Filomat 31:5 (2017), 1167–1173
DOI 10.2298/FIL1705167L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Topological Properties of the Hausdorff Fuzzy Metric Spaces

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Abstract. In the paper, necessary and sufficient conditions for two Hausdorff fuzzy metric spaces to be homeomorphic are studied. Also, several properties of the Hausdorff fuzzy metric spaces, as F-boundedness, separability and connectedness are explored.

1. Introduction

The concept of fuzzy metric has been introduced by many authors from different points of view [2, 3, 12, 14]. In particular, by generalizing the concept of probabilistic metric, Kramosil and Michalek [14] obtained the concept of fuzzy metric with the help of continuous t-norms. To make the topology generated by a fuzzy metric to be Hausdorff, George and Veeramani [3] modified in a slight but appealing way the concept given by Kramosil and Michalek. Whereafter, Gregori and Romaguera [10] proved that the topological space generated by a modified fuzzy metric is metrizable. The modified version of fuzzy metric is more restrictive, but it determines the class of spaces that are tightly connected with the class of metrizable topological spaces. So it is interesting to study the new version of fuzzy metric. Kočinac [13] studied some selection properties of fuzzy metric spaces. A common fixed point theorem in M-complete fuzzy metric spaces was proved by Kumar and Mihet [15]. Recently, Gregori et al. [6–11] gave much progress to the study of fuzzy metric spaces. Other more contributions to the study of fuzzy metric spaces can be found in [4, 5, 16–19, 21, 22].

In order to explore hyperspaces in given fuzzy metric spaces, Rodríguez-López and Romaguera [20] constructed the Hausdorff fuzzy metric on the family of nonempty compact sets and discussed precompactness, completeness and completion of the Hausdorff fuzzy metric spaces. Here, we construct another type of the Hausdorff fuzzy metric on the family of nonempty compact sets that coincides with the type due to Rodríguez-López and Romaguera [20]. Moreover, we investigate necessary and sufficient conditions for two Hausdorff fuzzy metric spaces on the family of nonempty compact sets to be homeomorphic. Finally, we explore several properties of the Hausdorff fuzzy metric spaces, as F-boundedness, separability and connectedness.

2010 *Mathematics Subject Classification*. Primary 54A40; Secondary 54B20, 54D05, 54D65, 54E35
Keywords. Fin(X), Comp(X), Hausdorff fuzzy metric, separable, connected, F-bounded
Received: 26 January 2015; Revised: 08 August 2015; Accepted: 09 August 2015
Communicated by Ljubiša D. R. Kočinac
This work was supported by Nation Natural Science Foundation of China (No. 11471153, No. 11571158, No. 11526109, No. 61379021), Natural Science Foundation of Fujian (No. 2016J01671, No. 2015J05011, No. Jk2014028), the Foundation of Fujian Province (No. JA14200)
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2. Preliminaries

Throughout the paper the letter $\mathbb{N}$ will denote the set of all natural numbers. Our basic reference for general topology is [1].

Definition 2.1. ([3]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$ for all $a \in [0,1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Obviously, $a * b = a \cdot b$ and $a * b = \min\{a, b\}$ are two common examples of continuous t-norms.

Definition 2.2. ([3]) A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:

1. $M(x, y, t) > 0$;
2. $M(x, y, t) = 1$ if and only if $x = y$;
3. $M(x, y, t) = M(y, x, t)$;
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
5. the function $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, $(M, *)$ will be called a *fuzzy metric on X*.

Definition 2.3. ([3]) Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0,1)$, $t > 0$ and $x \in X$. The set

$$B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$$

is called the *open ball with center x and radius r with respect to t*.

It is clear that $\{B_M(x, r, t) | x \in X, t > 0, r \in (0, 1)\}$ forms a base of a topology τ_M in X. $\{B_M(x, \frac{1}{n}, \frac{1}{n}) | n \in \mathbb{N}\}$ is a neighborhood base at x for the topology τ_M for all $x \in X$ (see [3]).

Definition 2.4. ([4]) A mapping f from a fuzzy metric space $(X_1, M_1, *_1)$ to a fuzzy metric space $(X_2, M_2, *_2)$ is called *uniformly continuous* if for each $r_2 \in (0, 1)$ and $t_2 > 0$, there exist $r_1 \in (0, 1)$ and $t_1 > 0$ such that $M_2(f(x), f(y), t_2) > 1 - r_2$ whenever $x, y \in X_1$ and $M_1(x, y, t_1) > 1 - r_1$.

Definition 2.5. ([1]) Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \to Y$ be a bijection. If both the mapping f and the inverse mapping $f^{-1} : Y \to X$ are continuous, then f is called a *homeomorphism*.

Definition 2.6. ([1]) A topological space (X, τ_X) is said to be *homeomorphic to* another topological space (Y, τ_Y) if there exists a homeomorphism $f : X \to Y$.

Definition 2.7. ([3]) A fuzzy metric space $(X, M, *)$ is said to be *F-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in X$.

Definition 2.8. ([5]) A fuzzy metric space $(X, M, *)$ is said to be *separable* if (X, τ_M) is separable.

3. Topological Construction of the Hausdorff Fuzzy Metric on Comp(X)

Given a fuzzy metric space $(X, M, *)$, we will denote by $\mathcal{P}(X)$, Comp(X) and Fin(X), the set of nonempty subsets, the set of nonempty compact subsets and the set of nonempty finite subsets of (X, τ_M), respectively. Let $M(a, B, t) := \sup_{b\in B} M(a, b, t)$, $M(B, a, t) := \sup_{b\in B} M(b, a, t)$ for all $a \in X$, $B \in \mathcal{P}(X)$ and $t > 0$ (see Definition 2.4 of [22]). Observe that $M(a, B, t) = M(B, a, t)$. In the following, for any $A \subset X$, the cardinality of A shall be denote by $|A|$.

Definition 3.1. ([20]) Let $(X, M, *)$ be a fuzzy metric space. For every $A, B \in$ Comp(X) and $t > 0$, define H_M: Comp(X)× Comp(X) × $(0, \infty) \to [0, 1]$ by

$$H_M(A, B, t) = \min\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(A, b, t)\}.$$

Then (Comp(X),H_M, $*$) is a fuzzy metric space. $(H_M, *)$ is called *the Hausdorff fuzzy metric on Comp(X).*

Lemma 3.2. ([20]) *Let $(X, M, *)$ be a fuzzy metric space. Then, for each $a \in X$, $B \in$ Comp(X) and $t > 0$, there exists a $b_a \in B$ such that $M(a, B, t) = M(a, b_a, t)$.*

Lemma 3.3. ([16]) *Let $(X, M, *)$ be a fuzzy metric space. Then $H_M(A, B, t) = 1 - \inf\{r | A \subset B_M(B, r, t), B \subset B_M(A, r, t)\}$ for all $A, B \in$ Comp(X) and $t > 0$, where $B_M(A, r, t) = \bigcup_{a\in A} B_M(a, r, t)$.*

Let $(X, M, *)$ be a fuzzy metric space. For each $n \in \mathbb{N}$, put $\mathrm{Fin}_n(X)=\{A \subset X | 1 \le |A| \le n\}$, which we regard as a subspace of Comp(X).

Proposition 3.4. *Let $(X, M, *)$ be a fuzzy metric space. Then $\mathrm{Fin}_n(X)$ is a closed subset of Comp(X).*

Proof. Let $A \in$ Comp(X)\ $\mathrm{Fin}_n(X)$ and $t > 0$. Then A contains at least $n + 1$ points. Now we choose $A_1 \subset A$ with $|A_1| = n + 1$. Put $\varepsilon_0 = \max\{M(x, y, 2t) | x, y \in A_1\}$. Then there exists $\varepsilon_1 \in (\varepsilon_0, 1)$ such that $\varepsilon_1 * \varepsilon_1 > \varepsilon_0$. We claim that $B_{H_M}(A, 1 - \varepsilon_1, t) \cap \mathrm{Fin}_n(X) = \emptyset$. Indeed, otherwise, we can choose $B \in B_{H_M}(A, 1 - \varepsilon_1, t) \cap \mathrm{Fin}_n(X)$. Then $1 \le |B| \le n$. Note that $H_M(A, B, t) > \varepsilon_1$, according to Lemma 3.3, we have that $A \subset B_M(B, 1 - \varepsilon_1, t)$. Hence $A_1 \subset B_M(B, 1 - \varepsilon_1, t)$, i.e., $A_1 \subset \bigcup_{b\in B} B_M(b, 1 - \varepsilon_1, t)$. Since $|B| < |A_1|$, there exist $a_1, a_2 \in A_1$ and $b_1 \in B$ such that $a_1, a_2 \in B_M(b_1, 1 - \varepsilon_1, t)$. Then

$$M(a_1, a_2, 2t) \ge M(a_1, b_1, t) * M(b_1, a_2, t) \ge \varepsilon_1 * \varepsilon_1 > \varepsilon_0 \ge M(a_1, a_2, 2t),$$

which is a contradiction. So $\mathrm{Fin}_n(X)$ is a closed subset of Comp(X). □

Theorem 3.5. *Let $(X, M, *)$ be a fuzzy metric space. Then Fin(X) is an F_σ-set of Comp(X).*

Proof. Observe that Fin(X)$= \bigcup_{n=1}^{\infty} \mathrm{Fin}_n(X)$, it follows from Proposition 3.4 that Fin(X) is an F_σ-set of Comp(X). □

Theorem 3.6. *Let $(X, M, *)$ be a fuzzy metric space. Then X is homeomorphic to $\mathrm{Fin}_1(X)$.*

Proof. Let $f : X \to \mathrm{Fin}_1(X)$ be a mapping defined by $f(x) = \{x\}$ for every $x \in X$. Note that $H_M(\{x\}, \{y\}, t) = M(x, y, t)$ for all $x, y \in X$ and $t > 0$. It is straightforward to show that f is a homeomorphism. □

According to the above theorem, we can regard $(X, M, *)$ as a subspace of (Comp(X),H_M, $*$).

Corollary 3.7. *Let $(X_i, M_i, *_i)(i = 1, 2)$ be two fuzzy metric spaces. Then X_1 is homeomorphic to X_2 if and only if $\mathrm{Fin}_1(X_1)$ is homeomorphic to $\mathrm{Fin}_1(X_2)$.*

Let $(X_i, M_i, *_i)(i = 1, 2)$ be two fuzzy metric spaces and let φ be a continuous mapping from X_1 to X_2. Define a mapping $\varphi^* :$ Comp(X_1) $\to$ Comp(X_2) by $\varphi^*(A) = \varphi(A)$ for every $A \in$ Comp(X_1).

Theorem 3.8. *If both the mapping φ and the inverse mapping φ^{-1} are uniformly continuous. Then the following are equivalent.*

(i) *$\varphi : X_1 \to X_2$ is a homeomorphism.*
(ii) *$\varphi^* : Comp(X_1) \to Comp(X_2)$ is a homeomorphism.*
(iii) *$\varphi^*|_{Fin_1(X_1)} : Fin_1(X_1) \to Fin_1(X_2)$ is a homeomorphism, where $\varphi^*|_{Fin_1(X_1)}$ is the restriction of φ^* on $Fin_1(X_1)$.*

Proof. (ii)⇒(iii) Suppose that $\varphi^* : \mathrm{Comp}(X_1) \to \mathrm{Comp}(X_2)$ is a homeomorphism. Since $\varphi^*(\{x_1\}) = \varphi(x_1)$ for every $x_1 \in X_1$, it is easy to see that $\varphi^*|_{\mathrm{Fin}_1(X_1)} : \mathrm{Fin}_1(X_1) \to \mathrm{Fin}_1(X_2)$ is a homeomorphism.

(iii)⇒(i) Suppose that $\varphi^*|_{\mathrm{Fin}_1(X_1)} : \mathrm{Fin}_1(X_1) \to \mathrm{Fin}_1(X_2)$ is a homeomorphism. Since $\varphi^*(\{x_1\}) = \varphi(x_1)$ for every $x_1 \in X_1$, we get that φ is a bijection. Let $x \in X_1$ and $n \in \mathbb{N}$. Since $\varphi^*|_{\mathrm{Fin}_1(X_1)}$ is continuous, we can find $k \in \mathbb{N}$ such that

$$\varphi^*|_{\mathrm{Fin}_1(X_1)}(B_{H_{M_1}}(\{x\}, \frac{1}{k}, \frac{1}{k}) \cap \mathrm{Fin}_1(X_1)) \subseteq B_{H_{M_2}}(\varphi^*(\{x\}), \frac{1}{n}, \frac{1}{n}) \cap \mathrm{Fin}_1(X_2).$$

Observe that, for each $i \in \{1, 2\}$, $H_{M_i}(\{y\}, \{z\}, t) = M_i(y, z, t)$ whenever $y, z \in X_i$ and $t > 0$. We have that $\varphi(B_{M_1}(x, \frac{1}{k}, \frac{1}{k})) \subseteq B_{M_2}(\varphi(x), \frac{1}{n}, \frac{1}{n})$. Thus φ is continuous. To prove that φ^{-1} is continuous we use the similar argument as above.

(i)⇒(ii) Suppose that $\varphi : X_1 \to X_2$ is a homeomorphism. Let $A, B \in \mathrm{Comp}(X_1)$ with $A \neq B$. Then

$$\varphi^*(A) = \varphi(A) \neq \varphi(B) = \varphi^*(B).$$

Also, for any $C \in \mathrm{Comp}(X_2)$,

$$(\varphi^*)^{-1}(C) = \varphi^{-1}(C) \in \mathrm{Comp}(X_1).$$

Hence φ^* is a bijection. Let $r \in (0, 1)$ and $t > 0$. Since φ is uniformly continuous, then there exist $r_0 \in (0, 1)$ and $t_0 > 0$ such that $M_2(\varphi(a), \varphi(b), t) > 1 - r$ whenever $a, b \in X_1$ and $M_1(a, b, t_0) > 1 - r_0$. Let $A \in \mathrm{Comp}(X_1)$. Then

$$\varphi^*(B_{H_{M_1}}(A, r_0, t_0)) \subseteq B_{H_{M_2}}(\varphi^*(A), r, t).$$

In fact, for each $B \in B_{H_{M_1}}(A, r_0, t_0)$, we get that $H_{M_1}(A, B, t_0) > 1 - r_0$. According to Proposition 3.3, we obtain that $A \subseteq B_{M_1}(B, r_0, t_0)$ and $B \subseteq B_{M_1}(A, r_0, t_0)$. Now, consider $A \subseteq B_{M_1}(B, r_0, t_0)$. Then, for each $a \in A$, there exists $b \in B$ such that $M_1(a, b, t_0) > 1 - r_0$. Hence $M_2(\varphi(a), \varphi(b), t) > 1 - r$. It follows that $\varphi(A) \subseteq B_{M_2}(\varphi(B), r, t)$, i.e., $\varphi^*(A) \subseteq B_{M_2}(\varphi^*(B), r, t)$. Analogously, we get that $\varphi^*(B) \subseteq B_{M_2}(\varphi^*(A), r, t)$. Hence $H_{M_2}(\varphi^*(A), \varphi^*(B), t) > 1-r$. Consequently, $\varphi^*(B) \in B_{H_{M_2}}(\varphi^*(A), r, t)$. So φ^* is continuous. Since φ^{-1} is uniformly continuous, a similar argument shows that $(\varphi^*)^{-1}$ is continuous. Thus $\varphi^* : \mathrm{Comp}(X_1) \to \mathrm{Comp}(X_2)$ is a homeomorphism. □

4. Some Properties of the Hausdorff Fuzzy Metric on Comp(X)

In the section, we will study F-boundedness, separability and connectedness of the Hausdorff fuzzy metric spaces on Comp(X).

Lemma 4.1. ([20]) *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous functions on $X \times X \times (0, \infty)$.*

Lemma 4.2. *Let $(X, M, *)$ be a fuzzy metric space, $B \in Comp(X)$ and $t > 0$. Then $x \mapsto M(x, B, t)$ is a continuous function on X.*

Proof. Let $x_0 \in X$ and $t > 0$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X with x_n converging to x_0. Since $\{M(x_n, B, t)\}_{n\in\mathbb{N}}$ is a sequence in [0,1], there is a subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that the sequence $\{M(x_{n_m}, B, t)\}_{m\in\mathbb{N}}$ converges to some point of [0,1]. Note that $\{x_{n_m}\}_{m\in\mathbb{N}}$ converges to x_0, we claim that $x \mapsto M(x, B, t)$ is continuous function on X. □

Remark 4.3. Observe that Lemma 4.2 also shows that $x \mapsto M(A, x, t)$ is a continuous function on X for all $A \in \text{Comp}(X)$ and $t > 0$.

Proposition 4.4. *Let $(X, M, *)$ be a fuzzy metric space. Then, for each $A, B \in Comp(X)$ and $t > 0$, there exist $a_0 \in A$ and $b_0 \in B$ such that $H_M(A, B, t) = M(a_0, b_0, t)$.*

Proof. Without loss of generality, we suppose that $H_M(A, B, t) = \inf_{a \in A} M(a, B, t)$. Due to Lemma 4.2, we deduce that $\{M(a, B, t) | a \in A)\}$ is a compact subset of [0,1]. Then there exists $a_0 \in A$ such that $M(a_0, B, t) = \inf_{a \in A} M(a, B, t)$. Also, according to Lemma 3.2, we can find a $b_0 \in B$ such that $M(a_0, b_0, t) = M(a_0, B, t)$. So $H_M(A, B, t) = M(a_0, b_0, t)$. □

Theorem 4.5. *Let $(X, M, *)$ be a fuzzy metric space. Then $(Comp(X), H_M, *)$ is F-bounded if and only if $(X, M, *)$ is F-bounded.*

Proof. Suppose that $(\text{Comp}(X), H_M, *)$ is F-bounded. Then there exist $r \in (0, 1)$ and $t > 0$ such that $H_M(A, B, t) > 1 - r$ for all $A, B \in \text{Comp}(X)$. Let $x, y \in$ X. Observe that $M(x, y, t) = H_M(\{x\}, \{y\}, t) > 1 - r$, we conclude that $(X, M, *)$ is F-bounded.

Conversely, suppose that $(X, M, *)$ is F-bounded. Then there exist $r \in (0, 1)$ and $t > 0$ such that $M(x, y, t) > 1 - r$ for all $x, y \in X$. Let $A, B \in \text{Comp}(X)$. According to Proposition 4.4, we can find $a_0 \in A$ and $b_0 \in B$ such that $H_M(A, B, t) = M(a_0, b_0, t)$. Hence $H_M(A, B, t) > 1 - r$. We are done. □

Lemma 4.6. ([1]) *Let (X, τ) be a metrizable topological space and S a subspace of X. If X is separable, then so is S.*

Lemma 4.7. ([20]) *Let Y be a dense subset of a fuzzy metric space $(X, M, *)$. Then Fin(Y) is dense in (Comp(X),H_M, $*$).*

Theorem 4.8. *Let $(X, M, *)$ be a fuzzy metric space. Then $(Comp(X), H_M, *)$ is separable if and only if $(X, M, *)$ is separable.*

Proof. Assume that $(\text{Comp}(X), H_M, *)$ is separable. Since X is a subspace of $\text{Comp}(X)$, it follows from Lemma 4.6 that $(X, M, *)$ is separable.

Conversely, assume that $(X, M, *)$ is separable. Let Y be a countable dense subset of X. Then, according to Lemma 4.7, $\text{Fin}(Y)$ is dense in $(\text{Comp}(X),H_M, *)$. Since $\text{Fin}(Y)$ is countable, we conclude that $(\text{Comp}(X), H_M, *)$ is separable. □

Definition 4.9. A fuzzy metric space $(X, M, *)$ is said to be *connected* if (X, τ_M) is connected.

Theorem 4.10. *Let $(X, M, *)$ be a fuzzy metric space. Then $(Fin_n(X), H_M, *)$ is connected for every $n \in \mathbb{N}$ if and only if $(X, M, *)$ is connected.*

Proof. Suppose that $(\text{Fin}_n(X), H_M, *)$ is connected for every $n \in \mathbb{N}$. Then $\text{Fin}_1(X)$ is connected. Due to Theorem 3.6, we deduce that $(X, M, *)$ is connected.

Conversely, suppose that $(X, M, *)$ is connected. Then, according to Theorem 3.6, we conclude that $\text{Fin}_1(X)$ is connected. Assume that $\text{Fin}_k(X)(k \geq 1)$ is connected. To complete our proof, it suffices to prove that $\text{Fin}_{k+1}(X)$ is connected. Put $\text{Fin}_k^k(X) = \{A \subset X | |A| = k\}$. Let $F \in \text{Fin}_k^k(X)$ and $\mathcal{A}_F = \{F \cup \{x\} | x \in X\}$. We claim that $\mathcal{A}_F$ is a connected subset of $\text{Fin}_{k+1}(X)$. In fact, otherwise, we can find two nonempty disjoint closed subsets $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}_F$ such that $\mathcal{A}_F = \mathcal{B} \cup \mathcal{C}$. Without loss of generality, we may assume that $F \in \mathcal{C}$. Put $Y = \bigcup\{B \backslash F | B \in \mathcal{B}\}$ and $Z = (\bigcup\{C \backslash F | C \in \mathcal{C}\}) \cup F$. Then Y and Z are two nonempty disjoint subsets of X with $X = Y \cup Z$. Next, we shall show that Y and Z are both closed subsets of X. Let $y \in Y$. Then $F \cup \{y\} \in \mathcal{B}$. Thus, for each $t > 0$, there exists an $r_0 \in (0, 1)$ such that

$$B_{H_M}(F \cup \{y\}, r_0, t) \cap \mathcal{C} = \emptyset.$$

Therefore, for each $z \in Z$, we have that

$$H_M(F \cup \{z\}, F \cup \{y\}, t) = \min\{\inf_{a \in F \cup \{z\}} M(a, F \cup \{y\}, t), \inf_{b \in F \cup \{y\}} M(F \cup \{z\}, b, t)\} \leq r_0.$$

Since

$$M(z, y, t) \leq M(z, F \cup \{y\}, t) = \inf_{a \in F \cup \{z\}} M(a, F \cup \{y\}, t)$$

and

$$M(z, y, t) \leq M(F \cup \{z\}, y, t) = \inf_{b \in F \cup \{y\}} M(F \cup \{z\}, b, t),$$

we get that

$$M(z, y, t) \leq H_M(F \cup \{z\}, F \cup \{y\}, t) \leq r_0.$$

So $B_M(y, r_0, t) \cap Z = \emptyset$, which implies that Z is a closed subset of X. To prove that Y is a closed subset of X we use a similar argument. Hence $(X, M, *)$ fails to be connected, a contradiction occurs. Consequently, $\mathcal{A}_F$ is a connected subset of $\mathrm{Fin}_{k+1}(X)$. Since

$$\mathrm{Fin}_k(X) \cap \mathcal{A}_F = \{F\} \neq \emptyset,$$

we deduce that $\mathrm{Fin}_k(X) \cup \mathcal{A}_F$ is a connected subset of $\mathrm{Fin}_{k+1}(X)$. Observe that

$$\mathrm{Fin}_{k+1}(X) = \bigcup_{F \in \mathrm{Fin}_k^k(X)} (\mathrm{Fin}_k(X) \cup \mathcal{A}_F)$$

and

$$\bigcap_{F \in \mathrm{Fin}_k^k(X)} (\mathrm{Fin}_k(X) \cup \mathcal{A}_F) = \mathrm{Fin}_k(X) \neq \emptyset,$$

we conclude that $\mathrm{Fin}_{k+1}(X)$ is connected. The proof is finished. □

Lemma 4.11. ([1]) *The union of collection of connected subspaces of a topological space (X, τ_X) that have a point in common is connected.*

Lemma 4.12. ([1]) *Let A be a connected subspace of a topological space (X, τ_X). Then the closure $\overline{A}$ of A is also connected.*

Theorem 4.13. *Let $(X, M, *)$ be a fuzzy metric space. If $(X, M, *)$ is connected, then so is (Comp(X),$H_M, *$).*

Proof. Suppose that $(X, M, *)$ is connected. According to Theorem 4.10, we have that $(\mathrm{Fin}_n(X), H_M, *)$ is connected for every $n \in \mathbb{N}$. Note that

$$\mathrm{Fin}(X) = \bigcup_{n=1}^{\infty} \mathrm{Fin}_n(X)$$

and

$$\bigcap_{n=1}^{\infty} \mathrm{Fin}_n(X) = \mathrm{Fin}_1(X) \neq \emptyset.$$

It follows from Lemma 4.11 that Fin(X) is connected. Thanks to Lemma 4.7 and Lemma 4.12, we deduce that (Comp(X),$H_M, *$) is connected. □

Question 4.14. *Let $(X, M, *)$ be a fuzzy metric space. If (Comp(X),$H_M, *$) is connected, then is $(X, M, *)$ connected?*

5. Conclusion

In this work, we have studied necessary and sufficient conditions for two Hausdorff fuzzy metric spaces on the family of nonempty compact sets to be homeomorphic. Moreover, we have investigated some properties of the Hausdorff fuzzy metric spaces, as F-boundedness, separableness and connectedness.

Since some fixed point theorems for contractions in fuzzy metric spaces have been proved, a natural question arises:

Can we give some contraction theorems in Hausdorff fuzzy metric spaces?

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