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On Nonoscillatory Solutions of Emden-Fowler Dynamic Systems on Time Scales

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Abstract.

We study the existence and asymptotic behavior of nonoscillatory solutions of Emden-Fowler dynamic sytems on time scales. In order to show the existence, we use Schauder, Knaster and Tychonoff Fixed Point Theorems. Some examples are illustrated as well.

1. Introduction

In this paper, we deal with the classification of nonoscillatory solutions of the Emden-Fowler system of first order dynamic equations

$$\begin{cases} x^{\Delta}(t) = \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} |y(t)|^{\frac{1}{\alpha}} \, sgny(t) \\ y^{\Delta}(t) = -b(t)\, |x^{\sigma}(t)|^{\beta} \, sgnx^{\sigma}(t), \end{cases} \tag{1}$$

where $\alpha, \beta > 0$ and $a, b \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$. Whenever we write $t \geq t_1$, we mean that $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. A time scale $\mathbb{T}$, a nonempty closed subset of real numbers, is introduced by Bohner and Peterson in [7] and [8]. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above. We call (x, y) a *proper solution* if it is defined on $[t_0, \infty)_{\mathbb{T}}$ and $sup\{|x(s)|, |y(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$. A solution (x, y) of (1) is said to be nonoscillatory if the component functions x and y are both nonoscillatory, i.e., either eventually positive or

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eventually negative. Otherwise it is said to be oscillatory. Throughout this paper without loss of generality we assume that x is eventually positive in our proofs. Our results can be obtained similarly for the case that x is eventually negative.

System (1) can be easily derived from the Emden Fowler dynamic equation

$$\left(a(t)|x^{\Delta}(t)|^{\alpha} sgn x^{\Delta}(t)\right)^{\Delta} + b(t)|x(t)|^{\beta} sgn x^{\sigma}(t) = 0 \tag{2}$$

by letting $x = x$ and $y = |x^{\Delta}|^{\alpha} sgn x^{\Delta}$ in (2). If $\alpha = \beta$ in (2), then it is called a half-linear dynamic equation.

If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, equation (2) reduces to the Emden Fowler differential equation

$$(a(t)|x'(t)|^{\alpha} sgn x'(t))' + b(t)|x(t)|^{\beta} sgn x(t) = 0,$$

see [12], and the Emden-Fowler difference equation

$$\Delta\left(a_n|\Delta x_n|^{\alpha} sgn \Delta x_n\right) + b_n|x_{n+1}|^{\beta} sgn x_{n+1} = 0,$$

see [9], respectively.

This paper is motivated by the papers [9], [14] and [11]. The related oscillation and nonoscillation results for two and three dimensional dynamic systems are given in [6], [4], [5], and [2], respectively. The setup of this paper is as follows: In Section 1, we give preliminary lemmas playing an important role in the further sections. In Sections 2 and 3, we show the existence and asymptotic properties of nonoscillatory solutions of system (1) by using certain improper integrals and fixed point theorems. In Section 4, we obtain some conclusions. And finally, the paper concludes with some examples.

Let M be the set of all nonoscillatory solutions of system (1). One can easily show that any nonoscillatory solution (x, y) of system (1) belongs to one of the following classes:

$$M^{+} := \{(x, y) \in M : \ x(t)y(t) > 0 \ \text{ eventually}\}$$
$$M^{-} := \{(x, y) \in M : \ x(t)y(t) < 0 \ \text{ eventually}\}.$$

Lemma 1.1. *[6, Lemma 2.1] Let (x, y) be a solution of system (1). Then the component functions x and y are themselves nonoscillatory if (x, y) is a nonoscillatory solution of system (1).*

Remark 1.1. Let (x, y) be a nonoscillatory solution of system (1). If $x(t)$ is nonoscillatory for $t \geq t_0$, then the other component function $y(t)$ is also nonoscillatory for sufficiently large t.

For convenience, let us set

$$Y_a = \int_{t_0}^{\infty} A(t)\Delta t \quad \text{ and } \quad Z_b = \int_{t_0}^{\infty} b(t)\Delta t,$$

where $A = \left(\frac{1}{a}\right)^{\frac{1}{\alpha}}$.

The following lemma gives some sufficient conditions for oscillation and nonoscillation of system (1).

Lemma 1.2. (a)[6, Lemma 2.3] If $Y_a < \infty$ and $Z_b < \infty$, then system (1) is nonoscillatory.
(b)[6, Lemma 2.2] If $Y_a = \infty$ and $Z_b = \infty$, then system (1) is oscillatory.

In the next two lemmas we show that M^{+} and M^{-} can be empty.

Lemma 1.3. *If $Y_a = \infty$ and $Z_b < \infty$, then any nonoscillatory solution (x, y) of system (1) belongs to M^{+}, i.e $M^{-} = \emptyset$.*

Proof. Suppose that $Y_a = \infty$ and $Z_b < \infty$. The proof is by contradiction. So assume that there exists a solution (x, y) of system (1) such that $(x, y) \in M^{-}$. Without loss of generality assume that $x(t) > 0$ for $t \geq t_1$. Then by integrating the first equation of system (1) from t_1 to t and the monotonicity of y, we have

$$x(t) = x(t_1) - \int_{t_1}^{t} A(s)\left(-y(s)\right)^{\frac{1}{\alpha}} \Delta s \leq x(t_1) - \left(-y(t_1)\right)^{\frac{1}{\alpha}} \int_{t_1}^{t} A(s)\Delta s.$$

As $t \to \infty$, $x \to -\infty$. But this contradicts the positivity of x. Note that the proof can be done without the condition $Z_b < \infty$. However in order for nonoscillatory solutions to exist, we need the assumption $Z_b < \infty$ by Lemma 1.2 (b). □

Lemma 1.4. *If $Y_a < \infty$ and $Z_b = \infty$, then any nonoscillatory solution (x, y) of system (1) belongs to M^-, i.e., $M^+ = \emptyset$.*

Proof. Suppose that $Y_a < \infty$ and $Z_b = \infty$. The proof is by contradiction. So assume that there exists a nonoscillatory solution (x, y) of system (1) such that $xy > 0$ eventually. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_1$. So by integrating the second equation of system (1) from t_1 to t and the monotonicity of x give us

$$y(t) \leq y(t_1) - (x^\sigma(t_1))^\beta \int_{t_1}^{t} b(s)\Delta s.$$

As $t \to \infty$, it follows that $y(t) \to -\infty$. But this contradicts that y is eventually positive. □

The discrete version of the following lemmas can be found in [14].

Lemma 1.5. *Let (x, y) be a nonoscillatory solution of system (1).*

(a) If $Y_a < \infty$, then the component function x has a finite limit.

(b) If $Y_a = \infty$ or $Z_b < \infty$, then the component function y has a finite limit.

Proof. (a) Suppose that $Y_a < \infty$ and (x, y) is a nonoscillatory solution of system (1). Then by Lemma 1.1, x and y are themselves nonoscillatory. Without loss of generality, assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. If $(x, y) \in M^-$, then by the first equation of system (1), $x^\Delta(t) < 0$ for $t \geq t_1$. Therefore, limit of x exists. So let us show that the assertion follows if $(x, y) \in M^+$. From the first equation of system (1), we have $x^\Delta(t) > 0$ for $t \geq t_1$. Hence two things might happen: The limit of the component function x exists or blows up. Now let us show that $\lim_{t\to\infty} x(t) = \infty$ cannot happen. Assume $x(t) \to \infty$ as $t \to \infty$. By integrating the first equation of system (1) from t_1 to t and using the monotonicity of y, we get

$$x(t) \leq x(t_1) + y^{\frac{1}{\alpha}}(t_1) \int_{t_1}^{t} A(s)\Delta s.$$

Taking the limit as $t \to \infty$, it follows that $Y_a = \infty$, which is a contradiction. This completes the proof.

(b) Suppose that $Y_a = \infty$ or $Z_b < \infty$ and (x, y) is a nonoscillatory solution of system (1). The case $Z_b < \infty$ can be proved similar to part (a). For $Y_a = \infty$, assume that x is eventually positive. Then proceeding as in the proof of Lemma 1.3, it can be shown that y is eventually positive. Then by the second equation of system (1), it follows that y has a finite limit. □

In the following lemmas, we find upper and lower bounds for the component function x of a nonoscillatory solution (x, y) of system (1).

Lemma 1.6. *Let $Y_a < \infty$. If (x, y) is a nonoscillatory solution of system (1), then there exist $c, d > 0$ and $t_1 \geq t_0$ such that*

$$c \int_{t}^{\infty} A(s)\Delta s \leq x(t) \leq d$$

or

$$-d \leq x(t) \leq -c \int_{t}^{\infty} A(s)\Delta s$$

for $t \geq t_1$.

Proof. Suppose that $Y_a < \infty$ and (x, y) is a nonoscillatory solution of system (1). Without loss of generality, let us assume that x is eventually positive. Then by Lemma 1.5 (a), we have $x(t) \leq d$ for $t \geq t_1$. If $y(t) > 0$ for $t \geq t_1$, then x is eventually increasing by the first equation of system (1). So for large t, the assertion follows. If $y(t) < 0$ for $t \geq t_1$, then integrating the first equation of system (1) from t to ∞ and the monotonicity of y give

$$\begin{aligned} x(t) &= x(\infty) + \int_t^{\infty} A(s)(-y(s))^{\frac{1}{\alpha}} \Delta s \geq \int_t^{\infty} A(s)(-y(s))^{\frac{1}{\alpha}} \Delta s \\ &\geq (-y(t_1))^{\frac{1}{\alpha}} \int_t^{\infty} A(s)\Delta s. \end{aligned}$$

Setting $c = (-y(t_1))^{\frac{1}{\alpha}}$ in the last inequality proves the assertion. Assuming x is eventually negative gives the second part of the proof. □

Lemma 1.7. *Let $Y_a = \infty$ and $Z_b < \infty$. If (x, y) is a nonoscillatory solution of system (1), then there exist $k_1, k_2 > 0$ and $t_1 \geq t_0$ such that*

$$k_1 \leq x(t) \leq k_2 \int_{t_1}^{t} A(s)\Delta s$$

or

$$-k_2 \int_{t_1}^{t} A(s)\Delta s \leq x(t) \leq -k_1$$

for $t \geq t_1$.

Proof. Suppose that $Y_a = \infty$ and $Z_b < \infty$, and (x, y) is a nonoscillatory solution of system (1). Then by Lemma 1.1, x and y are themselves nonoscillatory. Without loss of generality let us assume that $x(t) > 0$ for $t \geq t_1$. Then by Lemma 1.3, (x, y) must be in M^+. Hence, there is a constant $k_1 > 0$ such that $x(t) \geq k_1$ for $t \geq t_1$. Integrating the first equation of system (1) and the monotonicity of y give

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^{t} A(s)y^{\frac{1}{\alpha}}(s)\Delta s \leq x(t_1) + y^{\frac{1}{\alpha}}(t_1) \int_{t_1}^{t} A(s)\Delta s \\ &= \left(\frac{x(t_1)}{\int_{t_1}^{t} A(s)\Delta s} + y^{\frac{1}{\alpha}}(t_1) \right) \int_{t_1}^{t} A(s)\Delta s. \end{aligned}$$

Since $Y_a = \infty$, we can choose $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} A(t)\Delta t \geq 1 \quad \text{for} \quad t \geq t_2.$$

So this implies that

$$x(t) \leq \left(x(t_1) + y^{\frac{1}{\alpha}}(t_1) \right) \int_{t_1}^{t} A(s)\Delta s$$

and the assertion follows by letting $k_2 = x(t_1) + y^{\frac{1}{\alpha}}(t_1)$. Assuming that x is eventually negative proves the second part of the proof. □

2. The Case $Y_a = \infty$ and $Z_b < \infty$

In this section, we show that M^+ can be divided into some sub-classes under the case $Y_a = \infty$. By Lemma 1.2(b), in order to obtain the existence of nonoscillatory solutions, we also have to assume $Z_b < \infty$. So throughout this section, we suppose that $Y_a = \infty$ and $Z_b < \infty$ hold. Then by Lemma 1.3, $(x, y) \in M^+$. Without loss of generality we suppose that $x > 0$ eventually. Then by the second equation of system (1), y is positive and decreasing eventually. In addition to that, by using the first equation of system (1) and taking Lemma 1.5(b) into consideration we have that $x(t) \to c$ or ∞, and $y(t) \to d$ or 0 as $t \to \infty$ for $0 < c < \infty$ and $0 < d < \infty$.

Lemma 2.1. *If $x(t) \to c$, then $y(t) \to 0$ as $t \to 0$ for $c < 0 < \infty$.*

Proof. Suppose that $x(t) \to c$ as $t \to \infty$. Assume the contrary. So $y(t) \to d$ for $0 < d < \infty$ as $t \to \infty$. Then since $y(t) > 0$ and decreasing eventually, there exists $t_1 \geq t_0$ such that $y(t) \geq d$ for $t \geq t_1$. By the first equation of system (1), we have

$$x^{\Delta}(t) = A(t)y^{\frac{1}{\alpha}}(t) \geq A(t)d^{\frac{1}{\alpha}} \quad \text{for} \quad t \geq t_1. \tag{3}$$

Integrating (3) from t_1 to t yields

$$x(t) \geq x(t_1) + d^{\frac{1}{\alpha}} \int_{t_1}^{t} A(s)\Delta s.$$

As $t \to \infty$, this gives us a contradiction to the fact $x(t) \to c$. So the assertion follows. □

In light of Lemma 2.1 and the explanation above, we have the following lemma.

Lemma 2.2. *For $0 < c < \infty$ and $0 < d < \infty$, any nonoscillatory solution in M^+ must belong to one of the following sub-classes:*

$$\begin{aligned}
M^+_{B,0} &= \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = 0\right\}, \\
M^+_{\infty,B} &= \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = d\right\}, \\
M^+_{\infty,0} &= \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = 0\right\}.
\end{aligned}$$

In the literature, solutions in $M^+_{B,0}$, $M^+_{\infty,B}$ and $M^+_{\infty,0}$ are called *subdominant solutions*, *dominant solutions* and *intermediate solutions*, respectively.

The following theorems show the existence of nonoscillatory solutions in sub-classes mentioned above by using the improper integrals:

$$\begin{aligned}
J_\alpha &= \int_{t_0}^{\infty} A(t) \left(\int_{t}^{\infty} b(s)\Delta s \right)^{\frac{1}{\alpha}} \Delta t \\
K_\beta &= \int_{t_0}^{\infty} b(t) \left(\int_{t_0}^{\sigma(t)} A(s)\Delta s \right)^{\beta} \Delta t.
\end{aligned}$$

Theorem 2.1. *$M^+_{B,0} \neq \emptyset$ if and only if $J_\alpha < \infty$.*

Proof. Suppose that $M^+_{B,0} \neq \emptyset$. Then there exists $(x, y) \in M^+$ such that $|x(t)| \to c > 0$ and $|y(t)| \to 0$ as $t \to \infty$. Without loss of generality let us assume that $x(t) > 0$ for $t \geq t_1$. Integrating the second equation of system (1) from t to ∞ gives us

$$y(t) = \int_t^\infty b(s)\,(x^\sigma(s))^\beta\,\Delta s. \tag{4}$$

Solving the first equation of system (1) for y, substituting the resulting equation into (4) and by the monotonicity of y, we obtain

$$x^\Delta(t) \geq A(t)x^{\frac{\beta}{\alpha}}(t)\left(\int_t^\infty b(s)\Delta s\right)^{\frac{1}{\alpha}}. \tag{5}$$

Integrating (5) from t_1 to t gives

$$\begin{aligned} x(t) &\geq x(t_1) + \int_{t_1}^t A(s)x^{\frac{\beta}{\alpha}}(s)\left(\int_s^\infty b(\tau)\Delta\tau\right)^{\frac{1}{\alpha}}\Delta s \\ &\geq x^{\frac{\beta}{\alpha}}(t_1)\int_{t_0}^t A(s)\left(\int_s^\infty b(\tau)\Delta\tau\right)^{\frac{1}{\alpha}}\Delta s. \end{aligned}$$

As $t \to \infty$, the assertion follows.

Conversely, suppose that $J_\alpha < \infty$. Choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^\infty A(t)\left(\int_t^\infty b(s)\Delta s\right)^{\frac{1}{\alpha}}\Delta t < \left(\frac{c}{2}\right)\frac{1}{c^{\frac{\beta}{\alpha}}} \tag{6}$$

for arbitrarily given $c > 0$. Let X be the set of all bounded, continuous, real valued functions with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}}\{|x(t)|\}$. It is clear that X is a Banach Space, see [10]. Let us define a subset Ω of X such that

$$\Omega := \{x \in X : \ \frac{c}{2} \leq x(t) \leq c, \quad t \geq t_1\}.$$

It is clear that Ω is closed, bounded and convex. Define an operator $F : \Omega \to X$ by

$$(Fx)(t) = c - \int_t^\infty A(s)\left(\int_s^\infty b(\tau)\,(x^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}}\Delta s \quad \text{for} \quad t \geq t_1. \tag{7}$$

By inequality (6), we have

$$\begin{aligned} c \geq (Fx)(t) &= c - \int_t^\infty A(s)\left(\int_s^\infty b(\tau)\,(x^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}}\Delta s \\ &\geq c - c^{\frac{\beta}{\alpha}}\int_t^\infty A(s)\left(\int_s^\infty b(\tau)\Delta\tau\right)^{\frac{1}{\alpha}}\Delta s \geq \frac{c}{2}, \end{aligned}$$

and so $F : \Omega \to \Omega$. Since

$$\begin{aligned} &\|(Fx_n)(t) - (Fx)(t)\| \\ &\quad\leq \int_{t_1}^\infty A(s)\left|\left(\int_s^\infty b(\tau)\,(x_n^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}} - \left(\int_s^\infty b(\tau)\,(x^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}}\right|\Delta s, \end{aligned}$$

where x_n is a sequence of functions converging to x. Hence, the Lebesque Dominated Convergence Theorem yields

$$\|(Fx_n)(t) - (Fx)(t)\| \to 0,$$

which implies the continuity of F on Ω. Also

$$0 \le -[F(x)(t)]^{\Delta} = A(t)\left(\int_t^{\infty} b(\tau)\,(x^{\sigma}(\tau))^{\beta}\,\Delta\tau\right)^{\frac{1}{\alpha}} \le c^{\frac{\beta}{\alpha}} A(t)\left(\int_t^{\infty} b(\tau)\Delta\tau\right)^{\frac{1}{\alpha}} < \infty$$

implies that F is equibounded and equicontinuous. Therefore by Schauder's Fixed Point Theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. Then

$$\bar{x}(t) = c - \int_t^{\infty} A(s)\left(\int_s^{\infty} b(\tau)\,(\bar{x}^{\sigma}(\tau))^{\beta}\,\Delta\tau\right)^{\frac{1}{\alpha}} \Delta s. \tag{8}$$

So as $t \to \infty$, $\bar{x}(t) \to c$. Note that $\bar{x}^{\Delta}(t) > 0$ for $t \ge t_1$. So it is eventually monotone, i.e., $\bar{x}$ is nonoscillatory. Therefore, taking the derivative of (8) and using the first equation of system (1) give us

$$\bar{y}(t) = \int_t^{\infty} b(\tau)\,(\bar{x}^{\sigma}(\tau))^{\beta}\,\Delta\tau.$$

It follows that $\bar{y}(t) > 0$ for $t \ge t_1$, i.e., $(\bar{x}, \bar{y})$ is nonoscillatory and then by Remark 1.1 and Lemma 1.3, $(\bar{x}, \bar{y}) \in M^+$. Taking the limit as $t \to \infty$ yields $\bar{y}(t) \to 0$. Hence $M^+_{B,0} \ne \emptyset$. □

Theorem 2.2. *$M^+_{\infty,B} \ne \emptyset$ if and only if $K_\beta < \infty$.*

Proof. Suppose that $M^+_{\infty,B} \ne \emptyset$. Then there exists $(x, y) \in M^+$ such that $|x(t)| \to \infty$ and $|y(t)| \to d$, for $0 < d < \infty$. Without loss of generality assume that $x(t) > 0$ for $t \ge t_1$ Integrating the first equation from t_1 to $\sigma(t)$ and the second equation from t_1 to t of system (1) give us

$$x^{\sigma}(t) = x^{\sigma}(t_1) + \int_{t_1}^{\sigma(t)} A(s) y^{\frac{1}{\alpha}}(s)\Delta s > d^{\frac{1}{\alpha}} \int_{t_1}^{\sigma(t)} A(s)\Delta s. \tag{9}$$

and

$$y(t_1) - y(t) = \int_{t_1}^{t} b(s)\,(x^{\sigma}(s))^{\beta}\,\Delta s, \tag{10}$$

respectively. Then by (9) and (10), we have

$$\begin{aligned}\int_{t_1}^{t} b(s)\left(\int_{t_1}^{\sigma(s)} A(\tau)\Delta\tau\right)^{\beta} \Delta s &< d^{\frac{-\beta}{\alpha}} \int_{t_1}^{t} b(s)\,(x^{\sigma}(s))^{\beta}\,\Delta s \\ &< d^{\frac{-\beta}{\alpha}}\,(y(t_1) - y(t))\end{aligned}$$

So as t goes to ∞, it follows that $K_\beta < \infty$.

Conversely, suppose that $K_\beta < \infty$. Choose $t_1 \ge t_0$ so large that

$$\int_{t_1}^{\infty} b(s)\left(\int_{t_1}^{\sigma(s)} A(\tau)\Delta\tau\right)^{\beta} \Delta s < \frac{d}{(2d)^{\beta}} \tag{11}$$

for arbitrarily given $d > 0$. Let X be the partially ordered Banach Space of all real-valued continuous functions with the norm $\|x\| = \sup_{t>t_1} \frac{|x(t)|}{\int_{t_1}^{t} A(s)\Delta s}$ and the usual pointwise ordering $\leq$. Define a subset Ω of X as follows:

$$\Omega : \{x \in X : \quad d^{\frac{1}{\alpha}} \int_{t_1}^{t} A(s)\Delta s \leq x(t) \leq (2d)^{\frac{1}{\alpha}} \int_{t_1}^{t} A(s)\Delta s \text{ for } t > t_1\}.$$

First since every subset of Ω has a supremum and infimum in Ω, $(\Omega, \leq)$ is a complete lattice. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \int_{t_1}^{t} A(s) \left(d + \int_{s}^{\infty} b(\tau) \left(x^{\sigma}(\tau)\right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s. \tag{12}$$

It can be shown that $F : \Omega \to \Omega$ is an increasing mapping for $t \geq t_1$.

So by the Knaster Fixed Point Theorem, we have that there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = \int_{t_1}^{t} A(s) \left(d + \int_{s}^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau)\right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s \text{ for } t > t_1. \tag{13}$$

Hence $\bar{x}$ is eventually positive, and hence nonoscillatory. Then by taking the derivative of (13) and using the first equation of system (1) give us

$$\bar{y}(t) = \left(\bar{x}^{\Delta}(t)\right)^{\alpha} a(t) = d + \int_{t}^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau)\right)^{\beta} \Delta\tau. \tag{14}$$

Then it follows that $\bar{y}$ is eventually positive, i.e., nonoscillatory. Hence, $(\bar{x}, \bar{y})$ is a nonoscillatory solution of system (1) and by Lemma 1.3 we have $(\bar{x}, \bar{y}) \in M^{+}$. For $\bar{x} \in \Omega$, we also have

$$\bar{x}(t) \geq \int_{t_1}^{t} A(s) \left[d + \int_{s}^{\infty} b(\tau) \left(d^{\frac{1}{\alpha}} \int_{t_1}^{\sigma(\tau)} A(\lambda)\Delta\lambda \right)^{\beta} \Delta\tau \right]^{\frac{1}{\alpha}} \Delta s$$

As $t \to \infty$, the right hand side of the last inequality goes to ∞ since $Y_a = \infty$. Therefore $\bar{x}(t) \to \infty$ as $t \to \infty$. Taking the limit as $t \to \infty$ of (14) gives that y has a finite limit. Therefore $M^{+}_{\infty,B} \neq \emptyset$. $\square$

Theorem 2.3. *If $J_\alpha = \infty$ and $K_\beta < \infty$, then $M^{+}_{\infty,0} \neq \emptyset$.*

Proof. Suppose that $J_\alpha = \infty$ and $K_\beta < \infty$. Since $Y_a = \infty$, we can choose $t_1, t_2 \geq t_0$ so large that

$$\int_{t_2}^{\infty} b(t) \left(\int_{t_0}^{\sigma(t)} A(s)\Delta s \right)^{\beta} \Delta t \leq 1 \tag{15}$$

and

$$\int_{t_1}^{t_2} A(s)\Delta s \geq 1. \tag{16}$$

Let X be the Fréchet Space of all continuous functions on $[t_1, \infty)_{\mathbb{T}}$ endowed with the topology of uniform convergence on compact subintervals of $[t_1, \infty)_{\mathbb{T}}$. Set

$$\Omega := \{x \in X : \quad 1 \leq x(t) \leq \int_{t_1}^{t} A(s)\Delta s \text{ for } t \geq t_1\}$$

and define an operator $T : \Omega \to X$ by

$$(Tx)(t) = 1 + \int_{t_2}^{t} A(s)\left(\int_s^\infty b(\tau)\,(x^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}}. \tag{17}$$

We can show that $T : \Omega \to \Omega$ is continuous on $\Omega \subset X$ by the Lebesque Dominated Convergence Theorem. Since

$$\begin{aligned} 0 \le [(Tx)(t)]^\Delta &= A(t)\left(\int_t^\infty b(\tau)\,(x^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}} \\ &\le A(t)\left(\int_t^\infty b(\tau)\left(\int_{t_1}^{\sigma(\tau)} A(\lambda)\Delta\lambda\right)^\beta \Delta\tau\right)^{\frac{1}{\alpha}} < \infty, \end{aligned}$$

it follows that T is equibounded and equicontinuous. Then by Tychonoff Fixed Point Theorem, there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (T\bar{x})(t) = 1 + \int_{t_2}^{t} A(s)\left(\int_s^\infty b(\tau)\,(\bar{x}^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}} \quad \text{for } t \ge t_2. \tag{18}$$

Therefore, it follows that $\bar{x}$ is eventually positive, i.e nonoscillatory. Then integrating (18) and by the first equation of system (1), we have

$$\bar{y}(t) = a(t)\left(x^\Delta(t)\right)^\alpha = \int_t^\infty b(\tau)\,(\bar{x}^\sigma(\tau))^\beta\,\Delta\tau. \tag{19}$$

It follows that $\bar{y}$ is eventually positive, and hence (x, y) is a nonoscillatory solution of system (1). So by Lemma 1.3 it follows that $(\bar{x}, \bar{y}) \in M^+$. Also by monotonicity of $\bar{x}$, we have

$$\bar{x}(t) = 1 + \int_{t_2}^{t} A(s)\left(\int_s^\infty b(\tau)\,(\bar{x}^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}} \ge (\bar{x}(t_2))^\beta \int_{t_2}^{t} A(s)\left(\int_s^\infty b(\tau)\Delta\tau\right)^{\frac{1}{\alpha}}.$$

Hence as $t \to \infty$, it follows that $\bar{x}(t) \to \infty$. And by (19), we have $\bar{y}(t) \to 0$ as $t \to \infty$. Therefore $M^+_{\infty,0} \neq \emptyset$. □

Next we give the integral relationships between J_α, K_β, Y_a, and Z_b and obtain a conclusion for the existence and non-existence of solution (x, y) of system (1) based on α and β. The proof of the following lemma is similar to the proofs of Lemma 1.1, Lemma 3.2, Lemma 3.3, Lemma 3.6 and Lemma 3.7 in [3].

Lemma 2.3. (a) If $J_\alpha < \infty$ or $K_\beta < \infty$ then $Z_b < \infty$.
(b) If $K_\beta = \infty$, then $Y_a = \infty$ or $Z_b = \infty$.
(c) If $J_\alpha = \infty$, then $Y_a = \infty$ or $Z_b = \infty$.
(d) Let $\alpha \ge 1$. If $J_\alpha < \infty$, then $K_\alpha < \infty$.
(e) Let $\beta \le 1$. If $K_\beta < \infty$, then $J_\beta < \infty$.
(f) Let $\alpha < \beta$. If $K_\beta < \infty$, then $J_\alpha < \infty$ and $K_\alpha < \infty$.
(g) Let $\alpha > \beta$. If $J_\alpha < \infty$, then $K_\beta < \infty$ and $J_\beta < \infty$.

The following corollaries give the existence and nonexistence of nonoscillatory solutions (x, y) of system (1) in our subclasses by Lemma 2.3 and our main theorems presented in this section.

Corollary 2.1. *Suppose that $Y_a = \infty$ and $Z_b < \infty$. Then*

(a) $M^+_{B,0} \neq \emptyset$ *if any of the followings hold:*
  (i) $J_\alpha < \infty$,
  (ii) $\alpha < \beta$ *and* $K_\beta < \infty$,
  (iii) $\alpha < \beta$, $\beta \ge 1$ *and* $J_\beta < \infty$,

(iv) $\alpha \le 1$ *and* $K_\alpha < \infty$.

(b) $M^+_{\infty,B} \neq \emptyset$ *if any of the followings hold:*

(i) $K_\beta < \infty$,

(ii) $\alpha > \beta$ *and* $J_\alpha < \infty$,

(iii) $\alpha \ge 1$ *and* $J_\beta < \infty$.

(c) $M^+_{B,0} = \emptyset$ *if any of the followings hold:*

(i) $J_\alpha = \infty$,

(ii) $\alpha > \beta$ *and either* $J_\beta = \infty$ *or* $K_\beta = \infty$,

(iii) $\alpha \ge 1$ *and* $K_\alpha = \infty$.

(d) $M^+_{\infty,B} = \emptyset$ *if any of the followings hold:*

(i) $K_\beta = \infty$,

(ii) $\alpha < \beta$ *and either* $J_\alpha = \infty$ *or* $K_\alpha = \infty$,

(iii) $\beta \le 1$ *and* $J_\beta = \infty$.

3. The Case $Y_a < \infty$ and $Z_b < \infty$

In this section, we show the existence of a solution (x, y) of system (1) by assuming $Y_a < \infty$. Since we investigate a solution (x, y) in M^+, we also have to assume that $Z_b < \infty$ because of Lemma 1.4. Recall that M^+ is the set of nonoscillatory solutions (x, y) such that x and y have the same sign. Without loss of generality let us assume that $x > 0$ eventually. Then by the first equation of system (1), x is eventually increasing and by Lemma 1.5 the limit of x approaches a positive constant and the limit of y exists. Also by the second equation of system (1) y is eventually decreasing and approaches a nonnegative constant.

In light of this information, one can easily prove the following lemma.

Lemma 3.1. *For* $0 < c < \infty$ *and* $0 < d < \infty$, *any nonoscillatory solution in* M^+ *belongs to the following subclasses:*

$$M^+_{B,B} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = d\right\}$$
$$M^+_{B,0} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = 0\right\}.$$

The following theorems show the existence of nonoscillatory solutions (x, y) in these subclasses of M^+.

Theorem 3.1. (a) $M^+_{B,B} \neq \emptyset$ if $Y_a < \infty$ and $Z_b < \infty$.

(b) If $M^+_{B,B} \neq \emptyset$, then $J_\alpha < \infty$.

Proof. (a) Suppose that $Y_a < \infty$ and $Z_b < \infty$. Then $J_\alpha < \infty$ by Lemma 2.3 (c). Since $Y_a < \infty$, for arbitrarily given $c, d > 0$ there exists $t_1 \ge t_0$ such that

$$\int_{t_1}^{t} A(s)\left(d + \int_s^\infty c^\beta b(s)\Delta s\right)^{\frac{1}{\alpha}} \le \frac{c}{2} \quad \text{for} \quad t \ge t_1. \tag{20}$$

Let X be the Banach space of all real-valued continuous functions endowed with the norm $\|x\| = \sup_{t\in[t_1,\infty)_\mathbb{T}} |x(t)|$ and with the usual pointwise ordering $\le$. Define a subset Ω of X such that

$$\Omega := \left\{x \in X : \ \frac{c}{2} \le x(t) \le c \quad \text{for} \quad t \ge t_1\right\}.$$

For any subset $\tilde{\Omega} \in \Omega$, it is obvious that $\inf \tilde{\Omega} \in \Omega$ and $\sup \tilde{\Omega} \in \Omega$. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \frac{c}{2} + \int_{t_1}^{t} A(s)\left(d + \int_s^\infty b(\tau)\,(x^\sigma(\tau))^\beta\,\Delta\tau\right)^{\frac{1}{\alpha}} \Delta s.$$

One can show that $F : \Omega \to \Omega$ and F is an increasing mapping. So by the Knaster Fixed point theorem [13], there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (F\bar{x})(t) = \frac{c}{2} + \int_{t_1}^{t} A(s)\left(d + \int_{s}^{\infty} b(\tau)\,(\bar{x}^{\sigma}(\tau))^{\beta}\,\Delta\tau\right)^{\frac{1}{\alpha}} \Delta s. \tag{21}$$

Therefore, it follows that $\bar{x}(t) > 0$ for $t \geq t_1$. So by the first equation of system (1), we have $\bar{y}(t) > 0$ for $t \geq t_1$, i.e., $(\bar{x}, \bar{y}) \in M^+$. From (21), we have

$$\bar{x} \leq \frac{c}{2} + \int_{t_1}^{t} A(s)\left(d + c^{\beta}\int_{s}^{\infty} b(\tau)\Delta\tau\right)^{\frac{1}{\alpha}} \Delta s.$$

So as $t \to \infty$, it follows that the limit of $\bar{x}$ is finite. By taking the derivative of (21) and the first equation of system (1), we have

$$\bar{y}(t) = \left(\bar{x}^{\Delta}(t)\right)^{\alpha} a(t) = d + \int_{t}^{\infty} b(\tau)\,(\bar{x}^{\sigma}(\tau))^{\beta}\,\Delta\tau. \tag{22}$$

Taking the limit of (22) as $t \to \infty$ yields that $\bar{y}(t) \to d$. Therefore, we conclude that $(\bar{x}, \bar{y}) \in M^+_{B,B} \neq \emptyset$.
(b) Suppose that $M^+_{B,B} \neq \emptyset$. Without loss of generality assume that x is eventually positive. Then there exists $t_1 \geq t_0$ and $(x, y) \in M^+$ such that $x \to c$ and $y \to d$ as $t \to \infty$ for $0 < c < \infty$ and $0 < d < \infty$. Integrating the second equation of system (1) from t to ∞ and using the monotonicity of x give us

$$y(t) > (x(t))^{\beta} \int_{t}^{\infty} b(s)\Delta s \quad \text{for} \quad t \geq t_1$$

or

$$y^{\frac{1}{\alpha}}(t) > (x(t))^{\frac{\beta}{\alpha}} \left(\int_{t}^{\infty} b(s)\Delta s\right)^{\frac{1}{\alpha}} \quad \text{for} \quad t \geq t_1. \tag{23}$$

Substituting (23) into the first equation of system (1) yields

$$x^{\Delta}(t) > A(t)x^{\frac{\beta}{\alpha}} \left(\int_{t}^{\infty} b(s)\Delta s\right)^{\frac{1}{\alpha}}. \tag{24}$$

Integrating (24) from t_1 to t and by the monotonicity of x give us

$$x(t) > x^{\frac{\beta}{\alpha}}(t_1) \int_{t_1}^{t} A(s)\left(\int_{s}^{\infty} b(\tau)\Delta\tau\right)^{\frac{1}{\alpha}} \Delta s \tag{25}$$

As $t \to \infty$, the assertion follows. □

The following theorem can be proved similar to Theorem 2.1.
Theorem 3.2. (a) $M^+_{B,0} \neq \emptyset$ if $Y_a < \infty$ and $Z_b < \infty$.
(b) If $M^+_{B,0} \neq \emptyset$, then $J_\alpha < \infty$.

By Lemma 2.1 and from our main results in Sections 2 and 3, one can have the following corollaries.
Corollary 3.1. *If $Y_a < \infty$ and $Z_b < \infty$, then any nonoscillatory solution in M^+ of system (1) belongs to $M^+_{B,B}$ or $M^+_{B,0}$, i.e., $M^+_{\infty,B} = M^+_{\infty,0} = \emptyset$.*
Corollary 3.2. *If $Y_a = \infty$ and $Z_b < \infty$, then $M^+_{B,B} = \emptyset$.*

4. Examples

In this section, we give three examples to illustrate Theorem 2.2 and Theorem 2.3.

Example 4.1. Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $\alpha = 1$, $A(t) = \frac{t}{1+2t}$, $b(t) = \frac{1}{q^{1+\beta}t^{\beta+2}}$, $s = q^m$ and $t = q^n$, where $m, n \in \mathbb{N}_0$, in system (1). It is easy to show that $Y_a = \infty$ and $Z_b < \infty$. Let us show that $K_\beta < \infty$.

$$\int_{t_0}^{T} b(t)\left(\int_{t_0}^{\sigma(t)} A(s)\Delta s\right)^{\beta} \Delta t = \sum_{t=1}^{\rho(T)} \frac{1}{q^{1+\beta}t^{\beta+2}}\left(\sum_{s=1}^{t} \frac{s^2(q-1)}{1+2s}\right)^{\beta} (q-1)t$$
$$< \frac{(q-1)^{\beta+1}}{q^{1+\beta}} \sum_{t=1}^{\rho(T)} \frac{1}{t^{1+\beta}}\left(\sum_{s=1}^{t} s\right)^{\beta} < \frac{q-1}{q}\sum_{t=1}^{\rho(T)} \frac{1}{t}.$$

We also have

$$\lim_{T\to\infty} \sum_{t=1}^{\rho(T)} \frac{1}{t} = \sum_{n=0}^{\infty} \frac{1}{q^n} < \infty$$

by the geometric series test. So we have that $K_\beta < \infty$. It can be verified that $(t, \frac{1}{t} + 2)$ is a nonoscillatory solution of

$$\begin{cases} x^{\Delta} = \frac{t}{1+2t}\left|y\right| sgny \\ y^{\Delta} = -\frac{1}{q^{1+\beta}t^{\beta+2}}\left|x^{\sigma}\right|^{\beta} sgnx \end{cases}$$

in M^+ such that $\lim_{t\to\infty} t = \infty$ and $\lim_{t\to\infty}(\frac{1}{t} + 2) = 2$, i.e., $M^+_{\infty,B} \neq \emptyset$.

Example 4.2. Let $\mathbb{T} = \mathbb{R}$, $\alpha > \beta$ with $\beta < 1$, $A(t) = e^{2t}$ and $b(t) = \alpha e^{-t(\alpha+\beta)}$ in system (1). Clearly, $Y_a = \infty$ and $Z_b < \infty$. One can show that

$$J_\alpha = \int_{t_0}^{\infty} e^{2t}\left(\int_{t}^{\infty} \alpha e^{-s(\alpha+\beta)} ds\right)^{\frac{1}{\alpha}} dt = \infty$$

and

$$K_\beta = \int_{t_0}^{\infty} \alpha e^{-t(\alpha+\beta)}\left(\int_{t_0}^{t} e^{2s} ds\right)^{\beta} dt < \infty.$$

It is easy to verify that $(e^t, e^{-\alpha t})$ is a nonoscillatory solution of

$$\begin{cases} x' = e^{2t}\left|y\right|^{\frac{1}{\alpha}} sgny \\ y' = -\alpha e^{-t(\alpha+\beta)}\left|x\right|^{\beta} sgnx \end{cases}$$

in M^+ such that $\lim_{t\to\infty} e^t = \infty$ and $\lim_{t\to\infty} e^{-\alpha t} = 0$, i.e., $M^+_{\infty,0} \neq \emptyset$.

Example 4.3. Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $\alpha = 1$, $\beta < 1$, $A(t) = 1 + t$, $b(t) = \frac{1}{(1+t)(1+tq)^{\beta+1}}$ in system (1). It is easy to verify that $Y_a = \infty$ and $Z_b < \infty$. Letting $s = q^m$ and $t = q^n$, where $m, n \in \mathbb{N}_0$, gives

$$\int_{t_0}^{T} A(t)\left(\int_{t}^{T} b(s)\Delta s\right)^{\frac{1}{\alpha}} \Delta t = \sum_{t=1}^{\rho(T)}(1+t)\left(\sum_{s=t}^{\rho(T)} \frac{(q-1)s}{(1+s)(1+sq)^{\beta+1}}\right)(q-1)t$$
$$\geq (q-1)^2 \sum_{t=1}^{\rho(T)}(1+t)\left(\frac{t}{(1+t)(1+tq)^{\beta+1}}\right)t = (q-1)^2 \sum_{t=1}^{\rho(T)} \frac{t^2}{(1+tq)^{\beta+1}}.$$

So we have

$$\lim_{T\to\infty}\sum_{t=1}^{\rho(T)}\frac{t^2}{(1+tq)^{\beta+1}}=\sum_{n=0}^{\infty}\frac{q^{2n}}{(1+q^{n+1})^{\beta+1}}=\infty$$

by the Test for Divergence and $\beta<1$. Now let us show that $K_\beta<\infty$. One can show that

$$\int_{t_0}^{\sigma(t)}A(s)\Delta s=\sum_{s=1}^{t}(1+s)(q-1)s\le tq(1+tq)$$

and so we have

$$\int_{t_0}^{T}b(t)\left(\int_{t_0}^{\sigma(t)}A(s)\Delta s\right)^{\beta}\Delta t\le\sum_{t=1}^{\rho(T)}\frac{1}{(1+t)(1+tq)^{\beta+1}}\left(tq(1+tq)\right)^{\beta}t(q-1)q^{\beta}(q-1)\sum_{t=1}^{\rho(T)}\frac{t^{\beta}}{1+t}.$$

Therefore,

$$\lim_{T\to\infty}q^{\beta}(q-1)\sum_{t=1}^{T}\frac{t^{\beta}}{1+t}=q^{\beta}(q-1)\sum_{n=0}^{\infty}\frac{(q^n)^{\beta}}{(1+q^n)}<\infty$$

by the Ratio Test and $\beta<1$. It can also be verified that $\left(1+t,\frac{1}{t+1}\right)$ is a nonoscillatory solution of

$$\begin{cases}x^{\Delta}=(1+t)\left|y\right|^{\frac{1}{\alpha}}sgny\\ y^{\Delta}=-\frac{1}{(1+t)(1+tq)^{\beta+1}}\left|x^{\sigma}\right|^{\beta}sgnx\end{cases}$$

in M^+ such that $\lim_{t\to\infty}(1+t)=\infty$ and $\lim_{t\to\infty}\dfrac{1}{t+1}=0$, i.e., $M^+_{\infty,0}\neq\emptyset$.

References

[1] R. P. Agarwal and J. V. Manojlović. On the Existence and the Asymptotic Behavior of Nonoscillatory Solutions of Second Order Quasilinear Difference Equations. Funkcialaj Ekvacioj, 56 (2013) 81-109.

[2] A. Akgül, E. Akın. Almost Oscillatory Three-Dimensional Dynamical Systems of First Order Delay Dynamic Equations. Nonlinear Dynamics and Systems Theory, 14(3) (2014), 208-222.

[3] Ö. Öztürk, E. Akın. Classification of Nonoscillatory Solutions of Emden-Fowler Dynamic Equations on Time Scales. Dynamic Systems and Appli- cations, 25(2016) 219-236.

[4] E. Akın, Z Došlá and B. Lawrence. Oscillatory Properties for Three-Dimensional Dynamic Systems. Nonlinear Analysis 69 (2008) 483-494.

[5] E. Akın, Z Došlá and B. Lawrence. Almost Oscillatory Three-Dimensional Dynamical Systems. Advances in Difference Equations 2012, 2012:46.

[6] D. R. Anderson. Oscillation and Nonoscillation Criteria for Two-dimensional Time-Scale Systems of First Order Nonlinear Dynamic Equations. Electronic Journal of Differential Equations, Vol. 2009 (2009), No. 24, pp 1-13.

[7] M. Bohner and A. Peterson. Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston, 2001.

[8] M. Bohner and A. Peterson. Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston, 2003.

[9] M. Cecchi, Z Došlá, and M. Marini. On oscillation and nonoscillation properties of Emden-Fowler difference equations. *Cent. Eur. J. Math.*, 7(2), 322-334, 2009.

[10] P. G. Ciarlet. Linear and Nonlinear Functional Analysis with Applications. Siam, 2013.

[11] Z. Došla and M. Marini. On Super-linear Emden-Fowler Type Differential Equations. Elsevier, J. Math. Anal. Appl. 416 (2014) 497-510.

[12] Z. Došlá and I. Vrkoč. On an extension of the Fubini theorem and its applications in ODEs. Elsevier, Nonlinear Analysis 57 (2004) 531-548.

[13] B. Knaster. Un th*é*or*è*me sur les fonctions d'ensembles. Ann. Soc. Polon. Math. 6 (1928) 133—134.

[14] Wan-Tong Li. Classification Schemes for Nonoscillatory Solutions of Two-Dimensional Nonlinear Difference Systems. Computer and Mathematics with Applications 42 (2001) 341-355.