Filomat 31:5 (2017), 1129–1139
DOI 10.2298/FIL1705129B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Kato Linear Relations

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Abstract. For a Banach space the notions of generalized Kato linear relation and the corresponding spectrum are introduced and studied. We show that the symmetric difference between the generalized Kato spectrum and the Goldberg spectrum of multivalued linear operators in Banach spaces is at most countable. The obtained results are used to describe the generalized Kato spectrum of the inverse of the left shift operator regarded as a linear relation.

1. Introduction

The generalized Kato decomposition for operators in Banach spaces was introduced by Mbekhta [26] as an extension of the Kato decomposition which arises from the classical treatment of perturbation theory of Kato [19]. In the last decades it has greatly benefited from the work of many authors, in particular from the work of Mbekhta [26–28], Aiena [1], Bouamama [7], Benharrat-Messirdi [5], Jiang-Zhong [16, 17]. The operators which satisfy this property form a class which includes many important classes of operators as for example, the class of quasi-Fredholm, regular, Kato type, semi-Fredholm and B-Fredholm operators.

Linear relations made their appearance in Functional Analysis in J. von Neumann [30] motivated by the need to consider adjoints of non-densely defined operators used in applications to the theory of generalized equations [8], and also by the need to consider the inverses of certain operators used in the study of some Cauchy problems associated with parabolic type equations in Banach spaces [12]. The investigation of linear relations in the last years, becomes more significance since they have applications in problems in Physics and other areas of applied mathematics. We cite some of them,

- The treatment of degenerate boundary value problems (see, for instance, [10] and the references therein).
- The development of fixed point theory for linear relations to the existence of mild solutions of quasi-linear differential inclusions of evolution and also to many problems of fuzzy theory, game theory and mathematical economics, discontinuous differential equations which occur in the biological sciences, optimal control, computing homology of operators, computer assisted proofs in dynamics and digital imaging, (see [18] and the references therein).

2010 *Mathematics Subject Classification*. Primary 47A06; Secondary 47A53
Keywords. Regular linear relation, Generalized Kato linear relation, Generalized Kato spectrum, Goldberg spectrum.
Received: 16 January 2015; Accepted: 09 December 2015
Communicated by Dragan S. Djordjević
The work of T. Álvarez was supported by MICINN (Spain) Grant MTM2010-20190.
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- The applications of the spectral theory of linear relations to the study of many problems of operators as, for example, the spectral theory of ordered pair of operators and of linear bundles. For a deeper information on these and other applications of the spectral theory of linear relations we refer to [4] and the references therein.
- The applications to the invariant subspace problem, see [14].

The purpose of this paper is to show that many of the results of [1] and [27] concerning to the generalized Kato operators and the results of [16, 17] concerning the the generalized Kato spectrum of bounded operators remain valid in the context of closed linear relations. Our paper is organized as follows: In the next section, we set up some give some notations and present some auxiliary results which are needed in the following sections. In section 3, we introduce and study the class of a generalized Kato linear relations. We investigate the relationship between a generalized Kato linear relation and its adjoint and we also give a decomposition of a generalized Kato linear relation T as an operator-like sum $T = D + Q$ where D is a regular linear relation and Q is an everywhere defined quasi-nilpotent operator with certain additional properties. Section 4, contains the main results concerning the the generalized Kato spectrum of a linear relation. In particular, we extended to the case of closed linear relations, the result concerning the symmetric difference between the generalized Kato spectrum and the Goldberg spectrum of an operator proved by M. Benharrat and B. Messirdi [5]. Finally, we apply the results obtained in this section to calculate the generalized Kato, regular and Goldberg spectra of the left shift operator and its inverse regarded as a linear relation.

2. Preliminary Results

We adhered to the notation and terminology of the monographs [9] and [31]. Let E be a linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation T or multivalued linear operator in E is any mapping having domain $\mathcal{D}(T)$ a nonempty subspace of E, and taking values in the collection of nonempty subsets of E such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all nonzero scalars α, β and $x_1, x_2 \in \mathcal{D}(T)$. For $x \in E \setminus \mathcal{D}(T)$ we define $Tx = \emptyset$. With this convention, we have $\mathcal{D}(T) = \{x \in E : Tx \neq \emptyset\}$. The class of all linear relations in E is denoted by $LR(E)$.

Let $T \in LR(E)$. Then T is uniquely determined by its graph which is defined by $G(T) := \{(x, y) \in E \times E : x \in \mathcal{D}(T), y \in Tx\}$. See that in the sequel we identify T with its graph. The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) := \{(y, x) \in E \times E : (x, y) \in G(T)\}$. The subpaces $T(0)$, $T^{-1}(0) := N(T)$ and $R(T) := T(\mathcal{D}(T))$ are called the multivalued part, the null space and the range of T, respectively. We say that T is an operator if T maps the points of its domain to singletons equivalently if $T(0) = \{0\}$, injective if $N(T) = \{0\}$ and T is called surjective if $R(T) = E$.

For $T, S \in LR(E)$ the linear relations $T + S$, $T \hat{+} S$, $T \oplus S$ and ST are defined by

$$G(T + S) := \{(x, y + z) : (x, y) \in G(T), (x, z) \in G(S)\},$$

$$G(T \hat{+} S) := \{(x + u, y + v) : (x, y) \in G(T), (u, v) \in G(S)\},$$

the last sum is direct when $G(T) \cap G(S) = \{(0, 0)\}$. In this case we write $T \oplus S$.

$$G(ST) := \{(x, z) \in E \times E : (x, y) \in G(T), (y, z) \in G(S) \text{ for some } y \in E\}.$$

It is easy to see that

$$\mathcal{D}(ST) = \{x \in \mathcal{D}(T) : Tx \cap \mathcal{D}(S) \neq \emptyset\}.$$

For $\lambda \in \mathbb{K}$ and $T \in LR(E)$, the linear relation λT is defined by

$$G(\lambda T) := \{(x, \lambda y) : (x, y) \in G(T)\},$$

while $\lambda - T$ stands for $= \lambda I - T$ where I is the identity operator on E and since the product of linear relations is clearly associative, if $n \in \mathbb{Z}$, T^n is defined as usual with $T^0 = I$ and $T^1 = T$. The resolvent set of T is the set

$$\rho(T) := \{\lambda \in \mathbb{K} : \lambda - T \text{ is injective and surjective }\},$$

and the spectrum of T,

$$\sigma(T) := \mathbb{K} \setminus \rho(T).$$

Let $T \in LR(E)$, if M is a subspace of E then the restriction T_M of T in M is defined by

$$G(T_M) := G(T) \cap (M \times M).$$

Note that $R(T_M) \subset M$ and $\mathcal{D}(T_M) \subset M$ by definition.

Definition 2.1. *Let M and N are two subspaces of a linear space E such that $E = M \oplus N$ (that is $E = M + N$ and $M \cap N = 0$). We say that $T \in LR(E)$ is completely reduced by the pair (M, N), denoted as $(M, N) \in Red(T)$, if $T = T_M \oplus T_N$.*

As an immediate consequence of Definition 2.1 we obtain that if $(M, N) \in Red(T)$ then $N(T_M) + N = N(TP_M)$ and $R(T_M) = R(TP_M)$ where P_M and P_N denote the projections of E onto M and N, respectively.

Lemma 2.2. *[31, Lemma 2.1 and Corollary 2.2] Let M and N be subspaces of a linear space E and let $T \in LR(E)$, with $(M, N) \in Red(T)$. Then, for each $n \in \mathbb{N}$ we have:*

(i) $T^n = T^n_M \oplus T^n_N$, $\mathcal{D}(T^n) = \mathcal{D}(T^n_M) \oplus \mathcal{D}(T^n_N)$, $N(T^n) = N(T^n_M) \oplus N(T^n_N)$, $R(T^n) = R(T^n_M) \oplus R(T^n_N)$ *and* $T^n(0) = T^n_M(0) \oplus T^n_N(0)$.

(ii) $T^n = T^n_M P_M + T^n_N P_N$ *and* $R(T^n) = R(T^n_M P_M) + R(T^n_N P_N)$.

(iii) $T^n_M P_M = (TP_M)^n$ *and* $T^n_N P_N = (TP_N)^n$.

Suppose that E is a normed space and let $T \in LR(E)$. If Q_T denotes the quotient map from E onto $E/\overline{T(0)}$, then it easy to see that $Q_T T$ is an operator and thus we can define $\|Tx\| := \|Q_T Tx\|$, $x \in \mathcal{D}(T)$ and $\|T\| := \|Q_T T\|$ called the norm of Tx and T respectively. We say that T is closed if its graph is a closed subspace of $E \times E$, continuous if $\|T\| < \infty$, bounded if T is everywhere defined and continuous, and T is called open if T^{-1} is continuous equivalently if $\gamma(T) > 0$ where

$$\gamma(T) = \begin{cases} +\infty & \text{if} \quad \mathcal{D}(T) \subset \overline{N(T)} \\ \inf\left\{ \frac{\|Tx\|}{\operatorname{dist}(x, N(T))} : x \in \mathcal{D}(T) \setminus \overline{N(T)} \right\} & \quad \text{otherwise.} \end{cases}$$

Let M be a subspace of E and let E^* be the dual space of E. As it is usual, $M^\perp = \{x^* \in E^* : x^*(M) = 0\}$. Moreover, if M and N are closed linear subspaces of E then $(M + N)^\perp = M^\perp \cap N^\perp$. The dual relation $M^\perp + N^\perp = (M \cap N)^\perp$ is not always true, since $(M \cap N)^\perp$ is always closed but $M^\perp + N^\perp$ need not be closed. However, a classical theorem establishes that $M^\perp \cap N^\perp$ is closed in E^* if and only if $M + N$ is closed in E, (see [20, Theorem 4.8, Chapter IV]).

The adjoint T^* of a linear relation T is defined by

$$G(T^*) = G(-T^{-1})^\perp \subset E^* \times E^*.$$

This means that $(y^*, x^*) \in G(T^*)$ if and only if $y^*(y) - x^*(x) = 0$ for all $(x, y) \in G(T)$. Recall that if T is a linear relation in E then T^* is a closed linear relation in E^* such that

$$\mathcal{D}(T^*) = \{y^* \in E^* : y^* T \text{ is continuous operator }\}.$$

Note that if T is a closed linear relation in a Banach space E then $R(T)$ is closed if and only if $R(T^*)$ is closed if and only if T is open and furthermore if T is everywhere defined then T is bounded. The proofs of these properties can be found in [9, Chapter III].

The following result concerning the behaviour of the adjoint in products will be very useful in the sequel.

Lemma 2.3. *Let E be a Banach space and let $S, T \in LR(E)$ be closed and everywhere defined. Then*

(i) $(ST)^* = T^* S^*$.

(ii) If $\rho(T) \neq \emptyset$, then $(T^n)^* = (T^*)^n$, for all $n \in \mathbb{N}$.

Proof. (i) Since $\mathcal{D}(S) = \mathcal{D}(T) = E$, we have that ST is everywhere defined so that $\mathcal{D}(ST)$ is a core of T (in the sense of [9, Definition IV. 4.1]. On the other hand, since S is continous and $S(0) \subset ST(0)$ we infer from [9, Proposition II. 1. 10] and [15, Theorem 2.10] that S is (ST)-co-continuous (in the sense of [15, Definition 2.3]) and that S^* is $(ST)^*$-bounded (in the sense of [15, Definition 2.7]). In this situation, it follows from [15, Theorem 3.1] that $(ST)^* = T^*S^*$.

(ii) Clearly $\mathcal{D}(T^n) = E$ and by virtue of [11, Lemma 3.1] one has that for all $n \in \mathbb{N}$, T^n is closed and hence bounded. Assume that $(T^n)^* = (T^*)^n$. Then by the statement (i) applied to T^n and T we obtain that $(T^nT)^* = (T^{n+1})^* = T^*(T^*)^n = (T^*)^{n+1}$. Thus (ii) has been proved. □

Recall that if T and S are linear relations such that $T \subset S$ then $S^* \subset T^*$. Indeed, $T \subset S$, so $-T^{-1} \subset -S^{-1}$, hence $G(S^*) = G(-S^{-1})^\perp \subset G(-T^{-1})^\perp = G(T^*)$.

Lemma 2.4. *Let E be a Banach space and let $T \in LR(E)$ be closed and everywhere defined such that $(M, N) \in Red(T)$ with T_N an operator. Then*

(i) The linear relations P_MT, TP_M, P_NT and TP_N are bounded, $P_MT = TP_M$ and $P_NT = TP_N$.

(ii) $(TP_M)^* = P_{N^\perp}T^* = T^*P_{N^\perp}$ and $(TP_N)^* = P_{M^\perp}T^* = T^*P_{M^\perp}$.

Proof. (i) Since T is everywhere defined by hypothesis it follows that $\mathcal{D}(P_MT) = \mathcal{D}(T) = E$. Furthermore, the boundedness of T and P_M combined with [9, Corollary II.3.13] ensures that P_MT and TP_M are both continuous.

Applying now Lemma 2.2 one has $TP_M(0) = T(0) = T_M(0) \oplus T_N(0) = T_M(0) = P_MT(0)$ because T_N is an operator.

Let $(x, y) \in P_MT$, then using again Lemma 2.2 it follows that $y \in P_MTx = P_M(T_MP_Mx + T_NP_Nx) = TP_Mx$, so which shows that $P_MT \subset TP_M$. A combination of the above properties and [9, Exercice I. 2.14 (b)] leads to $P_MT = TP_M$. Similarly, $P_NT = TP_N$ is a bounded linear relation.

(ii) This is an immediate consequence of Lemma 2.3 (i) and the statement (i). □

Let M, N be two closed linear subspaces of the Banach space E and let

$$\delta(M, N) = \sup\{\text{dist}(x, N) : x \in M, \|x\| = 1\},$$

in the case that $M \neq \{0\}$, otherwise we define $\delta(\{0\}, N) = 0$.

The gap between M and N is defined by $\widehat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}$.

$\widehat{\delta}$ is a metric on the set $\mathcal{F}(E)$ of all linear closed subspaces of E, and the convergence $M_n \longrightarrow M$ in $\mathcal{F}(E)$ is obviously defined by $\widehat{\delta}(M_n, M) \longrightarrow 0$ as $n \longrightarrow \infty$ in $\mathbb{R}$. Moreover, $(\mathcal{F}(E), \widehat{\delta})$ is a complete metric space (see [20]).

In the rest of this section E will be a complex Banach space and T will always denote a closed linear relation in E except where stated otherwise.

Lemma 2.5. *Let $\lambda, \mu \in \mathbb{C}$, Then,*

(i) $\gamma(\lambda - T)\delta(N(\mu - T), N(\lambda - T)) \leq |\mu - \lambda|$.

(ii) $\min\{\gamma(\mu - T), \gamma(\lambda - T)\}\widehat{\delta}(N(\mu - T), N(\lambda - T)) \leq |\mu - \lambda|$.

Proof. (i). The statement is trivial for $\lambda = \mu$. Suppose that $\lambda \neq \mu$, we have two cases. First, assume that $\gamma(\lambda - T) = \infty$. Then $D(\lambda - T) = D(T) = E \subset \overline{N(\lambda - T)}$. Since $\lambda - T$ is closed, by [9, Exercise II.5.16], $N(\lambda - T)$ is closed. Hence $N(\mu - T) \subset E = N(\lambda - T)$ which implies that

$$\delta(N(\mu - T), N(\lambda - T)) = \delta(N(\mu - T), E) = 0 \leq \frac{1}{\gamma(\lambda - T)} |\mu - \lambda| = 0.$$

Second, $\gamma(\lambda - T) < \infty$. In such case

$$\gamma(\lambda - T) = \inf\left\{\frac{\|(\lambda - T)x\|}{\text{dist}(x, N(\lambda - T))} : x \in E \setminus N(\lambda - T)\right\}.$$

If $x \in N(\mu - T) \cap N(\lambda - T)$ then (i) holds. Assume that $x \in N(\mu - T) \setminus N(\lambda - T)$. Then

$$\begin{aligned} \gamma(\lambda - T)\text{dist}(x, N(\lambda - T)) &\leq \|(\lambda - T)x\| \\ &= \|Q_{\lambda-T}(\lambda - T)x\| \\ &= \left\|Q_{\lambda-T}(\lambda - \mu + \mu - T)x\right\| \\ &= \left\|Q_{\mu-T}(\mu - T)x + Q_{\mu-T}(\lambda - \mu)x\right\| \\ &= \left\|Q_{\mu-T}(\mu - T)(0) + Q_{\mu-T}(\lambda - \mu)x\right\| \\ &\leq \left|\mu - \lambda\right| \|x\|, \end{aligned}$$

by [9, Proposition I.28], we infer that $x \in N(\mu - T)$ if and only if $(\mu - T)x = (\mu - T)(0)$ and clearly $(\lambda - T)(0) = (\mu - T)(0) = T(0)$. Therefore $\gamma(\lambda - T)\text{dist}(x, N(\lambda - T)) \leq \left|\mu - \lambda\right| \|x\|$ if $x \in N(\mu - T)$. Thus (i) holds.

(ii). Clearly, the inequality follows from (i) by interchanging λ and μ. $\square$

Lemma 2.6. *[1, Lemma 1.34] Let M and N be two closed subspaces of E. For every $x \in X$ and $0 < \epsilon < 1$ there exists $x_0 \in X$ such that $(x - x_0) \in M$ and*

$$dist(x_0, N) \geq \left((1 - \epsilon)\frac{1 - \delta(M, N)}{1 + \delta(M, N)}\right)\|x_0\|. \tag{1}$$

The following purely algebraic lemma helps to read Definition 2.8 below.

Lemma 2.7. *[23, Lemma 3.7] Let S be a linear operator in a linear space. The following statements are equivalent:*

(i) $N(S) \subseteq R(S^m)$, for all nonnegative integer m.

(ii) $N(S^n) \subseteq R(S)$, for all nonnegative integer n.

(iii) $N(S^n) \subseteq R(S^m)$, for all nonnegative integers n and m.

Definition 2.8. *[2, Definition 10] The linear relation T is called regular if $R(T)$ is closed and T verifies one of the equivalent conditions of Lemma 2.7.*

Proposition 2.9. *For $(M, N) \in Red(T)$, T is regular if and only if both T_M and T_N are regular.*

Proof. Let $(M, N) \in Red(T)$, then $T^n = T^n_M \oplus T^n_N$, $R(T^n) = R(T^n_M) \oplus R(T^n_N)$ and $N(T^n) = N(T^n_M) \oplus N(T^n_N)$ for every $n \in \mathbb{N}$. It easy to see that T^{-1} is continuous if and only both T^{-1}_M and T^{-1}_N are continuous. So that T^{-1}_M (resp. T^{-1}_N) is continuous if and only if T_M (resp. T_N) is open, this is equivalent to $R(T_M)$ (resp. $R(T_N)$) is closed. Hence $R(T_M)$ and $R(T_N)$ are closed if and only if $R(T)$ is closed. $\square$

The following results concerning the product of regular linear relations will be useful in the following section.

Theorem 2.10. *[2] Let T, S be two closed linear relations in E with $TS = ST$. If TS is regular, then both T and S are regular.*

Theorem 2.11. *Let T, S be two closed linear relations in E with $TS = ST$ and $0 \in \rho(S)$. If T is regular, then TS is regular.*

Proof. Clearly $R(TS) = R(T)$ is closed. On other hand, proceeding by induction we obtain that $(TS)^n = T^nS^n$ and since S^n is surjective it follows from [9, Proposition I.4.2 (a)] that $I \subset S^nS^{-n}$. The use of these properties together with the hypotheses gives $N(TS) = N(T) \subset R(T^n) \subset R(T^nS^nS^{-n}) \subset R(T^nS^n = R((TS)^n)$, for all $n \in \mathbb{N}$. $\square$

3. Generalized Kato Linear Relations in Banach Spaces

In the sequel E we be denote a complex space and T will be always an everywhere defined closed linear relation in E having a nonempty resolvent set. Now, we introduce an important class of linear relations which involves the concept of regularity.

Definition 3.1. *T is said to be a generalized Kato linear relation, if there exists a pair of closed subspaces (M,N) of E such that $(M,N) \in Red(T)$ with T_M a regular linear relation and T_N is a bounded quasi-nilpotent operator (that is $\sigma(T_N) = \{0\}$).*

The pair (M,N) is called the generalized Kato decomposition of the linear relation T, abbreviated as $GKD(M,N)$.

The above notion is introduced for closed operators in Hilbert space in [26], for bounded operators in Banach spaces in [7, 16]. If we assume in the definition above that T_N is nilpotent, that is, there exists $d \in \mathbb{N}$ for which $(T_N)^d = 0$. In this case T is said to be Kato linear relation of degree d.

Clearly, every regular relation is a generalized Kato linear relation with $M = E$ and $N = \{0\}$ and a quasi-nilpotent operator has a GKD with $M = \{0\}$ and $N = E$, as well as linear relation of Kato type, so the class of generalized Kato linear relations contains the class of semi-Fredholm linear relations.

Theorem 3.2. *Let T be a closed linear relation, everywhere defined with $\rho(T) \neq \emptyset$. If (M,N) is a GKD of T, then $(N^\perp, M^\perp)$ is a GKD of T^*.*

Proof. Suppose that (M,N) is a GKD of T. Since $E = M \oplus N$ we have that $E^* = N^\perp \oplus M^\perp$. We shall prove that $(N^\perp, M^\perp)$ is a GKD of T^*. By [9, Proposition III.1.2], T^* is closed. Further, both subspaces $N^\perp$ and $M^\perp$ are invariant under T^*. Indeed, by definition of T^* we have that $x^* \in T^*y^*$ if and only if $y^*(y) = x^*(x)$ for all $(x,y) \in T$. Let $x^* \in T^*y^*$ for some $y^* \in D(T^*) \cap M^\perp$. Then $(x,y) \in T = T_M \oplus T_N$, so $(P_M x, P_M y) \in T_M$ and $(P_N x, P_N y) \in T_N$, so that $0 = y^*(P_M y) = x^*(P_M x)$ which implies that $x^* \in M^\perp$. Hence $T^*M^\perp \subset M^\perp$. A similar proof for $T^*N^\perp \subset N^\perp$. We claim that

$$T^* = T^*P_{N^\perp} + T^*P_{M^\perp} = T^*_{N^\perp} \oplus T^*_{M^\perp}. \tag{2}$$

From the decomposition $T = T_M \oplus T_N$, by Lemma 2.2 (i)-(ii), we obtain that $T = TP_M + TP_N$, since TP_M and TP_N are bounded, [9, Proposition III.1.5 (b)] asserts that $T^* = (TP_M)^* + (TP_N)^*$ and therefore, by Lemma 2.4, $T^* = T^*P_{N^\perp} + T^*P_{M^\perp}$. Now, let $(y^*, x^*) \in T^*P_{N^\perp} + T^*P_{M^\perp}$, then $(y^*, x^*) = (y^*, u^* + v^*)$ with $(y^*, u^*) \in T^*P_{N^\perp}$ and $(y^*, v^*) \in T^*P_{M^\perp}$, so $(P_{N^\perp}y^*, u^*) \in T^*$ and $(P_{M^\perp}y^*, v^*) \in T^*$ and by the definition of T^* we have that $P_{N^\perp}y^*(P_N y) = u^*(P_N x)$ and $P_{M^\perp}y^*(P_M y) = v^*(P_M x)$ for all $(x,y) \in T$. Hence $u^*(P_N x) = 0$ and $v^*(P_M x) = 0$ so that $u^* \in N^\perp$ and $v^* \in M^\perp$. Therefore $(P_{N^\perp}y^*, u^*) \in T^*_{N^\perp}$ and $(P_{M^\perp}y^*, v^*) \in T^*_{M^\perp}$ so that $(y^*, x^*) = (P_{N^\perp}y^* + P_{M^\perp}y^*, u^* + v^*) \in T^*_{N^\perp} \oplus T^*_{M^\perp}$. Therefore, $T^*P_{N^\perp} + T^*P_{M^\perp} \subset T^*_{N^\perp} \oplus T^*_{M^\perp}$. Now if we apply the Lemma 2.2 to T^* and $(N^\perp, M^\perp)$, we deduce that

$$T^*_{N^\perp} \oplus T^*_{M^\perp} = T^*_{N^\perp}P_{N^\perp} + T^*_{M^\perp}P_{M^\perp} = T^*P_{N^\perp} + T^*P_{M^\perp}.$$

Therefore (2) holds.

$$T^*_{N^\perp} \text{ is regular.} \tag{3}$$

We first show that $R(T^*_{N^\perp})$ is closed. Since T_M is regular by hypothesis we have that the closed linear relation TP_M has closed range equivalently $R((TP_M)^*)$ is closed. This last property together with Lemma 2.4 leads to $R(T^*_{N^\perp})$ is closed. It only remains to see that $N(T^{*n}_{N^\perp}) \subset R(T^{*n}_{N^\perp})$ for all $n \in \mathbb{N}$. It follows from Lemma 2.3 and [9, Proposition III.1.4 (a)] and [9, Lemma III.3.5] that

$$\begin{aligned} N((T^*)^n_{N^\perp}) &= N((T^*)^n) \cap N^\perp = N((T^n)^*) \cap N^\perp \\ &= R(T^n)^\perp \cap N^\perp = (R(T^n) + N)^\perp. \end{aligned}$$

On the other hand, one deduces from Lemma 2.4 together with the fact that $R((TP_M)^*)$ is closed and [9, Proposition III.4.6] that

$$R(T^*_{N^\perp}) = R(T^*P_{N^\perp}) = R((TP_M)^*) = N(TP_M)^\perp = (N(T_M) + N)^\perp.$$

and now the regularity of T_M allow us to infer that $N(T_M) + N \subset R(T_M^n) + N \subset R(T^n) + N$, and hence $(R(T^n) + N)^\perp \subset (N(T_M) + N)^\perp$. Therefore (3) holds.

$$T^*_{M^\perp} \text{ is a bounded quasi-nilpotent operator.} \tag{4}$$

$T^*_{M^\perp}$ is an operator. Since T is everywhere defined we have that $T^*(0) = \mathcal{D}(T)^\perp = \{0\}$ and thus it follows trivially from (2) that $T^*_{M^\perp}(0) = \{0\}$ equivalently $T^*_{M^\perp}$ is an operator.

$\mathcal{D}(T^*_{M^\perp}) = M^\perp$. Since T is continuous we have by [9, Proposition III.4.6] that $\mathcal{D}(T^*) = T(0)^\perp$, so that it follows from Lemma 2.2 that

$$\begin{aligned}\mathcal{D}(T^*_{M^\perp}) &= \mathcal{D}(T^*) \cap M^\perp = T(0)^\perp \cap M^\perp \\ &= T_M(0)^\perp \cap M^\perp \quad (\text{ as } T_N \text{ is an operator}) \\ &= (T_M(0) \cap M)^\perp = M^\perp.\end{aligned}$$

$T^*_{M^\perp}$ is continuous. Since T is continuous one deduces from [9, Corollary III.1.13] that T^* is also continuous and hence $T^*_{M^\perp}$ is continuous.

$T^*_{M^\perp}$ is quasi-nilpotent. Indeed, as an immediate consequence of Lemma 2.2 and (2) we obtain that $\sigma(T_N) = \sigma(TP_N)$ and $\sigma(T^*_{M^\perp}) = \sigma(T^*P_{M^\perp})$ which implies that $\{0\} = \sigma(T_N) = \sigma(TP_N) = \sigma((TP_N)^*)$. Since $\sigma((TP_N)^*) = \sigma(T^*P_{M^\perp})$ by [9, Proposition VI.1.11], now by Lemma 2.4 we conclude that $\sigma(T^*P_{M^\perp}) = \{0\}$. The proof is completed. □

A proof of this theorem for bounded operators can be found in [1, Theorem 1.43].

Lemma 3.3. *Let T be a closed linear relation in E. If T is a generalized Kato relation then $T(M \cap \mathcal{D}(T)) + N$ is closed for every GKD(M, N) of T.*

Proof. By the same way of the proof of Lemme 2.3 in [27]. □

Theorem 3.4. *If T is a generalized Kato relation then there exist two linear relations D and Q in E such that:*

1. *$T = D + Q$ and $G(QD) = D(T) \times \{0\}$, $G(DQ) = E \times T(0)$.*
2. *The restriction of D on $R(D)$ is a regular relation.*
3. *Q is an everywhere defined quasi-nilpotent operator.*

Proof. If T is regular we take $D = T$ and $Q = 0$ and if T is quasi-nilpotent we take $D = 0$ and $Q = T$. Now suppose that T is not regular neither quasi-nilpotent, which admits a GKD(M, N), let P_M be the projection of E onto M along N and P_N be the projection of E onto N along M. Let $D = TP_M$ and $Q = TP_N$. The relations D and Q are closed linear relations. It follows that

$$D + Q = TP_M + TP_N = T,$$

Now, let $(x, y) \in QD$, so tht $(x, z) \in D$ and $(z, y) \in Q$ for some $z \in E$. Then, by definition, $(P_Mx, z) \in T_M$ and $(P_Nz, y) \in T_N$. Clearly $P_Mx \in \mathcal{D}(T)$, so that $x = P_Mx + P_Nx \in \mathcal{D}(T)$, and the fact that $N \subset \mathcal{D}(T)$, we have $z \in M$, so $P_Mz = 0$ and consequently, $y = 0$. Hence $QD \subset \mathcal{D}(T) \times \{0\}$. To show the converse inclusion, let $(x, 0) \in \mathcal{D}(T) \times \{0\}$. Then $P_Mx \in \mathcal{D}(T)$, so that $(P_Mx, z) \in T_M$ for some $z \in E$, hence $(x, z) \in D$. Furthermore, $(z, 0) \in T$ as $(P_Nz, 0) \in T_N$. Thus it follows that $\mathcal{D}(T) \times \{0\} \subset QD$, thus the equality $QD = \mathcal{D}(T) \times \{0\}$ is proved. Let us prove that $DQ = E \times T(0)$. Let $(x, y) \in DQ$, by definition, $(x, z) \in Q$ and $(z, y) \in D$ for some $z \in E$. Then $(P_Nx, z) \in T_N$ and $(P_Mz, y) \in T_N$, since $z \in N$, we have $P_Mz = 0$. Therefore, $y \in T_M(0) = T(0)$. Hence $DQ \subset E \times T(0)$. Conversely, let $(x, y) \in E \times T(0)$. Then $P_Nx \in N$, so that $(P_Nx, z) \in T_N$ for some $z \in E$. This implies $(P_Mz, y) = (0, y) \in T_M$. Thus $(x, z) \in T$, $(z, y) \in D$, so that $(x, y) \in DQ$. Hence $E \times T(0) \subset DQ$.

On other hand, since $N(D) = (M \cap N(T)) + N$ we have then

$$M \cap N(D) = M \cap N(T) = N(T_M).$$

We claim that that $R(T_M^n) \subseteq R(D^n)$ for all $n \geq 1$. In fact let $y \in R(T_M^n)$, then there exists $x \in \mathcal{D}(T^n) \cap M$ such that $y \in T^n x$, so $y \in T^n P_M x = (TP_M)^n x = D^n x$ and hence $R(T_M^n) \subseteq R(D^n)$ for all $n \geq 1$. From this inclusion and the regularity of T_M we have $R(D)$ is closed and

$$R(D) \cap N(D) \subseteq M \cap N(D) \subseteq R(T_M^n) \subseteq R(D^n),$$

for all $n \geq 1$, and hence $D_{R(D)}$ is a regular relation.

Finally, if we take into account that TP_N is quasi-nilpotent, we conclude that Q is quasi-nilpotent. □

Theorem 3.5. *If T is a generalized Kato linear relation, then there exists an open disc $\mathbb{D}(0, \epsilon)$ for which $\lambda - T$ is regular for all $\lambda \in \mathbb{D}(0, \epsilon) \setminus \{0\}$.*

Proof. If $M = \{0\}$, precisely, T is quasi-nilpotent, then for all $0 \neq \lambda \in \mathbb{C}$, $\lambda - T$ is invertible, hence $\lambda - T$ is regular. Now assume that $M \neq \{0\}$ and (M, N) is a GKD of T, so $(\lambda - T) = (\lambda - T)_M \oplus (\lambda - T)_N$. Since T_M is regular then $R(T_M)$ is closed and we have $\gamma(T_M) > 0$. Then by [2, Theorem 23], there exists $\nu > 0$ such that $(\lambda - T)_M$ is regular for all $|\lambda| < \nu$. As T_N is quasi-nilpotent, we know that $(\lambda - T)_N$ is invertible for all $0 \neq \lambda \in \mathbb{C}$, obviously $(\lambda - T)_N$ is regular. By Proposition 2.9, $\lambda - T$ is regular for all $\lambda \in \mathbb{D}(0, \nu) \setminus \{0\}$. □

Remark 3.6. *For operators the above Theorem 3.5 is proved in [16, Theorem 2.2].*

4. Generalized Kato Spectrum of Linear Relation

For the linear relation T, let us define the regular spectrum, the Kato spectrum and the generalized Kato spectrum as follows respectively:

$\sigma_{reg}(T) := \{\lambda \in \mathbb{C} : \lambda - T$ is not regular$\}$,

$\sigma_k(T) := \{\lambda \in \mathbb{C} : \lambda - T$ is not Kato linear relation$\}$,

$\sigma_{gk}(T) := \{\lambda \in \mathbb{C} : \lambda - T$ is not generalized Kato linear relation$\}$.

We have

$$\sigma_{gk}(T) \subseteq \sigma_k(T) \subseteq \sigma_{se}(T).$$

As a straightforward consequence of Theorem 3.5, we easily obtain the following theorem.

Theorem 4.1. *$\sigma_{gk}(T)$ is a closed subset of $\mathbb{C}$. Moreover, $\sigma_{reg}(T) \setminus \sigma_{gk}(T)$ and $\sigma_k(T) \setminus \sigma_{gk}(T)$ are at most countable.*

Proposition 4.2. *Assume that $0 \in \rho(T)$. Then $\lambda \in \sigma_i(T)$ if and only if $\lambda \neq 0$ and $\lambda^{-1} \in \sigma_i(T^{-1})$ for $i = gk, k$.*

Proof. By [3, Proposition 4.1.], we have

$$\lambda - T = -\lambda(\lambda^{-1} - T^{-1})T.$$

Since $0 \in \rho(T)$ and T commutes with $(\lambda^{-1} - T^{-1})$, it follows from Theorems 2.10 and 2.11 that $(\lambda^{-1} - T^{-1})_M$ is regular if and only if $(\lambda - T)_M$ is regular and $(\lambda^{-1} - T^{-1})_N$ is quasi-nilpotent (resp. nilpotent) if and only if $(\lambda - T)_N$ is quasi-nilpotent (resp. nilpotent). This is equivalent to the statement of the Theorem. □

Note that most of the classes of linear relations, for example in Fredholm theory, require that the linear relations have closed ranges. Thus it is natural to consider the Goldberg spectrum or closed-range spectrum of the linear relation T, as in the case of operators, as follows

$$\sigma_{ec}(T) = \{\lambda \in \mathbb{C}\,;\ R(\lambda - T) \text{ is not closed}\}.$$

Note that the Goldberg spectrum spectrum is a part of the regular spectrum has not good properties even for bounded operators T (see [6, p. 7-8]). However, the spectrum $\sigma_{ec}(T)$ can be used to obtain informations on the location in the complex plane of the various types of essential spectra, Fredholm, Weyl and Browder spectra etc..., for large classes of linear operators arising in applications. For example, integral, difference, and pseudo-differential operators (see [13] and the references therein).

Motivated by the relation between the generalized Kato spectrum and the Goldberg spectrum, in [5] M. Benharrat and B. Messirdi proved that the symmetric difference between them is at most countable. Now, we study this relation in the case of multivalued linear operators which admit a generalized Kato decomposition in Banach spaces. We begin with the following preparatory result proved in [3, Theorem 3.1] which is crucial for our purposes.

Theorem 4.3. *For α a nonzero positive real number, let*

$$\mathcal{R}(\alpha) = \{\lambda \in \mathbb{C} \,;\, \gamma(\lambda - T) \geq \alpha\}.$$

Then

(i) For all $\alpha > 0$, $\mathcal{R}(\alpha)$ is closed.

(ii) If λ_0 is an accumulation point of $\mathcal{R}(\alpha)$, then $\lambda_0 - T$ is regular.

Proposition 4.4. *If $\lambda \in \sigma_{ec}(T)$ is non-isolated point then $\lambda \in \sigma_{gk}(T)$.*

Proof. Let $\lambda \in \sigma_{ec}(T)$ be a non-isolated point. Assume that $\lambda - T$ is a generalized Kato linear relation. Then by Theorem 3.5 there exists an open disc $\mathbb{D}(\lambda, \epsilon)$ such that $\mu - T$ is regular in $\mathbb{D}(\lambda, \epsilon) \setminus \{\lambda\}$, so that $R(\mu - T)$ is closed for all $\mu \in \mathbb{D}(\lambda, \epsilon) \setminus \{\lambda\}$. This contradicts our assumption that λ is a non-isolated point. □

Theorem 4.5. *The symmetric difference $\sigma_{gk}(T)\Delta\sigma_{ec}(T)$ is at most countable.*

Proof. We have

$$\sigma_{gk}(T)\Delta\sigma_{ec}(T) = (\sigma_{gk}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T))) \cup (\sigma_{ec}(T) \cap (\mathbb{C} \setminus \sigma_{gk}(T))).$$

From Proposition 4.4 the set $\sigma_{ec}(T) \setminus \sigma_k(T)$ is at most countable, we have $\mathbb{C} \setminus \sigma_{ec}(T) = \bigcup_{m=1}^{\infty} \mathcal{R}(\frac{1}{m})$ and

$$\sigma_{gk}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T)) = \bigcup_{m=1}^{\infty} (\sigma_{gk}(T) \cap \mathcal{R}(\frac{1}{m})).$$

To finish the proof we prove that the set $\sigma_{gk}(T) \cap \mathcal{R}(\frac{1}{m})$ is at most countable. Let λ_0 be a non-isolated point of $\sigma_{gk}(T) \cap \mathcal{R}(\frac{1}{m})$. Then there exists $(\lambda_n)_n \subset \sigma_{gk}(T) \cap \mathcal{R}(\frac{1}{m})$ such that $\lambda_n \to \lambda_0$, by Theorem 4.3 $\lambda_0 - T$ is regular, hence $\lambda_0 \notin \sigma_{gk}(T)$. This contradicts that $\sigma_{gk}(T)$ is closed. □

The above results are now applied to obtain the generalize Kato, Kato, regular and Goldberg spectra of a linear relation.

Example 4.6. *Let $E = \ell^2$ be the Hilbert space of all square summable complex sequences*

$$x = (x_n)_n = (x_1, x_2, \ldots),$$

indexed by the a nonnegative integers, with the norm associated with the usual inner product $\langle ., . \rangle$. We define the right shift operator A and the left shift operator B in ℓ^2 by

$$A(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

and

$$B(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

We know that $\|A\| = \|B\| = 1$, $\sigma(A) = \sigma(B) = \mathbb{D} = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$ and $B = A^*$, where A^* is the operator defined by

$$G(A^*) = \{(0, u) \in \ell^2 \times \ell^2 : \langle u, x \rangle = \langle u, y \rangle \text{ for all } (x, y) \in \ell^2 \times \ell^2\}.$$

Furthermore, $N(B) = span\{e_1\}$ and $R(B) = \ell^2$, so that B^{-1} is a linear relation with $\mathcal{D}(B^{-1}) = R(B) = \ell^2$, $R(B^{-1}) = \mathcal{D}(B) = \ell^2$ and $N(B^{-1}) = B(0) = \{0\}$. Hence B^{-1} is a closed linear relation and $0 \in \rho(B^{-1})$. We shall prove that

Theorem 4.7. *We have*

(i) $\sigma_{reg}(B) = \sigma_{gk}(B) = \sigma_k(B) = \sigma_{ec}(B) = \mathbb{S} = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$,

(ii) $\sigma_{reg}(B^{-1}) = \sigma_{gk}(B^{-1}) = \sigma_k(B^{-1}) = \sigma_{ec}(B^{-1}) = \mathbb{S}$.

To this end, we need the following result.

Proposition 4.8. *[3, Corollary 4.10.] Assume that* $0 \in \rho(T)$*. Then, if* $\lambda \neq 0$

(i) $\lambda \in \sigma_{ec}(T)$ *if and only if* $\lambda^{-1} \in \sigma_{ec}(T^{-1})$,

(ii) $\lambda \in \sigma_{reg}(T)$ *if and only if* $\lambda^{-1} \in \sigma_{reg}(T^{-1})$.

Proof. [Proof of Theoerem 4.7] (i) We have $0 \notin \sigma_{reg}(B)$, $0 \notin \sigma_{ec}(B)$, $N(B) = span\{e_1\}$ and $R(B) = \ell^2$. Let $0 < |\lambda| < 1$, then for $\frac{1}{|\lambda|} > 0$, $\lambda^{-1} - A$ is an isomorphism and since

$$\lambda - B = \lambda BA - B = \lambda B(A - \lambda^{-1}),$$

we deduce that $R(\lambda - B) = R(B) = \ell^2$ and $N(\lambda - B) = N(B) = span\{e_1\}$ if $0 < |\lambda| < 1$. Hence $R(\lambda - B)$ is closed and $\lambda - B$ is regular if $0 < |\lambda| < 1$, so that $\sigma_{reg}(B)$ and $\sigma_{ec}(B)$ are contained in $\{\lambda \in \mathbb{C}; |\lambda| \geq 1\} \cap \sigma(B) = \mathbb{S}$. Conversely, it is clear that $\mathbb{S} = \partial\sigma(B) \subset \sigma_{reg}(B)$, where $\partial\sigma(B)$ is the boundary of $\sigma(B)$, hence $\sigma_{reg}(B) = \mathbb{S}$. Since B is regular if and only if $B^* = A$ is regular (see, for example [1, Theorem 1.19]), we have $\lambda \in \sigma_{reg}(B)$ if and only if $\overline{\lambda} \in \sigma_{reg}(B^*)$. This implies that $\sigma_{reg}(A) = \mathbb{S}$. Further, we note that if $|\lambda| = 1$ then $\lambda - A$ is injective, this and the fact that $\sigma_{reg}(A) = \mathbb{S}$ implies that $R(\lambda - A)$ is not closed if $|\lambda| = 1$, equivalently $R((\lambda - A)^*) = R(\overline{\lambda} - B)$ is not closed if $|\lambda| = 1$. Consequently $\sigma_{ec}(B) = \mathbb{S}$.

On the other hand, it is clear that $\sigma_{gk}(B) \subset \sigma_k(B) \subset \sigma_{reg}(B) = \mathbb{S}$. Conversely, $\sigma_{ec}(B) = \mathbb{S}$ so that $\partial\sigma_{ec}(B) = \mathbb{S}$ and if $\lambda \in \partial\sigma_{ec}(B)$ then λ is non-isolated and it follows from Proposition 4.4 that $\lambda \in \sigma_{gk}(B)$. Hence $\sigma_{gk}(B) = \sigma_k(B) = \mathbb{S}$.

(ii) B^{-1} is an everywhere defined closed linear relation such that $0 \in \rho(B^{-1})$. Then the use of Proposition 4.8 together with (i) ensures that $\sigma_{reg}(B^{-1}) = \sigma_{ec}(B^{-1}) = \mathbb{S}$. Further, it is clear that $\sigma_{gk}(B^{-1}) \subset \sigma_k(B^{-1}) \subset \sigma_{reg}(B^{-1}) = \mathbb{S}$. The converse inclusion follows now from Proposition 4.4. The proof is completed. □

Remark 4.9. *The preceding Theorem 4.7 shows that the generalized Kato spectrum of a linear relation may be a proper subset of its spectrum. However, we see in [17, Example 3.14] that an operator may have a generalized Kato spectrum that coincides with the whole spectrum. In contrast, it is not difficult to find examples of operators having empty generalized Kato spectrum. For example, a quasi-nilpotent operator and the operator T on* ℓ^2 *defined by*

$$T(x_1, x_2, \ldots) = (0, x_1, 0, \frac{1}{3}x_2, \ldots).$$

We also note that if T is a Riesz operator (that is, $\lambda - T$ *is Fredholm for every* $0 \neq \lambda \in \mathcal{K}$*) has infinite points in* $\sigma(T)$*, then* $\sigma_{gk}(T) = \{0\}$ *(see, [17] for details).*

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