimensional systems.

1. Introduction and Preliminaries

Three fundamental theorems concerning fixed points are Brouwer's, Schauder and Banach. Brouwer theorem states that every continuous function on a ball B in R^n into itself has a fixed point. This theorem simply guarantees the existence of a solution, but gives no information about the uniqueness of the solution. In Schauder fixed point theorem, if B is a compact, convex subset of a Banach space X and $f : B \to B$ is a continuous function, then f has a fixed point. A very interesting useful result in fixed point theory is due to Banach known as the Banach contraction principle and is one of the most important theorem in classical functional analysis. This theorem has several generalizations either by extending the contraction mapping, generalizing the completeness or occasionally even both.

On the other hand, normal structure is one of the basic concepts in metric fixed point theory. It was introduced by Brodskii and Milman in [6]. In 1980, Bynum [7] introduced the normal structure coefficient $N(X)$ which was applied by Casini and Maluta [9] to obtain a fixed point theorem for uniformly lipschitzian mappings. The important application of normal structure is in fixed point theory and other fields related to the existence of solutions of differential equations and integral equations etc.

Let C be a nonempty bounded subset of a Banach space X. Then a point $x_0 \in C$ is said to be
(i) a diametral point of C if

$$\sup\{\|x_0 - x\| : x \in C\} = \sup\{\|x - y\| : x, y \in C\} = \delta(C),$$

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where $\delta(C)$ denotes the diametral of C.
(ii) a nondiametral point of C if

$$\sup\{\|x_0 - x\| : x \in C\} < \sup\{\|x - y\| : x, y \in C\}.$$

A nonempty convex subset C of a Banach space X is said to have normal structure if each convex bounded subset D of C with at least two points contains a nondiametral point, i.e., there exists $x_0 \in D$ such that

$$\sup\{\|x_0 - x\| : x \in D\} < \sup\{\|x - y\| : x, y \in D\}.$$

Definition 1.1. [7] Let E be a nonempty subset of a Banach space X and let F be a nonempty family of subsets of E. The family F is said to be normal structure provided that $r_C(C) < \delta(C)$(where $r_C(C) = \inf_{x\in C} \sup_{y\in C} \|x-y\|$ is Chebyshev radius of C relative to it self) for every bounded set $C \in F$ with $\delta(C) > 0$. If there exists a constant $0 < c < 1$ such that $r_C(C) \leq c.\delta(C)$ for every bounded set $C \in F$ with $\delta(C) > 0$, then F is said to be uniformly normal structure.
Definition 1.2. (**Normal structure coefficient**) [7] Let X be a Banach space. Then the number $N(X)$ is said to be the normal structure coefficient if

$$N(X) = \inf\Big\{\frac{\delta(C)}{r_C(C)}\Big\},$$

Where the infimum is taken over all closed convex bounded subsets C of X with $r_C(C) > 0$. It is clear that $N(X) \geq 1$.
Remark 1.1. $N(X) > 1$ if and only if X has uniformly normal structure.
It is known that a Banach space with uniformly normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniformly normal structure (see, e.g., [28]). It is also been compucated that $N(H) = \sqrt{2}$ for a Hilbert spaces H. The computations of the normal structure coefficient $N(X)$ for general Banach spaces look however complicated. No exact values of $N(X)$ are known except for some special cases (e.g., Hilbert and L^p spaces). In general, we have the following lower bounded for $N(X)$ (see [3, 7, 19])

$$N(X) \geq \frac{1}{1 - \delta_X(1)}.$$

Other lower bounds for $N(X)$ in terms of some Banach space parameters or constants can be found in [15, 25].

Assume that X is a real Banach space with uniformly normal structure and C is a nonempty closed convex subset of X. A mapping $T : C \to C$ is said to be a Lipschitzian mapping if, for each integer $n \geq 1$, there exist a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C.$$

A Lipschitzian mapping is said to be a k-uniformly Lipschitzian mapping if $k_n = k$ for all $n \geq 1$.

In 1973, Goebel and Kirk [13] then posed the question whether or not the constant $\gamma > 1$ which solves the equation

$$(1 - \delta_X(1/\gamma))\gamma = 1, \tag{1}$$

is the largest number for which any k-uniformly Lipschitzian mapping T with $k < \gamma$ has a fixed point, δ_X being the modulus of convexity of X.

Particularly, in 1993, Tan and Xu [27] answered the question of Goebel and Kirk [13] mentioned above in the negative by proving the following theorem:
Theorem 1.1. ([27], Theorem 3.5) Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X, and $\tau = \{T_s : s \in G : G$ be an unbounded subset of $[0, \infty)$ $\}$ a k-uniformly Lipschitzian semigroup on C with $k < \alpha$, where $\alpha > 1$ is the unique solution of the equation

$$\frac{\alpha^2}{N(X)}\delta_X^{-1}(1 - \frac{1}{\alpha}) = 1, \tag{2}$$

where $N(X) > 1$ is the normal structure coefficient of X. Suppose there exists an $x_0 \in C$ such that the orbit $\{T_s x_0 : s \in G\}$ is bounded. then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

In 2010, Ceng, Xu and Yao [10], studied the existence of fixed points of uniformly Lipschitzian semigroups $\tau = \{T_s : s \in G\}$ in the setting of Banach spaces X under conditions weaker than uniform convexity. More precisely, they replaced the uniform convexity of X in Theorem 1.1 with the weaker condition of the uniformly normal structure of X.

In this paper we will introduce a new common fixed point theorem of uniformly continuous semigroups of self-mappings in Banach space with uniform normal structure. Our result extends the result due to Ceng Xu and Yao [10, Theorem 3.1]. Also, we will use this theorem to study asymptotic stability for a class of infinite-dimensional nonlinear uniformly continuous systems on Banach state space.

We begin with some notations and preliminaries.

Recall that X is strictly convex if its unit sphere does not contain any line segments, that is, X is strictly convex if and only if the following implication holds:

$$x, y \in X, \ \ \|x\| = \|y\| = 1 \text{ and } \|(x+y)/2\| = 1 \ \ \Rightarrow x = y.$$

In order to measure the degree of convexity of X, we define its modulus of convexity $\delta_X : [0,2] \to [0,1]$ by

$$\delta_X(\varepsilon) = \inf\{1 - \|(x+y)/2\| : \|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x-y\| \geq \varepsilon\}.$$

The characteristic of convexity of X is the number $\varepsilon_0(X) = \sup\{\varepsilon : \delta_X(\varepsilon) = 0\}$. It is easy to see [13] that X is uniformly convex iff $\varepsilon_0(X) = 0$; uniformly nonsquare iff $\varepsilon_0(X) < 2$; and strictly convex iff $\delta(2) = 1$. Moreover, if $\varepsilon_0(X) < 1$; then X has a normal structure.

The following properties of modulus of convexity X are quite well-known (see [14])

(a) δ_X is increasing on [0,2], and moreover strictly increasing on $[\varepsilon_0, 2]$;

(b) δ_X is continuous on $[0,2)$ (but not necessarily at $\varepsilon = 2$);

(c) $\delta_X(2) = 1$ iff X is strictly convex;

(d) $\delta_X(0) = 0$ and $\lim_{\varepsilon \to 2^-} \delta_X(\varepsilon) = 1 - \varepsilon_0/2$

(e) $[\|a-x\| \leq r, \|a-y\| \leq r \text{ and } \|x-y\| \geq \varepsilon] \ \Rightarrow \ \|a-(x+y)/2\| \leq r(1 - \delta_X(\varepsilon/r))$.

We need the notion of asymptotic centers, due to Edelstein [12]. Let C be a nonempty closed convex subset of a Banach space X and let $\{x_t : t \in G\}$ be a bounded net of elements of X. Then the asymptotic radius and asymptotic center of $\{x_t\}_{t\in G}$ with respect to C are the number

$$r_C\{x_t\} = \inf_{y \in C} \limsup_t \|x_t - y\|,$$

and respectively, the (possibly empty) set

$$A_C(\{x_t\}) = \{y \in C : \limsup_t \|x_t - y\| = r_C(\{x_t\}).$$

Lemma 1.1. ([27], Lemma 2.1) If C is a nonempty closed convex subset of a reflexive Banach space X, then for every bounded net $\{x_t\}_{t\in G}$ of elements of X, $A_C(\{x_t\})$ is a nonempty bounded closed convex subset of C. In particular, if X is a uniformly convex Banach space, then $A_C(\{x_t\})$ consists of a single point.

Lemma 1.2.([27], Lemma 2.2) Suppose X is a Banach space with uniformly normal structure. Then for every bounded net $\{x_t\}_{t\in G}$ of elements of X there exists $y \in \overline{co}(\{x_t : t \in G\})$ such that

$$\limsup_t \|x_t - y\| \leq \widetilde{N}(X) D(\{x_t\}),$$

where $\widetilde{N}(X) = 1/N(X)$, $\overline{co}(E)$ is the closure of the convex hull of a set $E \subset X$ and $D(\{x_t\}) = \lim_t(\sup\{\|x_i - x_j\| : t \leq i, j \in G\})$ is the asymptotic diameter of $\{x_t\}$.

For an early application of the notion of asymptotic centers to fixed point theory, the reader might be referred to the paper by S. Reich [26].

2. Fixed Point of Semigroup of Uniformly Continuous Mappings

In this section, we shall be concerned with a special kind of one parameter semigroup of self-mappings on a closed convex convex subset of a Banach space with normal structure, namely, the uniformly continuous semigroups.

Definition 2.1. Let C be a closed convex subset of a Banach space X. Then the collection $\tau = \{T_s : s \in G\}$ of mappings on C into itself is said to be uniformly continuous semigroup on C if the following conditions are satisfied:

(i) $T_{s+t}x = T_sT_tx$ for all $s, t \in G$ and $x \in C$;

(ii) $\forall\ x \in C$, the mapping $t \to T_tx$ from G into C is continuous when G has the relative topology of $[0, \infty)$;

(iii) for each $t \in G$, $T_t : C \to C$ is uniformly continuous on C.

Remark 2.1. Every Lipschitzian semigroup on C is uniformly continuous on C but the converse my not be true.

Example 2.1. Let $\{T_t : t \in \{0, 1\}\}$ be a family of mappings defined as follows:

$$T_tx = \begin{cases} \sqrt{x}, & t = 1, \\ x, & t = 0, \end{cases}$$

where $x \in [0, \infty)$. Then T_t is uniformly continuous but not Lipschitzian.

Theorem 2.1. Suppose X is a real Banach space with with uniformly normal structure, C is a nonempty closed convex subset of X, and $\tau = \{T_s : s \in G\}$ is a uniformly continuous semigroup on C and

$$N(X) > \frac{2c}{1-c}, \quad 0 < c < 1. \tag{3}$$

If $\{T_sx_0 : s \in G\}$ is bounded for some $x_0 \in C$, then there exists $z \in C$ such that $T_sz = z$ for all $s \in G$.

Proof. Put $\widetilde{N}(X) = N(X)^{-1}$. Since X has a uniformly normal structure, X is reflexive. Due to the boundedness of $\{T_sx_0 : s \in G\}$ and by Lemma 1.1, we get that $A_C(\{T_tx_0\}_{t\in G})$ is nonempty bounded closed convex subset of C. Then we can choose $x_1 \in A_C(\{T_tx_0\}_{t\in G})$ such that

$$\limsup_t \|T_tx_0 - x_1\| = \inf_{y\in C} \limsup_t \|T_tx_0 - y\|.$$

Consequently we can choose $x_2 \in A_C(\{T_tx_1\}_{t\in G})$ such that

$$\limsup_t \|T_tx_1 - x_2\| = \inf_{y\in C} \limsup_t \|T_tx_1 - y\|.$$

Continuing this process, we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in C with the properties:

(i) for each $n \geq 0$, $\{T_tx_n\}_{t\in G}$ is bounded;

(ii) for each $n \geq 0$, $x_{n+1} \in A_C(\{T_tx_n\}_{t\in G})$; that is x_{n+1} is a point in C such that

$$\lim_t \|T_tx_n - x_{n+1}\| = \inf_{y\in C} \lim_t \|T_tx_n - y\|.$$

Write $r_n = r_C(\{T_tx_n\}_{t\in G})$. Then by Lemma 1.2 we have

$$\begin{aligned} r_n &= \limsup_t \|T_tx_n - x_{n+1}\| \leq \widetilde{N}(X)D(\{T_tx_n\}_{t\in G}) \\ &= \widetilde{N}(X) \lim_t(\sup\{\|T_ix_n - T_jx_n\| : t \leq i, j \in G\}) \\ &\leq \widetilde{N}(X) \limsup_t\{\|x_n - T_ix_n\| + \|T_jx_n - x_n\| : t \leq i, j \in G\} \\ &\leq 2\widetilde{N}(X)d(x_n), \end{aligned}$$

that is

$$r_n \leq 2\widetilde{N}(X)d(x_n), \tag{4}$$

where $d(x_n) = \sup\{\|x_n - T_t x_n\| : t \in G\}$.
We may assume that $d(x_n) > 0$ for all $n \geq 0$ (since otherwise x_n is a common fixed point of the semigroup τ and the proof is finished). Let $n \geq 0$ be fixed and let $\varepsilon > 0$ be small enough. We can choose $j \in G$ such that

$$\|T_j x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \varepsilon$$

and then choose $s_0 \in G$ so large that

$$\|T_s x_n - x_{n+1}\| < r_n + \varepsilon, \ \ \forall \ \ s \geq s_0,$$

It turns out, for $s \geq s_0 + j$,

$$\|T_s x_n - T_j x_{n+1}\| \leq \|T_s x_n - x_{n+1}\| + \|x_{n+1} - T_j x_{n+1}\|]$$

Hence

$$\|T_s x_n - T_j x_{n+1}\| \leq r_n + \varepsilon + d(x_{n+1}).$$

Then it follows from property (e) that

$$\|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \leq (r_n + \varepsilon + d(x_{n+1}))\Big(1 - \delta_X\Big(\frac{d(x_{n+1}) - \varepsilon}{r_n + \varepsilon + d(x_{n+1})}\Big)\Big)$$

for $s \geq s_0 + j$ and hence

$$r_n \leq \limsup_s \|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \leq (r_n + \varepsilon + d(x_{n+1}))\Big(1 - \delta_X\Big(\frac{d(x_{n+1}) - \varepsilon}{r_n + \varepsilon + d(x_{n+1})}\Big)\Big).$$

Taking the limit as $\varepsilon \to 0$, we obtain

$$r_n \leq (r_n + d(x_{n+1}))\Big(1 - \delta_X\Big(\frac{d(x_{n+1})}{r_n + d(x_{n+1})}\Big)\Big).$$

This implies that

$$0 \leq \delta_X\Big(\frac{d(x_{n+1})}{r_n + d(x_{n+1})}\Big) \leq \frac{d(x_{n+1})}{r_n + d(x_{n+1})} < 1, \tag{5}$$

then there exist a real number $c < 1$ such that

$$\frac{d(x_{n+1})}{r_n + d(x_{n+1})} < c < 1.$$

Hence

$$d(x_{n+1}) < \frac{c}{1-c} r_n. \tag{6}$$

Therefore, utilizing (4) and (6), we obtain

$$d(x_{n+1}) < 2\widetilde{N}(X)\frac{c}{1-c} d(x_n). \tag{7}$$

Write $A = 2\widetilde{N}(X)\frac{c}{1-c}$. Then $A < 1$. Hence, it is follows from (7) that

$$d(x_n) < Ad(x_{n-1}) < ... < A^n d(x_0). \tag{8}$$

Since

$$\|x_{n+1} - x_n\| \leq \limsup_t \|T_t x_n - x_{n+1}\| + \limsup_t \|T_t x_n - x_n\| \leq r_n + d(x_n) < 3\, d(x_n).$$

We get from (8) that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| < \infty$, and hence $\{x_n\}$ is a norm-Cauchy. Let $z = \|.\| - \lim_n x_n$. Finally, we have for each $s \in G$ and by the continuity of T_s,

$$\|z - T_s z\| \leq \lim_{n\to\infty} d(x_n) \to 0 \text{ as } n \to \infty.$$

Hence, $T_s z = z$ for all $s \in G$ and the proof is complete.

Since every Lipschitzian mapping is uniformly continuous then one can obtain the following corollary:
Corollary 2.1. ([10], Theorem 3.1) Suppose X is areal Banach space with $N(X) > \max(1, \varepsilon_0)$, C is a nonempty closed convex subset of X, and $\tau = \{T_s; s \in G\}$ is a uniformly Lipschitzian semigroup on C with Lipschitz constant $k < \alpha_*$. Here ε_0 is the characteristic of convexity of X and

$$\alpha_* = \sup\left\{\alpha > 1 : \alpha^2 \delta_X^{-1}(1 - \frac{1}{\alpha})N(X)^{-1} \leq 1 \textit{ and } 1 - \frac{1}{\alpha} \in (0, 1 - \frac{1}{2}\varepsilon_0)\right\}.$$

If $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in C$, then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.
Example 2.2. Consider $X = [0, 1]$ equipped with the usual norm. Let F be the set of closed subsets of X. Define the function T_t by

$$T_t x = xe^{-t}, \ t \geq 0,$$

for all $x \in X$. Then the following statements are holds
1- F is uniformly normal;
2- the normal structure coefficient $N(X) > 1$;
3- T_t is bounded uniformly continuous semigroup on X.
Hence, all the conditions of Theorem 2.1 are satisfied and 0 is a fixed point of T_t.

3. An Application

Many research works are concerned with asymptotic stability of infinite dimensional systems and applications to many classes of partial differential equations: see e.g [4, 16, 20, 21, 23, 24] and many other references. The fundamental theory of stability was established by the Russian scientist Aleksander Mikhailovich Lyapunov, is extensively developed for finite-dimensional systems. Many results on the asymptotic behavior of nonlinear infinite-dimensional systems are known, for which the dissipativity property plays an important role, see e.g [5, 8, 11, 18, 22]. In 2007 and 2010, Aksikas et al.[1, 2] developed a theory concerning the asymptotic stability of a class of nonlinear and semilinear infinite-dimensional Banach state space (distributed parameter) systems. They used the strict dissipativity as additional condition of the state operator A_T to prove that the ω-limit set reduces to the singleton set.

In this section we present some preliminaries and basic concepts on nonlinear semigroup theory. Also, we use Theorem 2.1 to introduce asymptotic stability criterion of nonlinear uniformly continuous semigroup in Banach space X with uniform normal structure.

The infinitesimal generator A_T of the nonlinear uniformly continuous semigroup T_t on a nonempty closed subset D of X is defined on its domain

$$D(A_T) = \{x \in D : \lim_{t\to 0} t^{-1}[T_t x - x] \text{ exists }\}$$

by

$$A_T x = \lim_{t\to 0^+} t^{-1} \ [T_t x - x], \ \forall \ x \in D(A_T).$$

For any $x_0 \in D(A)$, $x(t, x_0) = T_t x_0$ is the unique solution of the following nonlinear abstract Cauchy problem

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & Ax(t), \quad t > 0 \\ x(0) & = & x_0, \end{array}\right. \tag{9}$$

Definition 3.1. Consider the system (9) and assume that A generates a nonlinear uniformly continuous semigroup T_t. Consider an equilibrium point $\bar{x}$ of (9), i.e. $\bar{x} \in D(A)$ and $A\bar{x} = 0$. $\bar{x}$ is an asymptotically stable equilibrium point of (9) on D if

$$\forall\ x_0 \in D \lim_{t\longrightarrow\infty} x(t, x_0) := \lim_{t\longrightarrow\infty} T_t x_0 = \bar{x}.$$

The $\omega-$limit set $\omega(x_0)$ of x_0 is the set of all states in D which are the limits at infinity of converging sequences in the orbit through x_0 i.e more specifically $x \in \omega(x_0)$ if and only if $x \in D$ and there exists a sequence $t_n \longrightarrow \infty$ such that

$$x = \lim_{n\longrightarrow\infty} T_{t_n} x_0.$$

The following propositions shall play an important role in our main application theorem.

Proposition 3.1. [1] For any $x_0 \in D$, if $\omega(x_0)$ is nonempty and if $\overline{x}$ is a fixed point of T_t, i.e $T_t\overline{x} = \overline{x}$ for all $t \geq 0$, then

$$\omega(x_0) \subset \{z :\| z - \overline{x} \|= r\},\ \ with\ \ r \leq\| x_0 - \overline{x} \| .$$

Remark 3.1. We note from the Definition 3.1 and Proposition 3.1 that if $\bar{x}$ fixed point of T_t then it is equilibrium point of A.

Example 3.1. Let $X = [0, 1]$ and $T_t x = xe^t$. Then $T_t 0 = 0$ and $A0 = 0$ i.e. $x = 0$ fixed point of T_t and equilibrium point of A.

Proposition 3.2. Let T_t be a nonlinear uniformly continuous on $D = \overline{D(A_T)}$ (closer of $D(A_T)$), generated by A_T. Then $\omega(x_0) = \omega(x)\ \forall\ x \in \omega(x_0)$.

Proof. Fix $x \in \omega(x_0)$, say $x = \lim_{n\to\infty} T_{t_n} x_0$ with $t_n \to \infty$ as $n \to \infty$. Suppose now $y \in \omega(x_0)$, say $y = \lim_{n\to\infty} T_{\omega_n} x_0$ with $\omega_n \to \infty$ as $n \to \infty$. We may assume without loss of generality that $s_n = \omega_n - t_n \geq n,\ n = 1, 2, ...,$ since

$$\|T_{s_n} x - y\| \leq \|T_{s_n} x - T_{s_n+t_n} x_0\| + \|T_{s_n+t_n} x_0 - y\| \to\ 0\ \ as\ \ n \to \infty$$

Hence, $y = \lim_{n\to\infty} T_{s_n} x$ which implies that $y \in \omega(x)$, then

$$\omega(x_0) \subset \omega(x). \tag{10}$$

Similarly , one can deduce that

$$\omega(x) \subset \omega(x_0). \tag{11}$$

From (10) and (11) we have

$$\omega(x_0) = \omega(x).$$

Proposition 3.3. Let us consider the system (9) and let $T(t)$ be the nonlinear uniformly continuous semigroup on $D = \overline{D(A_T)}$, generated by A_T. Then

$$\omega(x_0) \subset D(A_T).$$

Proof. Without loss of generality assume that $\overline{x} = 0$. Consider any $x_0 \in D(A_T)$. Let $x \in \omega(x_0)$. By Proposition 3.2, $\omega(x_0) = \omega(x)$. There is $s_n \to \infty$ as $n \to \infty$ such that $x = \lim_{n\to\infty} T_{s_n} x$. Since $\omega(x_0)$ is closed convex subset Banach space with uniform normal structure. Then applying Theorem 2.1 we obtain that

$$T_t x = x\ \ \forall\ \ t \geq 0. \tag{12}$$

Then, $\lim_{t\to 0^+} \frac{1}{t}[T_t x - x]$ exists. Hence, $x \in D(A_T)$.

Proposition 3.4. Let us consider the system (9) and assume that A generates a nonlinear uniformly continuous semigroup T_t. Consider an equilibrium point $\overline{x}$ of (9), i.e $\overline{x} \in D(A)$ and $A\overline{x} = 0$. Then

$$\omega(x_0) = \{0\}.$$

Proof. Suppose that $x, y \in \omega(x_0), x \neq y$, since

$$\|x - y\| = \|x - T_{t_n}x + T_{t_n}x - y\| \leq \|x - T_{t_n}x\| + \|T_{t_n}x - y\| \to 0 \quad as \quad n \to \infty \tag{13}$$

Which implies $x = y$ and since $\overline{x} \in \omega(x_0)$, then we have $x = y = \overline{x}$ i.e., $\omega(x_0) = \{0\}$.

Now, we are in a position to state the following asymptotic stability theorem which is our main goal in this section.

Theorem 3.1. Let us consider the system (9) and let T_t be the nonlinear uniformly continuous semigroup on a subset $D = \overline{D(A_T)}$ of a Banach space X with uniform normal structure, generated by A_T. Assume that $\overline{x}$ is an equilibrium point of (9) and $(I - \lambda A_T)^{-1}$ is compact for some $\lambda > 0$. Then $x(t, x_0) \to \overline{x}$ as $t \to \infty$ i.e. $\overline{x}$ is an asymptotically stable equilibrium point of (9) on D.

Proof. Without loss of generality assume that $\overline{x} = 0$. Suppose that $x_0 \in D(A_T)$. By Proposition 3.3, $\omega(x_0) \subset D(A_T)$. By Proposition 3.4, $\omega(x_0) = \{0\}$. Now let us consider $x_0 \in D$ (not necessarily in $D(A_T)$). Let $\epsilon > 0$ be arbitrarily fixed. By the density of $D(A_T)$ in D, there exists $y_0 \in D(A_T)$ such that

$$\|x_0 - y_0\| < \epsilon. \tag{14}$$

Since $y_0 \in D(A_T)$, it follows from Proposition 3.4 that $\omega(y_0) = \{0\}$, then

$$\lim_{t\to\infty} T_t y_0 = 0. \tag{15}$$

From Theorem 2.1, one can obtain that,

$$\lim_{t\to\infty} \|T_t x_0 - x_0\| = 0 \text{ and } \lim_{t\to\infty} \|T_t y_0 - y_0\| = 0, \tag{16}$$

It follows from (14) - (16) that,

$$\lim_{t\to\infty} T_t x_0 = 0.$$

Consequently, 0 is an asymptotically stable equilibrium point of (9) on D.

Example 3.2. Let C be a closed bounded convex subset of a Banach space X and let $A = -1$ be the infinitesimal generator of the uniformly continuous semigroup $(e^{-t})_{t\geq 0}$ on C of the following system

$$\begin{cases} \dot{x}(t) &= -x(t) \quad t > 0 \\ x(0) &= x_0, \end{cases} \tag{17}$$

then the solution $x_0 e^{-t}$ approaches 0 as $t \to \infty$ i.e. 0 is an asymptotically stable equilibrium point of (17) on C.

Open problem. It will be interesting to establish Theorem 3.1 for semilinear systems as in I. Aksikas and J. Winkin [1] and I. Aksikas and J. Fraser Forbes [2].

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