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Constructing a Basis from Systems of Eigenfunctions of one not Strengthened Regular Boundary Value Problem

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Abstract. We investigate a nonlocal boundary value spectral problem for an ordinary differential equation in an interval. Such problems arise in solving the nonlocal boundary value problem for partial equations by the Fourier method of variable separation. For example, they arise in solving nonstationary problems of diffusion with boundary conditions of Samarskii-Ionkin type. Or they arise in solving problems with stationary diffusion with opposite flows on a part of the interval. The boundary conditions of this problem are regular but not strengthened regular. The principal difference of this problem is: the system of eigenfunctions is comlplete but not forming a basis. Therefore the direct applying of the Fourier method is impossible. Based on these eigenfunctions there is constructed a special system of functions that already forms the basis. However the obtained system is not already the system of the eigenfunctions of the problem. We demonstrate how this new system of functions can be used for solving a nonlocal boundary value problem on the example of the Laplace equation.

1. Introduction

Investigations on spectral theory of ordinary differential operators begun from classical papers of J. Liouville and Sh. Sturm. Fundamental works in the spectral theory of differential operators were the papers by Birkhoff of 1908, where he introduced regular boundary conditions for the first time. The theory was significantly developed by Tamarkin and Stone. These works led to a new wide scientific direction having an enormous literature. We refer to [1, 2] for the extensive bibliography and the obtained results.

Despite the apparent simplicity, the spectral theory of ordinary differential operators is far from complete. This applies even to the case of a second-order operator

$$Lu = u''(x) + q(x)u$$

on the finite interval $x \in (a, b)$ which is called Sturm-Liouville operator. Brief survey of results in the spectral theory of the Sturm-Liouville operator is given in the recent paper by Makin [3].

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It is known that boundary conditions can be divided into three classes [4]:
- strengthened regular conditions;
- regular but not strengthened regular conditions;
- irregular conditions.

If the boundary conditions are strengthened regular then the system of root functions forms a Riesz basis in $L_2(a,b)$. This statement was proved in [5, 6] and [7, Chapter XIX].

In the other cases the basis property of the systems of root functions is not guaranteed. The final definition of classes of the boundary conditions for an operator of second order when the system of eigen- and associated functions forms the basis, was given in [8].

In the present work we consider one model spectral problem for an operator of multiple differentiation. Boundary conditions of the problem are regular but not strengthened regular. The system of eigenfunctions of the problem is complete, minimal, almost normed, but does not form a basis in L_2. On the basis of these eigenfunctions we construct a special system having basis property in L_2.

2. Statement of the Problem

Consider the spectral problem

$$\begin{aligned} &-u''(x) = \lambda u(x),\ 0 < x < \pi; \\ &u(0) = 0,\ u'(0) + u'(\pi) + \alpha u(\pi) = 0, \end{aligned} \tag{1}$$

where $\alpha > 0$ is a fixed parameter.

This problem arises while solving a nonlocal boundary value problem for the Laplace equation by the method of separation of variables. Let $D = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi\}$ be a half-disc. Our goal is to find a function $u(r,\varphi) \in C^0(\bar{D}) \cap C^2(D)$ satisfying in D the equation

$$\Delta U = 0 \tag{2}$$

with the boundary conditions

$$U(1,\theta) = f(\theta),\ 0 \le \theta \le \pi, \tag{3}$$

$$U(r,0) = 0, \quad r \in [0,1], \tag{4}$$

$$\frac{\partial U}{\partial \theta}(r,0) + \frac{\partial U}{\partial \theta}(r,\pi) + \alpha U(r,\pi) = 0,\ \ r \in (0,1). \tag{5}$$

For $\alpha = 0$ the problem (2) - (5) was considered in [9]. The difference of this problem ($\alpha > 0$) is the impossibility of direct applying of the Fourier method (separation of variables). Because the corresponding spectral problem (1) for the ordinary differential equation has the system of eigenfunctions not forming a basis.

One method of constructing the basis, based on the system of eigenfunctions of the problem

$$\begin{aligned} &-\vartheta''(x) = \lambda \vartheta(x),\ 0 < x < \pi; \\ &\vartheta(0) = 0,\ \vartheta'(0) = \vartheta'(\pi) + \alpha \vartheta(\pi) \end{aligned}$$

was suggested in [10]. The boundary conditions of this problem are regular but not strengthened regular conditions. And the system of its eigenfunctions does not form the basis. But a special system of functions built with help of these eigenfunctions will form the basis. And this fact is applied for the solution of a nonlocal initial-boundary problem for the heat equation. It is used in [11] for the solution of a nonlocal boundary value problem for the Helmholtz operator in a half-disc. And also it is used in [12] for the solution of an inverse nonlocal boundary value problem for the heat equation.

The goal of the present work is to construct the basis from the system of the eigenfunctions of the problem (1).

3. Preliminaries

Let us present briefly the main definitions and facts which will be used in what follows. Let B be a Banach space with the norm $\|\cdot\|_B$, and let B^* be its dual with the norm $\|\cdot\|_{B^*}$.

A system of elements $\{\varphi_k\}_{k=0}^{\infty}$ is said to be closed in B if the linear span of this system is everywhere dense in B; that is, any element of the space B can be approximated by a linear combination of elements of this system with any accuracy in the norm of the space B.

A system of elements $\{\varphi_k\}_{k=0}^{\infty}$ is said to be minimal in B if none of its elements belongs to the closure of the linear span of the other elements of this system.

It is well known that a system $\{\varphi_k\}_{k=0}^{\infty}$ is minimal if and only if there exists a biorthogonal system which is dual to it, that is, a system of linear functionals $\{\psi_k\}_{k=0}^{\infty}$ from B^* such that

$$(\varphi_k, \psi_j) = \delta_{k,j}$$

for all $k, j \in \mathbb{N}$. Moreover, if the initial system is simultaneously closed and minimal in B, then the biorthogonal dual system is uniquely defined.

We say that a system $\{\varphi_k\}_{k=0}^{\infty}$ is uniformly minimal in B, if there exists $\gamma > 0$ such that for all $k \in \mathbb{N}$,

$$dist\{\varphi_k, B_k\} \geq \gamma \|\varphi_k\|_B,$$

where B_k is the closure of the linear span of all elements φ_l with serial numbers $l \neq k$.

It is also well known that a closed and minimal system $\{\varphi_k\}_{k=0}^{\infty}$ is uniformly minimal in B if and only if:

$$\sup_{k\in\mathbb{N}} \|\varphi_k\|_B \|\psi_k\|_{B^*} < \infty.$$

A system $\{\varphi_k\}_{k=0}^{\infty}$ forms a basis of the space B if, for any element $f \in B$, there exists a unique expansion of it in the elements of the system, that is, the series $\sum_{k=0}^{\infty} f_k \varphi_k$ converges to f in the norm of the space B.

Any basis is a closed and minimal system in B, and, therefore, we can uniquely find its biorthogonal dual system $\{\psi_k\}_{k=0}^{\infty}$, and hence the expansion of any element of f with respect to the basis $\{\varphi_k\}_{k=0}^{\infty}$ coincides with its biorthogonal expansion, that is, $f_k = (f, \psi_k)$ for all $k \in \mathbb{N}$.

4. On Eigenvalues and Eigenfunctions of the Problem

In a whole the constructing eigenvalues and eigenfunctions of the problem (1) is a simple task. Therefore we omit some details of the calculations and present the main facts which we will use further.

We look for eigenvalues of the problem. Note that $\lambda = 0$ is not an eigenvalue, since problem (1) for this value of λ has only the trivial solution.

Let $\lambda \neq 0$. The eigenfunction should have the form $u(x) = \sin\left(\sqrt{\lambda}x\right)$. By taking into account the nonlocal boundary condition, we obtain two equations

$$\cos\left(\frac{\sqrt{\lambda}\pi}{2}\right) = 0, \qquad \cot\left(\frac{\sqrt{\lambda}\pi}{2}\right) = -\frac{\alpha}{\sqrt{\lambda}}.$$

Solutions of the first equation form a series of eigenvalues and eigenfunctions of the problem (1) of the form

$$\lambda_k^{(1)} = (2k+1)^2, \quad u_{k1}(x) = \sin\left((2k+1)\,x\right), \quad k = 0, 1, 2, ...$$

The second equations can be represented as

$$\cot(\beta\pi) = -\frac{\alpha}{2\beta}, \ \beta = \frac{\sqrt{\lambda}}{2}.$$

By β_k denote roots of this equation. It is easy to show that they satisfy the inequalities $2k+1 < 2\beta_k < 2k+2,\ k = 0,1,2,...$, and two-side estimates are carried out for $\delta_k = \beta_k - k - 1/2$, where k is large enough

$$\frac{\alpha}{\pi\,(2k+1)}\left(1-\frac{1}{2k+1}\right) < \delta_k = \alpha\left|O\left(\frac{1}{k}\right)\right| < \frac{\alpha}{\pi\,(2k+1)}. \tag{6}$$

Consequently there exists a second series of eigenvalues and eigenfunctions of the form

$$\lambda_k^{(2)} = (2\beta_k)^2, \quad u_{k2}(x) = \sin(2\beta_k x), \quad k = 0,1,2,...$$

Lemma 4.1. *The system of eigenfunctions* $\{u_{k1}, u_{k2}\}_{k=0}^{\infty}$ *of the problem* (1) *is complete and minimal, almost normed but does not form even an ordinary basis in* $L_2(0,\pi)$.

Proof. The completeness and minimality of the system follow from the regularity of boundary conditions of the spectral problem (1). The limitation of norms is easily checked by direct calculation. However the properties of the completeness and minimality are not enough for the basis property.

Really, consider scalar multiplications of pairs of eigenfunctions (u_{k1}, u_{k2}). By direct calculation, we find

$$(u_{k1}, u_{k2}) = \int_0^{\pi} \sin((2k+1)\,t)\sin(2\beta_k t)\,dt = \frac{\pi}{2}\frac{\sin(2\delta_k\pi)}{2\delta_k\pi}\frac{2k+1}{2k+1+\delta_k}.$$

Taking into account that $\|u_{k1}\| = \sqrt{\pi/2}$, and $\lim\limits_{k\to\infty}\|u_{k2}\| = \sqrt{\pi/2}$, we get that the angle between the normed eigenvectors tends to zero:

$$\lim_{k\to\infty}\left(\frac{u_{k1}}{\|u_{k1}\|}, \frac{u_{k2}}{\|u_{k2}\|}\right)_{L_2(0,\pi)} = 1. \tag{7}$$

Such systems can not form the unconditional basis. We show it more detailed.

The problem

$$\begin{gathered} -v''(x) = \overline{\lambda} v(\theta)\,,\ 0 < x < \pi; \\ v(0) + v(\pi) = 0, \quad v'(\pi) + \alpha v(\pi) = 0 \end{gathered} \tag{8}$$

is conjugated to the problem (1). The system of the eigenfunctions of this problem is biorthogonal to the system $\{u_{k1}, u_{k2}\}_{k=0}^{\infty}$:

$$\begin{aligned} v_{k1}(x) &= \tfrac{2}{\pi}\left\{\sin((2k+1)\,x) - \tfrac{2k+1}{\alpha}\cos((2k+1)\,x)\right\} \quad k = 0,1,2,.... \\ v_{k2}(x) &= C_{k2}\left\{\sin(2\beta_k x) - \tfrac{2\beta_k}{\alpha}\cos(2\beta_k x)\right\},\ k = 0,1,2,..., \end{aligned} \tag{9}$$

The constant C_{k2} are taken from the biorthogonal relations $(u_{k2}, v_{k2}) = 1$. Since we will not use the explicit form of the biorthogonal system, then we do not present here the explicit form of constant C_{k2}.

Due to biorthogonality of the system, the equations

$$(u_{k1}, v_{k1}) = 1, \quad (u_{k2}, v_{k1}) = 0, \quad k = 0,1,2,....$$

are valid.

It follows that $(u_{k1} - u_{k2}, v_{k1}) = 1$. Using the Cauchy-Bunyakovsky inequality, we get the estimate from the bottom

$$\|v_{k1}\| \geq (\|u_{k1} - u_{k2}\|)^{-1}.$$

Since $\|u_{k1}\| = \sqrt{\pi/2}$, and $\lim\limits_{k\to\infty}\|u_{k2}\| = \sqrt{\pi/2}$, then from here and from (7) it is easy to obtain

$$\lim_{k\to\infty}\|u_{k1}\|\,\|v_{k1}\| = \infty.$$

That is, the necessary condition of the basis property does not hold.

Lemma is proved. □

It is necessary to note the fact, that the system of eigenfunctions $\{u_{k1}, u_{k2}\}_{k=0}^{\infty}$ does not have the basis, also follows from more general facts [8].

5. Forming the Basis

Now from elements of the system $\{u_{k1}, u_{k2}\}_{k=0}^{\infty}$ we construct a new system which will be a basis in $L_2(0, \pi)$. We introduce new functions

$$\begin{aligned} \varphi_{2k}(x) &= u_{k1}(x), \\ \varphi_{2k+1}(x) &= (u_{k2}(x) - u_{k1}(x))(2\delta_k)^{-1}, \end{aligned} \quad k = 0, 1, 2, ... \tag{10}$$

Let us show that the constructed system is a Riesz basis in $L_2(0, \pi)$.

The biorthogonal system to (10) has the form:

$$\psi_{2k}(x) = v_{k2}(x) + v_{k1}(x),$$

$$\psi_{2k+1}(x) = 2\delta_k v_{k2}(x), \ k = 0, 1, 2, ...$$

This system is constructed from the eigenfunctions of the problem (8) conjugated to (1).

Let us show that the constructed additional system has the basis property.

Lemma 5.1. *The system of functions $\{\varphi_k(x)\}_{k=0}^{\infty}$ forms a Riesz basis in $L_2(0, \pi)$.*

Proof. Since this system is constructed from the eigenfunctions of the problem with regular boundary conditions and with the help of non-degenerated linear combinations, then the completeness and minimality of the system do not change.

Let us prove asymptotic quadratic closeness of the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ to the system forming the Riesz basis. As such we choose the system of eigen- and associated functions of a problem of the Samarskii-Ionkin type:

$$-w''(x) = \lambda w(x), \ 0 < x < \pi;$$

$$w(0) = 0, \ w'(0) + w'(\pi) = 0.$$

The boundary conditions of this problem are not strengthened regular. All the eigenvalues of this problem, except zero values, are multiple: $\lambda_k^{(1)} = \lambda_k^{(2)} = (2k+1)^2$, $k = 0, 1, 2, ...$ The eigenfunctions w_{2k} and the associated functions w_{2k+1} of the problem form the Riesz basis in $L_2(0, \pi)$ and have the form:

$$w_{2k}(x) = \sin((2k+1)x), \ k = 0, 1, 2, ...; \ w_{2k+1}(x) = x\cos((2k+1)x).$$

We need to show that the series converges

$$\sum_{k=0}^{\infty} \left\|\varphi_k - w_k\right\|^2 < \infty.$$

It is evident that $\varphi_{2k} - w_{2k} = 0$. For odd numbers we have:

$$\varphi_{2k+1}(x) = \frac{\sin(2\beta_k x) - \sin((2k+1)x)}{2\delta_k} = \frac{\sin(\delta_k x)}{\delta_k x} x\cos((2k+1+\delta_k)x).$$

Thus it is not difficult to get the estimate $\left|\varphi_{2k+1}(x) - w_{2k+1}(x)\right| \leq C\delta_k$. From here and from the asymptotics (6) for δ_k we have the asymptotic inequality $\left|\varphi_{2k+1} - w_{2k+1}\right| \leq C_1/k$, where C_1 does not depend on k.

The obtained inequality provides the quadratic closeness of the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ and the Riesz basis $\{w_k(x)\}_{k=0}^{\infty}$. Lemma is proved. □

Further on, by standard methods it is not difficult to justify that if the function $f(x) \in C^2[0,\pi]$ and satisfies the boundary conditions of the problem (1), then its Fourier series by the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ converges uniformly.

We can calculate that

$$\begin{aligned} -\varphi_{2k}''(x) &= \lambda_k^{(1)}\varphi_{2k}(x), \\ -\varphi_{2k+1}''(x) &= \lambda_k^{(2)}\varphi_{2k+1}(x) + \tfrac{\lambda_k^{(2)}-\lambda_k^{(1)}}{2\delta_k}\varphi_{2k}(x). \end{aligned} \tag{11}$$

Using these formulas, it is possible to apply the method of separation of variables for solving problems of the type (2) - (5).

6. Usage of the Obtained Results for Solving of the Nonlocal Boundary Equation

We can write any solution of problem (2) - (5) in the form of a biorthogonal series

$$u(r,\theta) = \sum_{k=0}^{\infty} R_k(r)\varphi_k(\theta), \tag{12}$$

where

$$R_k(r) = (u(r,\cdot), \psi_k(\cdot)) \equiv \int_0^{\pi} u(r,\theta)\psi_k(\theta)\,d\theta.$$

Functions (12) satisfy the boundary conditions (4) and (5).

Substituting (12) into equation (2) and the boundary conditions (3), taking into account (11), for finding unknown functions $R_k(r)$ we obtain following problems:

$$\begin{aligned} r^2R_{2k+1}''(r) + rR_{2k+1}'(r) - \lambda_k^{(2)}R_{2k+1}(r) &= 0, \\ r^2R_{2k}''(r) + rR_{2k}'(r) - \lambda_k^{(1)}R_{2k}(r) &= \tfrac{\lambda_k^{(2)}-\lambda_k^{(1)}}{2\delta_k}R_{2k+1}(r), \end{aligned} \tag{13}$$

with the boundary conditions $R_k(1) = f_k$, where f_k are the Fourier coefficients of the expansion of the function $f(\theta)$ into the biorthogonal series by $\{\varphi_k(\theta)\}_{k=0}^{\infty}$.

The regular solution of (13) exists, is unique and can be written in the explicit form:

$$\begin{aligned} R_{2k+1}(r) &= f_{2k+1}r^{\sqrt{\lambda_k^{(2)}}}, \\ R_{2k}(r) &= f_{2k}r^{\sqrt{\lambda_k^{(1)}}} + f_{2k+1}\tfrac{1}{2\delta_k}\left(r^{\sqrt{\lambda_k^{(2)}}} - r^{\sqrt{\lambda_k^{(1)}}}\right). \end{aligned} \tag{14}$$

Substituting (14) into (12), we obtain a formal solution of the problem:

$$u(r,\theta) = \textstyle\sum_{k=0}^{\infty} f_{2k}r^{2k+1}\sin((2k+1)\theta) + \sum_{k=0}^{\infty} f_{2k+1}\frac{1}{2\delta_k}[r^{2\beta_k}\sin(2\beta_k\theta) - r^{2k+1}\sin((2k+1)\theta)]. \tag{15}$$

Theorem 6.1. *If $f(\theta) \in C^2[0,\pi]$, $f(0) = 0$, $f'(0) = -f'(\pi) + \alpha f(\pi)$, then there exists a unique classical solution $u(r,\theta) \in C^0(\bar{D}) \cap C^2(D)$ of the problem (2)-(5).*

Proof. The uniqueness of the classical solution of the problem follows from the maximum principle and the Zaremba-Giraud principle for the Laplace equation. The formal solution of the problem is shown in the form of (15). In order to make sure that these functions are really the desired solutions, we need to verify the applicability of the superposition principle. For it we need to show the convergence of the series, the possibility of termwise differentiation, and to prove the continuity of these functions on the boundary of the half-disk.

The possibility of differentiating the series (15) any number of times at $r < 1$ is an obvious consequence of the convergence of power series and two-sided estimates (6) for δ_k. Let us justify the uniform convergence of the series (12) at $r \leq 1$. For this we use the sign of the uniform convergence of Weierstrass.

By direct calculation it is easy to see that the series (15) is majorized by the series $C_1(|f_0| + |f_1| + |f_2| + ...)$. This series converges due to the requirements of the theorem imposed on $f(\theta)$. Since all the terms of the series (15) are continuous functions, then the function $u(r, \theta)$ is continuous in the boundary domain $\bar{D}$.

The proof of the theorem is complete. □

7. Conclusion

Thus, in the present work we have investigated the nonlocal boundary value spectral problem (1) for an ordinary differential equation in an interval $(0, \pi)$. The boundary conditions of this problem are regular but not strengthened regular. The difference of this problem is: the system of eigenfunctions $\{u_{k1}, u_{k2}\}_{k=0}^{\infty}$ of the problem (1) is complete and minimal, almost normed but does not form even an ordinary basis in $L_2(0, \pi)$.

Based on these eigenfunctions $\{u_{k1}, u_{k2}\}_{k=0}^{\infty}$ we have constructed a special system of functions $\{\varphi_k(x)\}_{k=0}^{\infty}$ that already forms a Riesz basis in $L_2(0, \pi)$.

This fact is used for solving the nonlocal boundary value problem (2) - (5).

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