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A New Form of the Quintuple Product Identity and its Application

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Abstract. We give a new form of the quintuple product identity. As a direct application of this new form a simple proof of known identities of Ramanujan and also new identities for other well known continued fractions are given. We also give and prove a general identity for $(q^{3m}; q^{3m})_\infty$.

1. Introduction

While working on the integral representation for a continued fraction of Ramanujan [20] which is analogous to the famous Rogers-Ramanujan continued fraction $R(q)$, I had the quintuple product identity in an interesting new form. Later I found that this identity is very useful, as it unifies many identities of Ramanujan and also gives new identities and, I think, they are not in the literature. I will give the proof of this new form via Rogers-Fine identity. The well known identities of Ramanujan come as direct application of this identity and I also give a new identity for $(q^{3m}; q^{3m})_\infty$. Writing a paper on the quintuple product identity must have a brief history of the identity.

Brief History

The quintuple product identity has a long history and, as Berndt [7, p.83] points out, it is difficult to assign priority to it. Since the early 90s several authors gave different new proofs of the quintuple product identity [8,9,10,12]. In the earlier work of Weierstrass on elliptic functions the quintuple product identity was written implicitly in terms of sigma functions, see Schwarz's book [18, p.47]. In Fricke's book [13, pp.432-433] the quintuple product identity is written in terms of theta functions. Watson [21] while proving identities related to the Rogers-Ramanujan continued fraction and again while proving that $p(n)$, the number of partitions of n, satisfies certain congruences modulo powers of 5 and 7, proved the quintuple product identity. Bailey [6], who was conversant with Watson's work, gave a simple proof of the identity. Sears [19] in 1952 showed that the quintuple product identity follows from his earlier work. While proving the conjecture of Dyson on $p(n)$, Atkin and Swinnerton-Dyer [5] gave another proof of the identity. Andrews [2] using ${}_6\psi_6$ summation formula of Bailey gave another proof. Bhargava et al. [9] using Ramanujan's ${}_1\psi_1$ summation formula gave yet another proof.

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Hirschhorn [15] gave a generalization of the quintuple product identity. Very recently, Kongsiriwong and Liu [16] gave a proof using the cube root of unity. Detailed history was given by Berndt [7, p.83], Hirschhorn [15] and very recently a comprehensive study by Cooper [11].

To start the work I need some notation :

If q and x are complex numbers with $|q| < 1$ and n an integer, let

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty}(1-q^i x)$$

and

$$(x)_n = (x;q)_n = \prod_{i=0}^{n-1}(1-q^i x).$$

If $\left|q\right| < 1$ and $x \neq 0$ then

$$j(x,q) = (x)_\infty \,(q/x)_\infty \,(q)_\infty .$$

For $m \geq 1$

$$J_m = J_{m,3m} = j(q^m, q^{3m}) = (q^m; q^m)_\infty .$$

I state some of continued fractions for which the identities will be derived :
Celebrated Rogers-Ramanujan continued fraction [3,(1.1), p.186]

$$\begin{aligned} R(q) &= 1 + \frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots.} \\ &= \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} \end{aligned} \tag{1.1}$$

Continued fraction of Ramanujan which is analogous to $R(q)$ [20]:

$$\begin{aligned} C(q) &= \frac{1}{1+}\frac{1+q}{1+}\frac{q^2}{1+}\frac{q+q^3}{1+}\frac{q^4}{1+\ldots.} \\ &= \frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty (q^3;q^4)_\infty}. \end{aligned} \tag{1.2}$$

Cubic Continued Fraction of Ramanujan [4,(6.2.37), p.154]

$$\begin{aligned} G(q) &= \frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\ldots.} \\ &= \frac{(q;q^6)_\infty (q^5;q^6)_\infty}{(q^3;q^6)_\infty^2}. \end{aligned} \tag{1.3}$$

2. A New Form of the Quintuple Product Identity

I write the quintuple product identity in its new avtar as a theorem:

Theorem

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} x^{2n}}{(x;q)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n^2+3n+2)/2} x^{-2n-2}}{(q/x;q)_{n+1}} = \frac{(x^2;q)_\infty (q/x^2;q)_\infty (q;q)_\infty}{(x;q)_\infty (q/x;q)_\infty}. \tag{2.1}$$

Proof

The well known quintuple product identity [1, Th. 3.9] is as follows. For $|q| < 1$ and $x \neq 0$

$$\sum_{-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}x^{3n}(1+q^n x) = (-x)_\infty(-q/x)_\infty(q)_\infty(qx^2;q^2)_\infty(q/x^2;q^2)_\infty. \tag{2.2}$$

The five infinite product on the right-hand side of (2.2) justify the name.

The first step is very simple. From the definition of $j(x,q)$ it can be easily shown that the right-hand side of (2.2) equals

$$j(-x,q)\frac{j(qx^2,q^2)}{J_2}$$

and by [14, eq.(1.14), p. 642]

$$\begin{aligned} &= J_1\frac{j(x^2,q)}{j(x,q)} \\ &= \frac{(x^2)_\infty(q/x^2)_\infty(q)_\infty}{(x)_\infty(q/x)_\infty}, \end{aligned} \tag{2.3}$$

which is the right-hand side of (2.1).

For the left-hand side we use the Rogers-Fine identity [1, p.564]:

$$\sum_{n=0}^{\infty}\frac{(a)_n t^n}{(b)_{n+1}} = \sum_{n=0}^{\infty}\frac{(a)_n(at/b)_n b^n t^n q^{n^2}(1-atq^{2n})}{(b)_{n+1}(t)_{n+1}}. \tag{2.4}$$

Writing x for b and x^2/b for t in (2.4) and then taking limit as $a\to\infty$, we have

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2-n)/2}x^{2n}}{(x)_{n+1}} &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{(3n^2-n)/2}(x)_n x^{3n}(1-x^2q^{2n})}{(x)_{n+1}} \\ &= \sum_{n=0}^{\infty}(-1)^n q^{(3n^2-n)/2}(x)_n x^{3n}(1+xq^n). \end{aligned} \tag{2.5}$$

Writing q/x for b and q^2/ax^2 for t in (2.4) and then taking limit as $a\to\infty$, we have

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2+3n)/2}x^{-2n}}{(q/x)_{n+1}} &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{(3n^2+5n)/2}(q/x)_n(1-q^{2n+2}/x^2)x^{-3n}}{(q/x)_{n+1}} \\ &= \sum_{n=0}^{\infty}(-1)^n q^{(3n^2+5n)/2}(1+q^{n+1}/x)x^{-3n} \end{aligned}$$

or

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2+3n+2)/2}x^{-2n-2}}{(q/x)_{n+1}} = x\sum_{n=0}^{\infty}(-1)^n q^{(3n^2+5n+2)/2}(1+q^{n+1}/x)x^{-3n-3}. \tag{2.6}$$

The left-hand side of (2.2) is

$$\sum_{n=0}^{\infty}(-1)^n q^{n(3n-1)/2}x^{3n}(1+q^n x) + \sum_{n=-1}^{-\infty}(-1)^n q^{n(3n-1)/2}x^{3n}(1+q^n x) \tag{2.7}$$

$$= \sum_{n=0}^{\infty}(-1)^n q^{n(3n-1)/2}x^{3n}(1+q^n x) - x\sum_{n=0}^{\infty}(-1)^n q^{(3n^2+5n+2)/2}x^{-3n-3}(1+q^{n+1}/x), \tag{2.8}$$

where we have written $-n$ for n and then $n+1$ for n in the second summation.

Invoking (2.5) and (2.6) in (2.8), we have

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}x^{3n}(1+q^n x)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}x^{2n}}{(x)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2+3n+2)/2}x^{-2n-2}}{(q/x)_{n+1}}. \tag{2.9}$$

Now (2.3)and (2.9) prove the Theorem.

3. Applications of the Theorem

The quintuple product identity in its new form, stated as a Theorem in (2.1), unifies known identities of Ramanujan for his celebrated Rogers-Ramanujan continued fraction $R(q)$, which were proved by Andrews [3] and also gives new identities for his cubic continued fraction $G(q)$ and for the analogous continued fraction $C(q)$, also of Ramanujan.

(a) **Identities for** $R(q)$

(i) Making $q \to q^5$ and taking $x = q$ in (2.1), we have

$$(q^5;q^5)_\infty R(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(5n^2-n)/2}}{(q;q^5)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{(5n^2+11n+6)/2}}{(q^4;q^5)_{n+1}}, \tag{3.1}$$

which was proved by Andrews [3,(3.19),p 198].

(ii) Making $q \to q^5$ and taking $x = q^2$ in (2.1), we have

$$\frac{(q^5;q^5)_\infty}{R(q)}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(5n^2+3n)/2}}{(q^2;q^5)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{(5n^2+7n+2)/2}}{(q^3;q^5)_{n+1}}, \tag{3.2}$$

which was proved by Andrews [3, (3.18), p. 198].

(b) **Identity for Cubic Continued Fraction of Ramanujan [4, Cor.6.2.37, p.154]**

(i) Making $q \to q^6$ and taking $x = q$ in (2.1), we have

$$\frac{(q^2;q^2)_\infty(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty^2}\frac{1}{G(q)}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2-n}}{(q;q^6)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+7n+4}}{(q^5;q^6)_{n+1}}. \tag{3.3}$$

As far as I know this is a new identity for the cubic continued fraction $G(q)$.

(c) **Identities for** $C(q)$**, analogous to Rogers-Ramanujan Continued Fraction** $R(q)$

(i) Making $q \to q^4$ and taking $x = q$ in (2.1), we have

$$(q^4;q^4)_\infty C(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}}{(q;q^4)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+4n+2}}{(q^3;q^4)_{n+1}}. \tag{3.4}$$

As far as I know this is a new identity for $C(q)$.

(d) **A General Identity**

(i) Taking $x = q$ and making $q \to q^3$ in (2.1), we have

$$(q^3;q^3)_\infty=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(3n^2+n)/2}}{(q;q^3)_{n+1}}-q\sum_{n=0}^{\infty}\frac{(-1)^n q^{(3n^2+5n)/2}}{(q^2;q^3)_{n+1}}. \tag{3.5}$$

(ii) Taking $x = q^2$ and making $q \to q^6$ in (2.1), we have

$$(q^6;q^6)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+n}}{(q^2;q^6)_{n+1}} - q^2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+5n}}{(q^4;q^6)_{n+1}}. \tag{3.6}$$

(iii) We now give a general result.

Taking $x = q^m$ and making $q \to q^{3m}$ in (2.1), we have

$$(q^{3m};q^{3m})_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(3mn^2+mn)/2}}{(q^m;q^{3m})_{n+1}} - q^m \sum_{n=0}^{\infty} \frac{(-1)^n q^{(3mn^2+5mn)/2}}{(q^{2m};q^{3m})_{n+1}}. \tag{3.7}$$

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