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Affine Bessel Sequences and Nikishin's Example

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Abstract. We study affine Bessel sequences in connection with the spectral theory and the multishift structure in Hilbert space. We construct a non-Besselian affine system $\{u_n(x)\}_{n=0}^\infty$ generated by continuous periodic function $u(x)$. The result is based on Nikishin's example concerning convergence in measure. We also show that affine systems $\{u_n(x)\}_{n=0}^\infty$ generated by any Lipchitz function $u(x)$ are Besselian.

1. Introduction

Sequence of functions $\{u_n(x)\}_{n=0}^\infty \subset L^2(\Omega)$ is said to be *Besselian* if there exists $B > 0$ such that $\forall f \in L^2(\Omega)$ $\sum_{n=0}^\infty |(f, u_n)|^2 \le B\|f\|_2^2$.

It is known that there exist functional sequences $\{u_n(x)\}_{n=0}^\infty \subset L^\infty(\Omega)$ such that under the condition of

$$\|u_n\|_\infty \le A\|u_n\|_2, \qquad n = 0, 1, \dots, \tag{1}$$

we have that $\{u_n(x)\}_{n=0}^\infty$ are Besselian. On the other hand (1) is the neccessary condition for Besselian property or Riesz basicity in number of cases.

Example 1.1. *Every functional sequence from the following one-parameter family*

$$\left\{u_0(x) = x, \quad u_{k0}(x) = \sin 2\pi kx, \quad \tilde{u}_{k1}(x) = \frac{1}{4\pi k} x \cos 2\pi kx + C \sin 2\pi kx\right\}_{k=1}^\infty, \qquad C \in \mathbb{R},$$

consists of root functions from the Samarsky-Ionkin spectral problem (e.g. [1])

$$\begin{cases} u''(x) + \lambda u(x) = 0, & 0 < x < 1, \\ u(0) = 0, \quad u'(0) = u'(1). \end{cases}$$

Every sequence from the one-parameter family is Besselian and elements of each such sequence satisfy the uniform estimate $\|u_n\|_\infty \le A\|u_n\|_2$. If $C = 0$, the sequence forms an unconditional basis in $L^2(0,1)$, else if $C \ne 0$, it does

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not form a basis. As it follows from the results of [2], only uniformly bounded functional sequences can form Riesz bases consisting of eigenfunctions of one-dimensional self-adjoint Schrödinger operator with summable potential and discrete spectrum.

Example 1.2. *Let $H^2(\mathbb{T})$ be a Hardy space consisting of all functions $u \in L^2(\mathbb{T})$ in the form of $u(x) = \sum_{n=0}^{\infty} c_n e^{inx}$, $\sum_{n=0}^{\infty} |c_n|^2 < \infty$. An operator V is called shift if $Vu(x) = e^{ix}u(x)$. Assume that $u_n(x) = V^n u(x) = u(x)e^{inx}$, $n = 0, 1, \ldots$. It is well-known that a sequence of functions $\{u_n(x)\}_{n=0}^{\infty}$ is Besselian if and only if $u \in L^\infty(\mathbb{T})$.*

Additional conditions formulated in terms of $\{u_n(x)\}_{n=0}^{\infty}$ system coefficients space that provide Besselian property for general sequences of functions and for affine sequences (or affine systems) of functions in particular, are given in the paper [3]. In section 2 we show that additional conditions cannot be omitted in case of affine systems of functions; we also construct an affine system $\{u_n(x)\}_{n=0}^{\infty}$ consisting of continuous functions that does not form a Besselian sequence. The construction of this affine system is based on Nikishin's example [4] to Ulyanov's problem [5]. We consider the class of affine functions because there is a deep-rooted analogy to systems from the Example 2. Walsh type affine systems are introduced in the article [6]. We define them as follows:

Let the function $u(x)$, $x \in \mathbb{R}$, satisfy the following conditions:

$$u \in L^2(0,1), \qquad \int_0^1 u(x)\,dx = 0, \qquad u(x+1) = u(x).$$

We denote the space of such functions as $L_0^2(0,1)$, then we can define the following operators:

$$W_0 u(x) = u(2x), \qquad W_1 u(x) = r(x)u(2x), \tag{2}$$

where

$$r(x) = \begin{cases} 1, & x \in [m, m+1/2) \\ -1, & x \in [m+1/2, m+1) \end{cases}$$

is the periodic Haar–Rademacher–Walsh function.

For arbitrary multi-index $\alpha = (\alpha_0, \ldots, \alpha_{k-1})$, $k = 0, 1, \ldots$ consisting only of ones and zeros we consider the operator product $W^\alpha = W_{\alpha_0} \ldots W_{\alpha_{k-1}}$ ($W_{\alpha_{k-1}}$ is the first to act, W_{α_0} is the last; if $k = 0$, empty product is the identity operator). For any positive integer $n \in \mathbb{N}$ by its binary expansion $n = \sum_{\nu=0}^{k-1} \alpha_\nu 2^\nu + 2^k$ we assume

$$u_0(x) \equiv 1, \qquad u_n(x) = u_\alpha(x) = W^\alpha u(x) = r^{\alpha_0}(x) \ldots r^{\alpha_{k-1}}(2^{k-1}x)u(2^k x) = u(2^k x) \prod_{\nu=0}^{k-1} r_\nu^{\alpha_\nu}(x),$$

where $r_k(x) = r(2^k x)$, $k = 0, 1, \ldots$, are the Rademacher functions.

Definition 1.3. *The sequence of functions $\{u_n(x)\}_{n=0}^{\infty}$ is called Walsh type affine system of functions generated by $u(x)$.*

Applying these operators to the function $w(x) = r(x)$ we can obtain the classic Walsh-Paley system $\{w_n(x)\}_{n=0}^{\infty}$.

There is a structural analogy between affine systems and systems from the Example 2. A shift V in Hilbert space H can be defined as follows: *an isometry $V : H \to H$ is called a shift if there exists vector $e \in H$ such that system $\{V^n e\}_{n=0}^{\infty}$ forms an orthonormal basis in H.* Every shift $V : H \to H$ is unitary equivalent to the multiplication operator $Vu(x) = e^{ix}u(x)$ in Hardy space $H^2(\mathbb{T})$.

The *multishift* structure in Hilbert space is introduced in the paper [7]. A multishift is a shift extended into the case of two non-commutating operators and is defined as follows: *an isometry pair W_0 and W_1 from Hilbert space H is called a multishift if there exists vector $e \in H$ such that the family of vectors $\{W^\alpha e\}_{\alpha \in \mathbb{A}}$ forms an orthonormal basis in H, where $\mathbb{A} = \bigcup_{k=0}^{\infty}\{0,1\}^k$ is the family of all multi-indexes $\alpha = (\alpha_0, \ldots, \alpha_{k-1})$, $\alpha_\nu = 0$ or 1, $0 \le \nu \le k-1$, $W^\alpha = W_{\alpha_0} \ldots W_{\alpha_{k-1}}$.* Every multishift $W_0, W_1 : H \to H$ is unitary eqivalent to the pair of operators (2) in $L_0^2(0,1)$ space.

2. Main Result

Theorem 2.1. *There exists a continuous periodical function $u(x)$ that generates non-Besselian Walsh type affine system $\{u_n(x)\}_{n=0}^{\infty}$ in $L^2(0,1)$ space.*

Consider this in comparison to the following results:

Theorem 2.2. *Let a periodical function $u(x)$ satisfy the Lipchitz condition $|u(x)-u(y)| \le L|x-y|^{\alpha}$, $0<\alpha\le 1$. Then the Walsh type affine system $\{u_n(x)\}_{n=0}^{\infty}$ is Besselian in $L^2(0,1)$ space.*

Example 2.3. *A system $\{u_n(x)\}_{n=0}^{\infty}$ is called convergence system in measure for ℓ^2 if series $\sum_{n=0}^{\infty} a_n u_n(x)$ converges in measure for $\{a_n\}_{n=0}^{\infty} \in \ell^2$. E.M. Nikishin [4] constructed the example of function $u \in L_0^2(0,1) \cap C[0,1]$ such that the functional system $\{u(2^k x)\}_{k=0}^{\infty}$ generated by $u(x)$ is not a convergence system in measure for ℓ^2. There was also obtained the function $f \in L^2(0,1)$ such that $\sum_{k=0}^{\infty} |\int_0^1 f(x)u(2^k x)\,dx|^2 = \infty$. Thus $\{u(2^k x)\}_{k=0}^{\infty}$ is not Besselian in $L^2(0,1)$ space.*

This example proves the Theorem 1 because $u(2^k x) = W_0^k u(x)$, $k=0,1,\dots$ contains in Walsh type affine system $\{u_n\}_{n=0}^{\infty}$, so $\{u_n\}_{n=0}^{\infty}$ with its part is not Besselian. Here follows the proof of Theorem 2.

Proof. Consider Fourier-Walsh series $u = \sum_{n=1}^{\infty}(u,w_n)w_n$ for the function $u \in L_0^2(0,1)$. Using the natural bijection between the positive integer set $\mathbb{N}$ and the family $\mathbb{A}$, we represent the series in the form of $u = \sum_{\alpha\in\mathbb{A}}(u,w_\alpha)w_\alpha$. Let $|\alpha| = k$ be the length of a multi-index $\alpha = (\alpha_0,\dots,\alpha_{k-1})$. Assume that

$$U_k = \Big(\sum_{|\alpha|=k} |(u,w_\alpha)|^2\Big)^{1/2} = \Big(\sum_{n=2^k}^{2^{k+1}-1} |(u,w_n)|^2\Big)^{1/2}, \qquad k=0,1,\dots.$$

For any finite sequence $\{c_n\}_{n\in\mathbb{N}} = \{c_\beta\}_{\beta\in\mathbb{A}}$ we have

$$\sum_{n=1}^{\infty} c_n u_n = \sum_{\beta\in\mathbb{A}} c_\beta W^\beta u = \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\sum_{\beta\in\mathbb{A}} c_\beta (u,w_\alpha) W^\beta W^\alpha w.$$

Note that for any fixed $k=0,1,\dots,$ the $\{W^\beta W^\alpha w\}_{|\alpha|=k,\beta\in\mathbb{A}}$ family is orthonormal because it consists of pairwise different Walsh functions. Therefore, by Parseval's identity,

$$\Big\|\sum_{n=1}^{\infty} c_n u_n\Big\| \le \sum_{k=0}^{\infty}\Big\|\sum_{|\alpha|=k}\sum_{\beta\in\mathbb{A}} c_\beta(u,w_\alpha)W^\beta W^\alpha w\Big\| = \sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}|(u,w_\alpha)|^2\sum_{\beta\in\mathbb{A}}|c_\beta|^2\Big)^{1/2},$$

we obtain the estimate

$$\Big\|\sum_{n=1}^{\infty} c_n u_n\Big\| \le U\Big(\sum_{n=1}^{\infty}|c_n|^2\Big)^{1/2}, \tag{3}$$

where $U = \sum_{k=0}^{\infty} U_k$ is constant. Then we show that for a Lipchitz function $u(x)$ we have $U<\infty$. It is evident that Walsh functions $\{w_n\}_{n=2^k}^{2^{k+1}-1}$ and Haar functions $\{h_n\}_{n=2^k}^{2^{k+1}-1}$ from k-th block are connected by an orthogonal transform. Thus $\sum_{n=2^k}^{2^{k+1}-1}|(u,w_n)|^2 = \sum_{n=2^k}^{2^{k+1}-1}|(u,h_n)|^2$. For Fourier-Haar coefficients indexed by $n=2^k+j$, $j=0,\dots,2^k-1$ we have:

$$|(u,h_n)| \le 2^{k/2}\int_{j2^{-k}}^{(j+1/2)2^{-k}} |u(x)-u(x+2^{-k-1})|\,dx \le \frac{2^{k/2}L}{2^{k+1}2^{\alpha(k+1)}},$$

then $U_k \leq \frac{L/2}{2^{\alpha(k+1)}}$ and $U \leq \frac{L/2}{2^{\alpha}-1}$. Besselian property for the system $\{u_n\}_{n=0}^{\infty}$ follows from the estimate (3) in a common way:

$$\left(\sum_{n=1}^{\infty} |(f, u_n)|^2\right)^{1/2} = \sup \frac{|\sum_{n=1}^{\infty} c_n(f, u_n)|}{(\sum_{n=1}^{\infty} |c_n|^2)^{1/2}} \leq \|f\| \cdot \sup \frac{\|\sum_{n=1}^{\infty} c_n u_n\|}{(\sum_{n=1}^{\infty} |c_n|^2)^{1/2}} \leq U\|f\|.$$

Finally,

$$\sum_{n=0}^{\infty} |(f, u_n)|^2 \leq (1 + U^2)\|f\|^2.$$

□

Remark 2.4. *Walsh type affine systems are connected with Haar-type affine systems (systems of dilates and translates of one function). Conditions for Besselian property, Riesz basicity and basicity in $L^p(0,1)$ spaces, $1 \leq p < \infty$, are obtainted in papers [3], [8], [9].*

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