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Sturmian Comparison Theory for Half-Linear and Nonlinear Differential Equations via Picone Identity

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Abstract. In this paper, Sturmian comparison theory is developed for the pair of second order differential equations; first of which is the nonlinear differential equations of the form

$$(m(t)\Phi_\beta(y'))' + \sum_{i=1}^{n} q_i(t)\Phi_{\alpha_i}(y) = 0 \tag{1}$$

and the second is the half-linear differential equations

$$(k(t)\Phi_\beta(x'))' + p(t)\Phi_\beta(x) = 0 \tag{2}$$

where $\Phi_*(s) = |s|^{*-1}s$ and $\alpha_1 > \cdots > \alpha_m > \beta > \alpha_{m+1} > \cdots > \alpha_n > 0$. Under the assumption that the solution of Eq. (2) has two consecutive zeros, we obtain Sturm-Picone type and Leighton type comparison theorems for Eq. (1) by employing the new nonlinear version of Picone's formula that we derive. Wirtinger type inequalities and several oscillation criteria are also attained for Eq. (1). Examples are given to illustrate the relevance of the results.

1. Introduction

In this paper we are concerned with Sturmian type comparison of solutions of half-linear equations (2) and nonlinear equations of the form (1) where k, m, p and q_i's are continuous functions on $[0,\infty)$. We assume without further mention that the functions $k(t)$, $m(t)$ and $q_i(t)$, $i = 1,\dots n$, are positive and nonlinearities in Eq. (1) satisfy

$$\alpha_1 > \cdots > \alpha_m > \beta > \alpha_{m+1} > \cdots > \alpha_n > 0. \tag{3}$$

By a solution $x(t)$ of Eq. (2) on an interval $J \subset [t_0,\infty)$ we mean a nontrivial continuously differentiable function defined on J with $k(t)x' \in C^1(J)$ such that $x(t)$ satisfies Eq. (2). A solution $y(t)$ of Eq. (1) is defined in a similar manner.

It is well-known that the Sturmian theory plays an important role in the study of qualitative behavior of solutions of linear, half-linear and nonlinear equations. Sturmian type comparison theorems for linear equations are very classical and well-known (see [7, 8, 12, 14–16, 22, 23, 25, 26] and the references therein).

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In recent years, although the oscillation theory of nonlinear differential equations has been developed very rapidly, there are only a few papers with regard to the oscillation of their solutions as far as the Sturmian theory is concerned. Some pioneering works showed that there is a striking similarity between linear and half-linear [4, 9], forced super-linear [10], forced quasilinear [11], nonlinear equations [17, 29], linear and half-linear impulsive differential equations [18, 19]. Motivated by this, we attempt to obtain analogous comparison results for the pair of second order differential equations of the form (1) and (2).

The proof of the well-known Sturm-Picone comparison theorem [23] (see also [14, 15, 26]) for linear equations

$$\mathrm{L}_1[x] \equiv (k(t)x')' + p(t)x = 0;$$
$$\mathrm{L}_2[y] \equiv (m(t)y')' + q(t)y = 0$$

is based on employing the Picone's formula

$$\frac{x}{y}\,(ykx' - xmy')\Big|_a^b = \int_a^b \Big[(k-m)(x')^2 + (q-p)x^2 + m(x' - \frac{x}{y}\,y')^2 + \frac{x}{y}\,\{y\mathrm{L}_1[x] - x\mathrm{L}_2[y]\}\Big]\mathrm{d}t \tag{4}$$

which holds for all real valued functions x and y defined on an interval $[a, b]$ such that x, y, kx' and my' are differentiable on $[a, b]$ and $y \neq 0$ for $t \in [a, b]$. The formula (4) has also been used for establishing Wirtinger type inequalities for solutions of ordinary differential equations [14, 25], and generalized to nonselfadjoint equations [14, p. 11].

In 1999, Jaroš and Kusano [9] generalized the Sturm-Picone comparison theory for the pair of the half-linear equations

$$\mathrm{H}_{\mathrm{L}_1}[x] \equiv (k(t)\Phi_\beta(x'))' + p(t)\Phi_\beta(x) = 0;$$
$$\mathrm{H}_{\mathrm{L}_2}[y] \equiv (m(t)\Phi_\beta(y'))' + q(t)\Phi_\beta(y) = 0,$$

by employing the Picone type formula

$$\begin{aligned}
&\frac{x}{\Phi_\beta(y)}\Big[k\Phi_\beta(x')\Phi_\beta(y) - m\Phi_\beta(y')\Phi_\beta(x)\Big]\Big|_{t=a}^{t=b} \\
&= \int_a^b \Big\{(k-m)|x'|^{\beta+1} + (q-p)|x|^{\beta+1} + m\Big\{|x'|^{\beta+1} + \beta\Big|\frac{xy'}{y}\Big|^{\beta+1} - (\beta+1)x'\Phi_\beta(\frac{xy'}{y})\Big\} \\
&+ \frac{x}{\Phi_\beta(y)}\,\Big\{\Phi_\beta(y)\mathrm{H}_{\mathrm{L}_1}[x] - \Phi_\beta(x)\mathrm{H}_{\mathrm{L}_2}[y]\Big\}\Big\}\mathrm{d}t.
\end{aligned} \tag{5}$$

It is clear that a special case of Eq. (1) is the equation

$$(a(t)\Phi_\beta(y'))' + b(t)\Phi_\gamma(y) = 0, \qquad \gamma > 0. \tag{6}$$

When $\gamma \in (0, \beta)$, Eq. (6) is the sub-half-linear and when $\gamma \in (\beta, \infty)$, it is known as the super-half-linear equation. Particularly in last decade, there has been an increasing interest subject to the oscillation of the super-half-linear Eq. (6). However, there is little known for sub-half-linear equations (see the book by Agarwal, et al. [1]).

As far as the oscillation of less general equation

$$(a(t)y')' + b(t)f(y) = 0 \tag{7}$$

is considered, most of the results on oscillation of Eq. (6) are viable under the condition that $uf(u) > 0$, $u \neq 0$, and f satisfies some certain conditions of superlinearity and sublinearity, see [1, 3, 13, 24, 27, 28] and references therein.

The purpose of this paper is to show how Picone's formula can be used to extend the classical Sturmian theory to nonlinear equations of the form (1). Moreover, we also show that Picone's formula has also been used for proving Leighton type comparison results and setting Wirtinger type inequalities. By applying the comparison results, several oscillation criteria are established and examples are given to illustrate the importance of the results.

2. Main Results

Suppose that x and y are continuously differentiable functions defined on J_0 such that $kx', my' \in C^1(J_0)$. If $y(t) \neq 0$ for any $t \in J_0$, then we may define

$$\nu(t) := \frac{x(t)}{\Phi_\beta(y(t))}\Big\{k(t)\Phi_\beta(x'(t))\Phi_\beta(y(t)) - m(t)\Phi_\beta(y'(t))\Phi_\beta(x(t))\Big\}. \tag{8}$$

For clarity we suppress the variable t. In view of (1) and (2) it is not difficult to see, cf. [9] that

$$\nu' = (k-m)|x'|^{\beta+1} + \left(\sum_{i=1}^{n} q_i|y|^{\alpha_i-\beta} - p\right)|x|^{\beta+1} + m\left(|x'|^{\beta+1} + \beta\left|\frac{xy'}{y}\right|^{\beta+1} - (\beta+1)x'\Phi_\beta\left(\frac{xy'}{y}\right)\right). \tag{9}$$

The following lemmas are needed.

Lemma 2.1. *Let $u, v \in \mathbb{R}$ and $\gamma > 0$ be a constant, then*

$$\mathcal{H}[u,v] := u\Phi_\gamma(u) + \gamma v\Phi_\gamma(v) - (\gamma+1)u\Phi_\gamma(v) \geq 0, \tag{10}$$

with equality holding if and only if $u = v$.

Lemma 2.2. *Let $\{\alpha_j\}$, $j = 1, \dots, n$, be the n-tuple satisfying* (3). *Then there exists an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ satisfying*

(i) $\displaystyle\sum_{j=1}^{n} \alpha_j\eta_j = \beta$, and

(ii) $\displaystyle\sum_{j=1}^{n} \eta_j = 1, \quad 0 < \eta_j < 1.$

Lemma 2.1 is extracted from [6] and the proof of Lemma 2.2 can be obtained easily from that of [20, Lemma 1] by replacing α_i by α_i/β, see also [21, Lemma 1.2].

Note that if $n = 2$, we have $\alpha_1 > \beta > \alpha_2 > 0$. Then, in the case of Lemma 2.2, solving the system of equations in (i) and (ii), one easily gets

$$\eta_1 = \frac{\beta - \alpha_2}{\alpha_1 - \alpha_2}, \qquad \eta_2 = \frac{\alpha_1 - \beta}{\alpha_1 - \alpha_2}. \tag{11}$$

Employing the identity (9), we obtain the following comparison result.

Theorem 2.3. *(Sturm-Picone type comparison) Let $x(t)$ be a solution of Eq. (2) having two consecutive zeros $a, b \in J_0$. If*

$$k(t) \geq m(t); \tag{12}$$

$$\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \geq p(t) \tag{13}$$

for all $t \in [a,b]$, where $\eta_1, \dots, \eta_n$ are positive constants satisfying (i) *and* (ii) *of Lemma 2.2, then every solution $y(t)$ of Eq. (1) either has a zero in (a,b) or is a constant multiple of $x(t)$.*

Proof. Assume that $y(t)$ never vanishes on (a, b). Define $\nu(t)$ as in (8). Then differentiating both side of (8), we obtain (9). Recall the arithmetic-geometric mean inequality [2]

$$\sum_{i=1}^{n} c_i v_i \geq \prod_{i=1}^{n} v_i^{c_i} \tag{14}$$

where $c_i > 0$, $v_i \geq 0$ for $i = 1, 2, \ldots, n$. We can choose $c_i = \eta_i$ satisfying the conditions of Lemma 2.2 for the given α_i's satisfying (3), $i = 1, 2, \ldots, n$. Now, using Ineq. (14) with

$$v_i = \eta_i^{-1} q_i(t) |y(t)|^{\alpha_i - \beta}$$

we obtain

$$\sum_{i=1}^{n} q_i(t) |y(t)|^{\alpha_i - \beta} \geq \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \tag{15}$$

for all $t \in [a, b]$. Using (15), (9) turns out that

$$\nu'(t) \geq \big(k(t) - m(t)\big) |x'(t)|^{\beta+1} + \left(\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) - p(t) \right) |x(t)|^{\beta+1} + m(t) \mathcal{H}\Big[x'(t), \frac{x(t) y'(t)}{y(t)} \Big] \tag{16}$$

Clearly, the last term of (16) is integrable over (a, b) if $y(a) \neq 0$ and $y(b) \neq 0$. Moreover, $\nu(a) = \nu(b) = 0$ in this case. Suppose that $y(a) = 0$. The case $y(b) = 0$ is similar. Since $y'(a) \neq 0$ (otherwise, we have only the trivial solution) and

$$\lim_{t \to a^+} \frac{x(t)}{y(t)} = \lim_{t \to a^+} \frac{x'(t)}{y'(t)} < \infty, \tag{17}$$

we have

$$\lim_{t \to a^+} \Phi_\beta \left(\frac{x(t)}{y(t)} \right) < \infty. \tag{18}$$

It follows from (18) that

$$\lim_{t \to a^+} \mathcal{H}\Big[x'(t), \frac{x(t) y'(t)}{y(t)} \Big] = \lim_{t \to a^+} \left(|x'(t)|^{\beta+1} + \beta \left| \frac{x(t)}{y(t)} \right|^{\beta+1} \left| y'(t) \right|^{\beta+1} - (\beta + 1) x'(t) \Phi_\beta \left(y'(t) \right) \Phi_\beta \left(\frac{x(t)}{y(t)} \right) \right) < \infty \tag{19}$$

and

$$\lim_{t \to a^+} \nu(t) = \lim_{t \to a^+} x(t) \left[k(t) \Phi_\beta(x'(t)) - m(t) \Phi_\beta \left(\frac{x(t)}{y(t)} \right) \Phi_\beta(y'(t)) \right] = 0.$$

Integrating (16) from a to b, we see that

$$\int_a^b \left\{ \big(k(t) - m(t)\big) |x'(t)|^{\beta+1} + \left(\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) - p(t) \right) |x(t)|^{\beta+1} \right\} \mathrm{d}t \leq - \int_a^b m(t) \mathcal{H}\Big[x'(t), \frac{x(t) y'(t)}{y(t)} \Big] \mathrm{d}t \tag{20}$$

Using Ineq. (10) in Lemma 2.1 with $u = x'$, $v = xy'/y$ and $\gamma = \beta$, (20) is only possible under the assumptions (12) and (13) that either $y(t)$ has a zero in (a, b) or is a constant multiple of $x(t)$. $\square$

As an immediate consequence of Thm. 2.3, we have the following oscillation result.

Corollary 2.4. *(Sturm type oscillation) If the conditions* (12)-(13) *of Thm. 2.3 are satisfied for all* $t \in [t_0, \infty)$ *for some integer* $t_0 > 0$*, then Eq. (1) is oscillatory whenever Eq. (2) is oscillatory.*

Next, we provide a Leighton type comparison result between nontrivial solutions of (2) and (1), which may be considered as an extension of the classical comparison theorem of Leighton [16, Cor. 1].

Theorem 2.5. *(Leighton type comparison) Let $x(t)$ be a solution of Eq. (2) having two consecutive zeros $a, b \in J_0$. If*

$$L[x] := \int_a^b \left\{ \big(k(t) - m(t)\big)|x'(t)|^{\beta+1} + \left(\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) - p(t) \right) |x(t)|^{\beta+1} \right\} \mathrm{d}t > 0,$$

then every solution $y(t)$ of Eq. (1) has a zero in (a, b).

Proof. Assume that $y(t)$ has no zero in (a, b). Define the function $v(t)$ as in (8). As in the proof of Thm. 2.3, the functions under integral sign are all integrable regardless of the values of $y(a)$ or $y(b)$, and it follows from (16) that

$$0 = v(b) - v(a) \geq L[x] + \int_a^b m(t)\mathcal{H}\Big[x'(t), \frac{x(t)y'(t)}{y(t)}\Big]\mathrm{d}t > 0, \tag{21}$$

which is a contradiction. Therefore, $y(t)$ must have a zero in (a, b). □

If $L[x] \geq 0$, then we may conclude that either $y(t)$ has a zero in (a, b) or it is a constant multiple of $x(t)$.

Thm's. 2.3 and 2.5 give rise to the following oscillation result.

Corollary 2.6. *Suppose for any given $t_0 > 0$ there exists an interval $(a, b) \subset [t_0, \infty)$ for which either the conditions of Thm. 2.3 or Thm. 2.5 are satisfied, then Eq. (1) is oscillatory.*

In a similar manner we derive the following inequality.

Theorem 2.7. *(Wirtinger type inequality) If there exists a solution $y(t)$ of (1) such that $y \neq 0$ in (a, b), then*

$$W[h] := \int_a^b \left\{ \left(\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \right) |h(t)|^{\beta+1} - m(t)|h'(t)|^{\beta+1} \right\} \mathrm{d}t \leq 0 \tag{22}$$

for all $h \in \Omega_{ab}$, where

$$\Omega_{st} = \Big\{ h \in C^1[s, t] : h(s) = h(t) = 0 \Big\}.$$

Proof. Let y be a solution of (1) such that $y(t) \neq 0$ for any $t \in (a, b)$. We may define

$$u(t) = -m(t)\Phi_\beta\left(\frac{y'(t)}{y(t)}\right)|h(t)|^{\beta+1} \quad \text{for } h \in \Omega_{ab}. \tag{23}$$

In view of (1) and (15), it is not difficult to see that

$$\begin{aligned} u' &= \left(\sum_{i=1}^{n} q_i(t)|y(t)|^{\alpha_i - \beta} \right) |h(t)|^{\beta+1} - m(t)|h'(t)|^{\beta+1} + m(t)\mathcal{H}\Big[h'(t), \frac{h(t)y'(t)}{y(t)}\Big] \\ &\geq \left(\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \right) |h(t)|^{\beta+1} - m(t)|h'(t)|^{\beta+1} + m(t)\mathcal{H}\Big[h'(t), \frac{h(t)y'(t)}{y(t)}\Big] \end{aligned} \tag{24}$$

by suppressing the variable t. It is clear that if $y(a) \neq 0$ and $y(b) \neq 0$, then the last term in (24) is integrable over (a, b). If $y(a) = 0$, then since $y'(a) \neq 0$, it follows from

$$\lim_{t \to a^+} \frac{h(t)}{y(t)} = \lim_{t \to a^+} \frac{h'(t)}{y'(t)} < \infty$$

that

$$\lim_{t\to a^+} \mathcal{H}\Big[h'(t), \frac{h(t)y'(t)}{y(t)}\Big] = |h'(a)|^{\beta+1} + \beta|h'(a)|^{\beta+1} - (\beta+1)h'(a)\Phi_\beta(h'(a)) = 0$$

and

$$u(a) = -\lim_{t\to a^+}\left\{m(t)h(t)\Phi_\beta(y'(t))\Phi_\beta\left(\frac{h(t)}{y(t)}\right)\right\} = -m(a)h(a)\Phi_\beta(h'(a)) = 0.$$

The same argument applies if $y(b) = 0$. Thus, the last term in (24) is integrable on (a, b). Integrating (24) from a till b we see that

$$0 = u(b) - u(a) \geq W[h] + \int_a^b m(t)\mathcal{H}\Big[h'(t), \frac{h(t)y'(t)}{y(t)}\Big]\mathrm{d}t$$

and hence we prove that $W[h] \leq 0$ by using Ineq. (10) in Lemma 2.1 with $u = h'$, $v = hy'/y$ and $\gamma = \beta$. □

We have the following comparison criterion on the existence of a zero of a solution of (1) which can be considered as an extension of Lemma 1.3 in [25] and Cor. 1 in [9] to nonlinear equations.

Corollary 2.8. *If there exists a function $h \in \Omega_{ab}$ such that $W[h] > 0$, then every solution $y(t)$ of Eq. (1) has a zero in (a, b).*

As an immediate consequence of Cor. 2.8, we have the following oscillation result.

Corollary 2.9. *Suppose for any given $t_0 > 0$ there exists an interval $(a, b) \subset [t_0, \infty)$ and a function $h \in \Omega_{ab}$ for which $W[h] > 0$, then Eq. (1) is oscillatory.*

Remark 2.10. The oscillation criteria given in Cor. 2.9 for Eq. (1) can be regarded as an extension of the result given by Sun and Wong [20, Theorem 2] for which the case $q(t) \equiv 0$. In fact when $\beta = 1$, equations (2) and (1) reduce to

$$(k(t)x')' + p(t)x = 0 \tag{25}$$

and

$$(m(t)y')' + \sum_{i=1}^n q_i(t)|y|^{\alpha_i-1}y = 0, \tag{26}$$

respectively. In this case, Picone type formula (5) can be used properly as follows:

$$\frac{x}{y}\,(ykx' - xmy')\Big|_a^b = \int_a^b \left\{(k-m)(x')^2 + \Big(\sum_{i=1}^n q_i|y|^{\alpha_i-1} - p\Big)x^2 + \frac{m}{y^2}\big(x'y - xy'\big)^2\right\}\mathrm{d}t \tag{27}$$

which can be considered as an extension the well-known Picone's formula (4) to the pair of linear and mixed nonlinear equations of the form (25) and (26).

3. Examples

We first recall some basic facts about generalized trigonometric functions [5]. The generalized sine function $S(t)$ is defined as the unique solution of

$$(\Phi_\beta(x'))' + \beta\Phi_\beta(x) = 0, \qquad x(0) = 0, \quad x'(0) = 1, \tag{28}$$

where $\beta > 0$ is a fixed real number. As in the well-known case the generalized cosine function $C(t)$ is then defined by $C(t) = S'(t)$. The generalized tangent function $T(t)$ is defined by

$$T(t) = \frac{S(t)}{C(t)}, \quad t \neq \frac{\pi_\beta}{2} \pmod{\pi_\beta}, \quad \pi_\beta = \frac{2\pi}{\beta+1}\Big/ \sin\frac{\pi}{\beta+1}.$$

Moreover

$$|S(t)|^{\beta+1} + |C(t)|^{\beta+1} = 1 \quad \text{for all} \quad t \in \mathbb{R}. \tag{29}$$

Example 3.1. Consider the pair equations (1) and (28). As we mentioned above, $S(t)$ is an oscillatory solution of (28) with consecutive zeros at $t_n = n\pi_\beta$, $n \in \mathbb{N}$. Then we have the following results:

(**i**) If there exists an $n_0 \in \mathbb{N}$ such that

$$m(t) \le 1;$$
$$\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \ge \beta$$

for all $t \in [n_0\pi_\beta, (n_0+1)\pi_\beta]$, then every solution $y(t)$ of Eq. (1) must have a zero in each interval $(n_0\pi_\beta, (n_0+1)\pi_\beta)$ or $y(t)$ is a constant multiple of $S(t)$ by Thm. 2.3. Moreover, it is oscillatory by Cor. 2.4 or Cor. 2.6.

(**ii**) If there exists an $n_0 \in \mathbb{N}$ such that

$$L[S(t)] = \int_{n_0\pi_\beta}^{(n_0+1)\pi_\beta} \left\{ (1-m(t))|C(t)|^{\beta+1} + \left(\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) - \beta \right) |S(t)|^{\beta+1} \right\} \mathrm{d}t \ge 0,$$

then every solution $y(t)$ of Eq. (1) must have a zero in each interval $(n_0\pi_\beta, (n_0+1)\pi_\beta)$ or $y(t)$ is a constant multiple of $S(t)$ by Thm. 2.5. Moreover, it is oscillatory by Cor. 2.6.

(**iii**) Choosing $h(t) = S(t)$ in (22), we get

$$W[S(t)] = \int_{n_0\pi_\beta}^{(n_0+1)\pi_\beta} \left\{ \left(\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \right) |S(t)|^{\beta+1} - m(t))|C(t)|^{\beta+1} \right\} \mathrm{d}t.$$

If $W[S(t)] > 0$, then every solution $y(t)$ of Eq. (1) must have a zero in each interval $(n_0\pi_\beta, (n_0+1)\pi_\beta)$ by Cor. 2.8 and it is oscillatory by Cor. 2.9.

Example 3.2. Consider the pair equations (1) and the equation

$$(e^{-\beta t}\Phi_\beta(x'))' + \beta e^t \Phi_\beta(x) = 0. \tag{30}$$

It is clear that $S(e^t)$ is an oscillatory solution of (30) with consecutive zeros at $\sigma_n = \ln(n\pi_\beta)$, $n \in \mathbb{N}$. Then we have the following results:

(**i**) If there exists an $m_0 \in \mathbb{N}$ such that

$$m(t) \le e^{-\beta t};$$
$$\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \ge \beta e^t$$

for all $t \in [\sigma_{m_0}, \sigma_{m_0+1}]$, then every solution $y(t)$ of Eq. (1) must have a zero in each interval $(\sigma_{m_0}, \sigma_{m_0+1})$ or $y(t)$ is a constant multiple of $S(e^t)$ by Thm. 2.3. Moreover, it is oscillatory by Cor. 2.4 or Cor. 2.6.

(**ii**) If there exists an $n_0 \in \mathbb{N}$ such that

$$L[S(t)] = \int_{\sigma_{m_0}}^{\sigma_{m_0+1}} \left\{ (1-m(t)e^{\beta t})|C(e^t)|^{\beta+1} + \left(e^{-t} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) - \beta \right) |S(e^t)|^{\beta+1} \right\} e^t \mathrm{d}t \ge 0,$$

then every solution $y(t)$ of Eq. (1) must have a zero in each interval $(\sigma_{m_0}, \sigma_{m_0+1})$ or $y(t)$ is a constant multiple of $S(e^t)$ by Thm. 2.5. Moreover, it is oscillatory by Cor. 2.6.

(**iii**) Choosing $h(t) = S(t)$ in (22), we get

$$W[S(t)] = \int_{m_0\pi_\beta}^{(m_0+1)\pi_\beta} \left\{ \left(\prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \right) |S(t)|^{\beta+1} - m(t)|C(t)|^{\beta+1} \right\} \mathrm{d}t.$$

If $W[S(t)] > 0$, then every solution $y(t)$ of Eq. (1) must have a zero in each interval $(m_0\pi_\beta, (m_0+1)\pi_\beta)$ by Cor. 2.8 and it is oscillatory by Cor. 2.9.

Example 3.3. Consider the nonlinear equation

$$(\Phi_\beta(y'))' + \sum_{i=1}^{n} c_i \Phi_{\alpha_i}(y) = 0 \tag{31}$$

where c_i are positive constants, $i = 1, \dots n$. Using the oscillatory solution $S(t)$ of Eq. (28), we have the following results:

(**i**) If

$$c_n := \prod_{i=1}^{n} \eta_i^{-\eta_i} c_i^{\eta_i} \geq \beta \tag{32}$$

for all n, then every solution $y(t)$ of Eq. (31) must have a zero in each interval $(j\pi_\beta, (j+1)\pi_\beta)$, $j \in \mathbb{N}$, or $y(t)$ is a constant multiple of $S(t)$ by Thm. 2.3. Moreover, it is oscillatory by Cor. 2.4 or Cor. 2.6. On the other hand, we have

$$L[S(t)] = (c_n - \beta) \int_{j\pi_\beta}^{(j+1)\pi_\beta} |S(t)|^{\beta+1} \mathrm{d}t. \tag{33}$$

Since the integral on the left hand side of Ineq. (33) is positive, Ineq. (32) implies Ineq. (33).

(**ii**) Choosing $h(t) = S(t)$ in (22), we get

$$W[S(t)] = \int_{j\pi_\beta}^{(j+1)\pi_\beta} \left\{ c_n |S(t)|^{\beta+1} - |C(t)|^{\beta+1} \right\} \mathrm{d}t = (c_n + 1) \int_{j\pi_\beta}^{(j+1)\pi_\beta} |S(t)|^{\beta+1} \mathrm{d}t - \pi_\beta$$

where $j \in \mathbb{N}$. We can conclude that if

$$c_n > \pi_\beta \left(\int_{j\pi_\beta}^{(j+1)\pi_\beta} |S(t)|^{\beta+1} \mathrm{d}t \right)^{-1} - 1 \tag{34}$$

for all $n \in \mathbb{N}$, then every solution $y(t)$ of Eq. (31) must have a zero in each interval $(j\pi_\beta, (j+1)\pi_\beta)$ by Cor. 2.8 and it is oscillatory by Cor. 2.9. Note that the numbers $\eta_1, \dots, \eta_n$ in c_n are positive constants satisfying (i) and (ii) of Lemma 2.2.

When $\beta = 1$, then $\pi_\beta = \pi_1 = \pi$, $S(t) = \sin t$ and Ineq. (34) turns out to be $c_n > \pi^2/2 - 1$

Example 3.4. When $n = 2$, Eq. (1) reduces to the equation

$$(m(t)\Phi_\beta(y'))' + q_1(t)\Phi_{\alpha_1}(y) + q_2(t)\Phi_{\alpha_2}(y) = 0, \tag{35}$$

where $\alpha_1 > \beta > \alpha_2 > 0$. Choosing the constants η_1 and η_2 as in (11), we have the following comparison results between the solutions of the pair of equations (1) and (35):

(**i**) If there exists an $n_0 \in \mathbb{N}$ such that $m(t) \leq 1$ and

$$(\alpha_1-\alpha_2)^{\alpha_1-\alpha_2} q_1^{\beta-\alpha_2}(t) q_2^{\alpha_1-\beta}(t) \geq \beta^{\alpha_1-\alpha_2}(\beta-\alpha_2)^{\beta-\alpha_2}(\alpha_1-\beta)^{\alpha_1-\beta}$$

for all $t \in [n_0\pi_\beta, (n_0+1)\pi_\beta]$, then every solution $y(t)$ of Eq. (35) must have a zero in each interval $(n_0\pi_\beta, (n_0+1)\pi_\beta)$ or $y(t)$ is a constant multiple of $S(t)$ by Thm. 2.5.

(**ii**) If there exists an $n_0 \in \mathbb{N}$ such that

$$L[S(t)] = \int_{n_0\pi_\beta}^{(n_0+1)\pi_\beta} \left\{ (1-m(t))|C(t)|^{\beta+1} + \left(\Gamma_0 q_1^{\frac{\beta-\alpha_2}{\alpha_1-\alpha_2}}(t) q_2^{\frac{\alpha_1-\beta}{\alpha_1-\alpha_2}}(t) - \beta \right) |S(t)|^{\beta+1} \right\} \mathrm{d}t \geq 0,$$

then every solution $y(t)$ of Eq. (35) must have a zero in each interval $(n_0\pi_\beta, (n_0+1)\pi_\beta)$ or $y(t)$ is a constant multiple of $S(t)$ by Thm. 2.3, where

$$\Gamma_0 = (\alpha_1-\alpha_2)(\beta-\alpha_2)^{-\frac{\beta-\alpha_2}{\alpha_1-\alpha_2}}(\alpha_1-\beta)^{-\frac{\alpha_1-\beta}{\alpha_1-\alpha_2}}.$$

(**iii**) Choosing $h(t) = S(t)$ in (22), we get

$$W[S(t)] = \int_{m_0\pi_\beta}^{(m_0+1)\pi_\beta} \left\{ \left(\Gamma_0 q_1^{\frac{\beta-\alpha_2}{\alpha_1-\alpha_2}}(t) q_2^{\frac{\alpha_1-\beta}{\alpha_1-\alpha_2}}(t) \right) |S(t)|^{\beta+1} - m(t)|C(t)|^{\beta+1} \right\} \mathrm{d}t.$$

If $W[S(t)] > 0$, then every solution $y(t)$ of Eq. (35) must have a zero in each interval $(m_0\pi_\beta, (m_0+1)\pi_\beta)$ by Cor. 2.8 and it is oscillatory by Cor. 2.9.

4. Concluding Remarks

We note that the purpose of this paper is not only obtain some oscillation criteria for the nonlinear equations of the form (1) and (26) but also give some information about the zeros of the solutions of them. There are many oscillation results in the literature for these type of equations by using *Riccati technique* and based on obtaining some *Wirtinger type inequalities*. These results are known as "interval oscillation criteria". However, there is rarely any result about the place of the zeros and hence behavior of the oscillating solutions. In this paper, we gave an answer to the question that how fast the solutions oscillate and what is the distance between the zeros of the solutions roughly.

Finally, we present some open problems concerning possible extensions of Thm. 2.3. Consider the forced super-half-linear equation

$$(m(t)\Phi_\beta(z'))' + q(t)\Phi_\alpha(z) = f(t), \qquad \alpha > \beta > 0. \tag{36}$$

Jaroš et al. [11] derived a Picone's formula for the pair of equations (2) and (36), and they extend the classical Sturmian theory under the assumption $z(t)f(t) \leq 0$ by extending the Picone's type formula (5). It will be of interest to find analogous results for the pair of equations (2) and (36) when $f(t) = 0$. Another interesting problem is to obtain Sturmian like results for half-linear equation (2) and the sub-half-linear (36) (i.e. $\alpha \in (0, \beta)$) with or without forcing term $f(t)$. On the other hand, to dilate the similar results for the pair of half-linear equations with damping

$$(k(t)\Phi_\beta(x'))' + r(t)\Phi_\beta(x') + p(t)\Phi_\beta(x) = 0 \tag{37}$$

and sub(super)-half-linear equations with damping

$$(m(t)\Phi_\beta(y'))' + s(t)\Phi_\beta(y') + q(t)\Phi_\alpha(y) = 0 \tag{38}$$

is another question worth considering, where $\alpha \in (0, \beta)$ or $\alpha \in (\beta, \infty)$ respectively.

When $\alpha = \beta > 0$, the damping terms of Eq.'s (37) and (38) can be annihilated by multiplying them by the functions

$$\exp\left(\int^t r(\tau)/k(\tau)\mathrm{d}\tau\right) \quad \text{and} \quad \exp\left(\int^t s(\tau)/m(\tau)\mathrm{d}\tau\right)$$

respectively, and hence Picone type formula (5) can be employed directly. However, there are some difficulties to control the conditions of comparison results. So we present the last problem as is to find a direct Picone type formula for Eq.'s (37) and (38) without transforming them to half-linear equations. In fact, Picone type formula was first obtained by Kreith [14] in 1973 under some imposed conditions on the coefficients functions for the linear case i.e. $\alpha = \beta = 1$.

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