Filomat 31:4 (2017), 953–961
DOI 10.2298/FIL1704953E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Impulsive Fractional Differential Inclusions with Flux Boundary Conditions

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Abstract. In this work we investigate some existence results for solutions of a boundary value problem for impulsive fractional differential inclusions supplemented with fractional flux boundary conditions by applying Bohnenblust-Karlin's fixed point theorem for multivalued maps.

1. Introduction

This article is related to the existence of the solutions to the boundary value problem (BVP for short), for the following impulsive fractional differential inclusions,

$$^{C}D^{\alpha}y(t) \in F(t, y(t)), \;\; t \in J := [0, T], t \neq \theta_k, \; 1 < \alpha \leq 2, \tag{1}$$

$$\Delta y(\theta_k) = I_k(y(\theta_k^-)), \;\; \Delta y'(\theta_k) = I_k^*(y(\theta_k^-)), \; k = 1, 2, ..., p, \tag{2}$$

$$y'(0) = \lambda {}^{C}D^{\beta}y(T), \;\; y(0) + y(T) = \gamma g(y'), \tag{3}$$

where $^{C}D^{\alpha}$ is the Caputo fractional derivative, $F : J \times R \to \mathcal{P}(R)$ is compact convex valued multivalued map ($\mathcal{P}(R)$ is the family of all non-empty subsets of R), $I_k, I_k^*, g \in C(R, R)$, $\Gamma(2-\beta) \neq \lambda T^{1-\beta}$ with $\beta \in (0, 1]$ and γ, λ are real constants, $\Delta y(\theta_k) = y(\theta_k^+) - y(\theta_k^-)$ with $y(\theta_k^+) = \lim_{h\to 0^+} y(\theta_k + h)$ and $y(\theta_k^-) = \lim_{h\to 0^-} y(\theta_k + h)$, $\Delta y'(\theta_k)$ has a similar meaning for $y'(t)$, and $\{\theta_k\}_{k=1}^{p}$ is a finite strictly increasing sequence of impulse points θ_k such that $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_p < \theta_{p+1} = T$.

Here, $g(y')$ may be given by

$$g(y') = \sum_{i=1}^{p} \eta_i y^{'}(\xi_i)$$

where η_i, $i = 1, 2, ..., p$ are given constants and $\xi_i \in (\theta_l, \theta_{l+1})$; l is non-negative integer i.e. $0 \leq l \leq p$.

Plenty of studies have been dedicated to the issue of fractional order impulsive differential equations and inclusions by many scientists. That is why, it is owing to the fact that each of fractional calculus and impulsive theory serves very practical instruments for mathematical modeling of many concepts in different branches of science and engineering [1–7]. See [8–21] for some recent works on fractional differential

2010 *Mathematics Subject Classification*. Primary 26A33; Secondary 34A60, 34K10, 34K45
Keywords. Caputo fractional derivative, Existence and uniqueness, Fixed point theorem, Impulsive differential inclusion.
Received: 01 December 2015; Revised: 15 March 2016; Accepted: 16 March 2016
Communicated by Allaberen Ashyralyev (Guest Editor)
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equations and inclusions, and see [22–31] for the ones on impulsive fractional differential equations and inclusions.

Come to that, the authors in [32] and [33] have considered boundary value problems for fractional differential inclusions with flux boundary conditions recently. To be precise, the flux boundary condition $x'(0) = b^{C}D^{\beta}x(1)$ in [32] gives the proportional relationship between ordinary flux $x'(0)$ at the left end point of the given interval $[0,1]$ and a fractional flux ${}^{C}D^{\beta}x(1)$ at the right end point of it, where $\beta \in (0,1]$. Motivatedly, we shall be interested in a class of boundary value problems for the *impulsive* fractional differential inclusions with flux boundary conditions as in (1-3).

The rest of the paper is outlined as follows. In Section 2, we present some notations and preliminary results about fractional calculus and multivalued maps to be used in the sequent sections. In Section 3, we discuss some existence results for solutions of BVP (1-3) by means of Bohnenblust-Karlin's fixed point theorem for multivalued maps.

2. Preliminaries

Definition 2.1. *([1, 2]) The fractional (arbitrary) order integral of the function $h \in L^1(J, R_+)$ of order $\alpha \in R_+$ is defined by*

$$I_{0^+}^{\alpha}h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. *([1, 2]) For a function h given on the interval J, Caputo fractional derivative of order $\alpha > 0$ is defined by*

$${}^{C}D_{0^+}^{\alpha}h(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}h^{(n)}(s)ds, \ n = [\alpha]+1,$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.3. *([1]) Let $\alpha > 0$, $h(t) \in C^n[a,b]$, then*

$$I_{a^+}^{\alpha C}D_{a^+}^{\alpha}h(t) = h(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + ... + c_{n-1}(t-a)^{n-1},$$

for some $c_i \in R$, $i = 0,1,2,...,n-1$, $n = [\alpha]+1$.

Let us set $J_0 = [\theta_0, \theta_1]$, $J_1 = (\theta_1, \theta_2], ..., J_{k-1} = (\theta_{k-1}, \theta_k]$, $J_k = (\theta_k, \theta_{k+1}]$, $J' := [0,T] \backslash \{\theta_1, \theta_2, ..., \theta_p\}$ and define the set of functions:

$PC(J,R) = \{y : J \to R : y \in C((\theta_k, \theta_{k+1}], R),\ k = 0,1,2,...,p$ and there exist $y(\theta_k^+)$ and $y(\theta_k^-)$, $k = 1,2,...,p$ with $y(\theta_k^-) = y(\theta_k)\}$ and

$PC^1(J,R) = \{y \in PC(J,R),\ y' \in C((\theta_k, \theta_{k+1}], R),\ k = 0,1,2,...,p$ and there exist $y'(\theta_k^+)$ and $y'(\theta_k^-)$, $k = 1,2,...,p$ with $y'(\theta_k^-) = y'(\theta_k)\}$ which is a Banach space with the norm $\left\|y\right\| = \sup_{t\in J}\left\{\left\|y\right\|_{PC}, \left\|y'\right\|_{PC}\right\}$, where $\left\|y\right\|_{PC} := \sup\left\{\left|y(t)\right| : t \in J\right\}$.

Let $L^1(J,R)$ denote the Banach space of measurable functions $x : J \to R$ which are Lebesque integrable with the norm

$$\|x\|_{L^1} = \int_0^T |x(t)|\, dt \text{ for all } x \in L^1(J,R).$$

Now, we recall some basic facts of multivalued maps. See the books of Gorniewicz [34], Aubin and Frankowska[35], Deimling [36], and Hu and Papageorgiou[37]:

For a Banach space $(X, \|.\|)$, denote:

$$\mathcal{P}(X) = \{Y \subseteq X : Y \neq \emptyset\}, \mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\},$$

$$\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}, \mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is convex}\},$$

$$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}, \mathcal{P}_{cv,cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is convex and compact}\}.$$

A multivalued map $G : X \to \mathcal{P}(X)$ has convex(closed) values if $G(x)$ is convex(closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in B}\left\{\sup\left\{\left\|y\right\| : y \in G(x)\right\}\right\} < \infty$).

A multivalued map $G : [0,1] \to \mathcal{P}_{cl}(X)$ is said to be measurable if for every $x \in X$, the function $Y : [0,1] \to X$ defined by $Y(t) : dist(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\}$ is Lebesque measurable.

A multivalued map $F : J \times X \to \mathcal{P}(X)$ is said to be L^1-Caratheodory if

i) $t \to F(t, u)$ is measurable for each $u \in X$,

ii) $u \to F(t, u)$ is upper-semi continuous for almost all $t \in J$,

iii) For each $q > 0$, there exists $\phi_q \in L^1(J, R_+)$ such that

$$\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq \phi_q(t)$$

for all $\|u\| \leq q$ and for almost all $t \in J$.

For a function $x \in PC^1(J, R)$, we define the set of selections of F by

$$S_{F,x} = \left\{v \in L^1(J, R) : v(t) \in F(t, x) \text{ for a.e. } t \in J\right\}.$$

Lemma 2.4. *([38])Let $F : J \times R \to \mathcal{P}_{cv,\, cp}(R)$ be L^1-Caratheodory multivalued map with $S_{F,x} \neq \emptyset$ and let $\mathcal{L}$ be a linear continuous mapping from $L^1(J, R_+)$ to $C(J, R)$, then the operator*

$$\begin{aligned}\mathcal{L} \circ S_F \quad &: \quad C(J, R) \to \mathcal{P}_{cp,\, c}(C(J, R)) \\ & \qquad x \longmapsto (\mathcal{L} \circ S_F)(x) := \mathcal{L}(S_{F,x})\end{aligned}$$

is a closed graph operator in $C(J, R) \times C(J, R)$.

Proposition 2.5. *([34])Assume $\varphi : X \to Y$ is a multivalued map such that $\varphi(X) \subset K$ and the graph Γ_φ of φ is closed, where K is a compact set. Then φ is upper semi continuous.*

Lemma 2.6. *(Bohnenblust-Karlin [39]) Let X be a Banach space, D a nonempty subset of X, which is bounded, closed, and convex. Suppose $G : D \to \mathcal{P}(X)$ is u.s.c. with closed, convex values, and such that $G(D) \subset D$ and $\overline{G(D)}$ compact. Then G has a fixed point.*

3. Main Results

Definition 3.1. *A function $y \in PC^1(J, R)$ with its α-derivative existing on J' is said to be a solution of (1-3) if there exists a function $v(t) \in S_{F,y}$ which holds the equation ${}^C D^\alpha y(t) = v(t)$ for a.e. $t \in J'$ where the conditions*

$$\begin{aligned}\Delta y(\theta_k) &= I_k(y(\theta_k^-)),\ \Delta y'(\theta_k) = I_k^*(y(\theta_k^-)),\ k = 1, 2, ..., p, \\ y'(0) &= \lambda^C D^\beta y(T),\ y(0) + y(T) = \gamma g(y'),\end{aligned}$$

are satisfied for y.

Lemma 3.2. *Let $v \in L^1(J,R)$. A function $y(t) \in PC^1(J,R)$ is a solution of the fractional integral equation, for $t \in J_k$,*

$$
\begin{aligned}
y(t) \;=\;& \frac{\gamma}{2}g(y') + \frac{1}{2}\sum_{i=1}^{k}\left(\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + I_i(y(\theta_i^-))\right)\\
&-\frac{1}{2}\sum_{i=1}^{k}(T-\theta_i)\left(\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds + I_i^*(y(\theta_i^-))\right)\\
&-\frac{1}{2}\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + \int_{\theta_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\\
&+\sum_{i=1}^{k}(t-\theta_i)\left(\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds + I_i^*(y(\theta_i^-))\right) + \left(t-\frac{T}{2}\right)\frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta)-\lambda T^{1-\beta}}\\
&\times\left[\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}v(s)ds + \sum_{i=1}^{k}\frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)}\left(\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds + I_i^*(y(\theta_i^-))\right)\right]
\end{aligned}
\tag{4}
$$

if and only if $y(t)$ is a solution of the fractional BVP

$$
\begin{cases}
{}^{C}D^{\alpha}y(t) = v(t), \; t \in J := [0,T], \; t \neq \theta_k, \; 1 < \alpha \leq 2,\\
\Delta y(\theta_k) = I_k(y(\theta_k^-)), \; \Delta y'(\theta_k) = I_k^*(y(\theta_k^-)), \; k = 1,2,\ldots,p,\\
y(0) + y(T) = \gamma g(y'), \; y'(0) = \lambda\, {}^{C}D^{\beta}y(T).
\end{cases}
\tag{5}
$$

Proof. Assume that $y(t)$ is a solution of the problem (5). Then from Lemma 2.3 we have

$$
y(t) = \begin{cases}
\int_{\theta_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + \sum_{i=1}^{k}\left[\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + I_i(y(\theta_i^-))\right]\\
+\sum_{i=1}^{k}(t-\theta_i)\left[\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds + I_i^*(y(\theta_i^-))\right] + c_0 + c_1 t.
\end{cases}
$$

Moreover, we find the constants c_0 and c_1 by utilizing the boundary conditions $y(0) + y(T) = \gamma g(y')$ and $y'(0) = \lambda^C D^{\beta}y(T)$. And we obtain the fractional integral equation (4) by substituting c_0 and c_1.

On the other hand, let $y(t)$ satisfy the equation (4). Then, the fractional equation ${}^C D^{\alpha}y(t) = v(t)$ in (5) can be obtained in view of Definition (2.2) together with the property $\left({}^C D^{\alpha} I^{\alpha} v\right)(t) = v(t)$, $\alpha > 0$ ([1], Lemma 2.21). Further, it can be easily seen that Eq. (4) holds the impulse and boundary conditions in (5). Hence, the solution y given by (4) satisfies (5). The proof is complete. □

For the sake of convenience, we rewrite the integral equation (4): for $t \in J_k$,

$$
\begin{aligned}
y(t) \;=\;& M(t) + \frac{1}{2}\sum_{i=1}^{k}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + \sum_{i=1}^{k}\frac{2t-T-\theta_i}{2}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds\\
&-\frac{1}{2}\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + \int_{\theta_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + \left(t-\frac{T}{2}\right)\frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta)-\lambda T^{1-\beta}}\\
&\times\left(\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}v(s)ds + \sum_{i=1}^{k}\frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds\right)
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
M(t) \;=\;& \frac{\gamma}{2}g(y') + \frac{1}{2}\sum_{i=1}^{k}I_i(y(\theta_i^-)) + \sum_{i=1}^{k}\frac{2t-T-\theta_i}{2}I_i^*(y(\theta_i^-))\\
&+\left(t-\frac{T}{2}\right)\frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta)-\lambda T^{1-\beta}}\sum_{i=1}^{k}\frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)}I_i^*(y(\theta_i^-)).
\end{aligned}
$$

Theorem 3.3. *Assume that*

A1)The function $F : J \times R \to \mathcal{P}_{cv,cp}(R)$ is L^1-Caratheodory, A2)There exist a continuous non-decreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $m(t) \geq 0$, $\forall t \in J$ with $m^0 = \sup\{|m(t)| : t \in J\}$ such that $\|F(t,u)\| \leq m(t)\psi(|u|)$ for $\forall t \in J$, $\forall u \in R$ with

$$\lim_{x\to+\infty} \frac{\psi(x)}{x} = 0,$$

A3)There exist constants $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$ such that $|I_k(u)| \leq M_1$, $\left|I_k^*(u)\right| \leq M_2$ and $\left|g(u)\right| \leq M_3$ for all $u \in R$ and $k = 1, 2, ..., p$ with

$$M = \sup_{t\in J} |M(t)|$$

in view of M_1, M_2 and M_3.

Then, the BVP(1-3) has at least one solution.

Proof. Let us transform the problem (1)-(3) into a fixed point problem. By making use of (6) we consider the operator $\mathcal{N} : PC^1(J,R) \to \mathcal{P}\left(PC^1(J,R)\right)$ defined by $\mathcal{N}(y) = \left\{h \in PC^1(J,R)\right\}$ where, $t \in J_k$,

$$\begin{aligned} h(t) &= M(t) + \frac{1}{2}\sum_{i=1}^{k} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \sum_{i=1}^{k} \frac{2t - T - \theta_i}{2} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds \\ &\quad - \frac{1}{2}\int_{\theta_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \int_{\theta_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \left(t - \frac{T}{2}\right) \frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta) - \lambda T^{1-\beta}} \\ &\quad \times \left(\int_{\theta_k}^{T} \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)ds + \sum_{i=1}^{k} \frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds \right) \end{aligned}$$

for $v(t) \in S_{F,y}$. It is obvious that the fixed points of the operator $\mathcal{N}$ give solutions to the problem (1)-(3). Now, we shall apply Bohnenblust-Karlin's fixed point theorem (Lemma 2.6) in order to show that the operator $\mathcal{N}$ has fixed points. Then, let us try to satisfy the conditions of Lemma 2.6. The proof will be given in several steps for convenience.

***Step*1**: "$\mathcal{N}(y)$ is convex for each $y \in PC^1(J,R)$."

Let $h_1, h_2 \in \mathcal{N}(y)$ with $v_1, v_2 \in S_{F,y}$ such that, for $j = 1, 2$,

$$\begin{aligned} h_j(t) &= M(t) + \frac{1}{2}\sum_{i=1}^{k} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_j(s)ds + \sum_{i=1}^{k} \frac{2t - T - \theta_i}{2} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v_j(s)ds \\ &\quad - \frac{1}{2}\int_{\theta_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v_j(s)ds + \int_{\theta_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_j(s)ds + \left(t - \frac{T}{2}\right) \frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta) - \lambda T^{1-\beta}} \\ &\quad \times \left(\int_{\theta_k}^{T} \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v_j(s)ds + \sum_{i=1}^{k} \frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v_j(s)ds \right) \end{aligned}$$

then, for each $t \in J$ we have

$$\begin{aligned}[dh_1(t)+(1-d)h_2(t)] &= \frac{1}{2}\sum_{i=1}^{k}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)}[dv_1(s)+(1-d)v_2(s)]ds \\ &+\sum_{i=1}^{k}\frac{2t-T-\theta_i}{2}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}[dv_1(s)+(1-d)v_2(s)]ds \\ &-\frac{1}{2}\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}[dv_1(s)+(1-d)v_2(s)]ds \\ &+\int_{\theta_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[dv_1(s)+(1-d)v_2(s)]ds \\ &+\left(t-\frac{T}{2}\right)\frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta)-\lambda T^{1-\beta}} \\ &\times\left(\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}[dv_1(s)+(1-d)v_2(s)]ds\right. \\ &\left.+\sum_{i=1}^{k}\frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}[dv_1(s)+(1-d)v_2(s)]ds\right),\end{aligned}$$

where $0 \le d \le 1$.

Since $S_{F,y}$ is convex (i.e. $dv_1(s)+(1-d)v_2(s) \in S_{F,y}$ for $v_1, v_2 \in S_{F,y}$ and $0 \le d \le 1$) then $dh_1(t)+(1-d)h_2(t) \in \mathcal{N}(x)$.

***Step*2**: "There exists a constant $r_0 > 0$ such that we have $\mathcal{N}(\Omega_{r_0}) \subseteq \Omega_{r_0}$ for each $y \in \Omega_r = \{y(t) \in PC^1(J,R) : \|y\| \le r\}$ for any $r > 0$."

That is to say, Ω_r is a bounded closed convex set in $PC^1(J,R)$.

Let $y \in \Omega_r$ and $h \in \mathcal{N}(y)$ with $v \in S_{F,\ y}$, then for each $t \in J$ we obtain

$$\begin{aligned}\left|\mathcal{N}(y)(t)\right| &\le |M(t)|+\frac{1}{2}\sum_{i=1}^{k}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|\,ds+\sum_{i=1}^{k}\frac{|2t-T-\theta_i|}{2}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|v(s)|\,ds \\ &+\frac{1}{2}\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|\,ds+\int_{\theta_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|\,ds+\left|t-\frac{T}{2}\right|\frac{\left|\lambda\Gamma(2-\beta)\right|}{\left|\Gamma(2-\beta)-\lambda T^{1-\beta}\right|} \\ &\times\left(\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|v(s)|\,ds+\sum_{i=1}^{k}\frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|v(s)|\,ds\right)\end{aligned}$$

$$\begin{aligned}\left|\mathcal{N}(y)(t)\right| &\le M+m_0\psi(\|y\|)\left\{\frac{pT^\alpha}{2\Gamma(\alpha+1)}+\frac{pT^\alpha}{2\Gamma(\alpha)}+\frac{T^\alpha}{2\Gamma(\alpha+1)}+\frac{T^\alpha}{2\Gamma(\alpha+1)}\right. \\ &\left.+\frac{|\lambda|\,T\Gamma(2-\beta)}{2\left|\Gamma(2-\beta)-\lambda T^{1-\beta}\right|}\left(\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{pT^{\alpha-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}\right)\right\}\end{aligned}$$

$$\left\|\mathcal{N}(y)(t)\right\| \le M+\psi(\|y\|)\frac{m_0}{2}\left\{\frac{T^\alpha(p+p\alpha+2)}{\Gamma(\alpha+1)}+\frac{|\lambda|\,T^{\alpha-\beta+1}\left[(1-\beta)B(\alpha,1-\beta)+p\right]}{\left|\Gamma(2-\beta)-\lambda T^{1-\beta}\right|\Gamma(\alpha)}\right\}, \tag{7}$$

where $B(m,n)$ is Beta function.

Since $\lim_{x\to+\infty}\frac{\psi(x)}{x}=0$ from (A2), there exists a positive number r_0 which is sufficiently large such that

$$r_0 > M+\psi(r_0)\frac{m_0}{2}\left\{\frac{T^\alpha(p+p\alpha+2)}{\Gamma(\alpha+1)}+\frac{|\lambda|\,T^{\alpha-\beta+1}\left[(1-\beta)B(\alpha,1-\beta)+p\right]}{\left|\Gamma(2-\beta)-\lambda T^{1-\beta}\right|\Gamma(\alpha)}\right\}.$$

This, together with (7), implies that $\left\|\mathcal{N}(y)(t)\right\| < r_0$ whenever $\left\|y\right\| \le r_0$. Thus, $\mathcal{N}(\Omega_{r_0}) \subseteq \Omega_{r_0}$ now is shown.

***Step*3**: "$\mathcal{N}$ is a compact multi-valued map."

It is clear that $\mathcal{N}(\Omega_r)$ is uniformly bounded from (7). It remains to show that $\mathcal{N}(\Omega_r)$ is equicontinuous. Assume that $l_1, l_2 \in J$, $l_1 < l_2$. Let $y \in \Omega_r$ and $h \in \mathcal{N}(y)$ with $v \in S_{F,y}$ such that

$$\begin{aligned} h'(t) &= M'(t) + \sum_{i=1}^{k} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds \\ &+ \int_{\theta_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds + \frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta) - \lambda T^{1-\beta}} \\ &\times \left(\int_{\theta_k}^{T} \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)ds + \sum_{i=1}^{k} \frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds \right) \end{aligned}$$

where

$$\begin{aligned} |h'(t)| &\le |M'(t)| + \sum_{i=1}^{k} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |v(s)|\, ds \\ &+ \int_{\theta_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |v(s)|\, ds + \frac{|\lambda|\,\Gamma(2-\beta)}{\left|\Gamma(2-\beta) - \lambda T^{1-\beta}\right|} \\ &\times \left(\int_{\theta_k}^{T} \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |v(s)|\, ds + \sum_{i=1}^{k} \frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |v(s)|\, ds \right) \end{aligned}$$

$$\begin{aligned} |h'(t)| &\le |M'(t)| + m_0\psi(\left\|y\right\|) \left\{ \frac{pT^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \right. \\ &\left. + \frac{|\lambda|\,\Gamma(2-\beta)}{\left|\Gamma(2-\beta) - \lambda T^{1-\beta}\right|} \left(\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{pT^{\alpha-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)} \right) \right\} : K. \end{aligned}$$

Then for each $t \in J$ we have

$$\begin{aligned} \left|\mathcal{N}(y)(l_2) - \mathcal{N}(y)(l_1)\right| &\le \int_{l_1}^{l_2} |h'(s)|\, ds \\ &\le (l_2 - l_1)K \end{aligned}$$

implying that $\mathcal{N}$ is equicontinuous on J since the right-hand side of the inequality tends to zero as $l_1 \to l_2$. Thus, as a consequence of Arzela-Ascoli theorem, the operator $\mathcal{N} : PC^1(J,R) \to \mathcal{P}\left(PC^1(J,R)\right)$ is a compact multivalued map.

***Step*4**: "$\mathcal{N}$ has a closed graph."

Let $y_n \to x_*$, $h_n \to h_*$ and $h_n \in \mathcal{N}(y_n)$ with $v_n \in S_{F,\, y_n}$ such that for each $t \in J$

$$\begin{aligned} h_n(t) &= M_n(t) + \frac{1}{2}\sum_{i=1}^{k} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds + \sum_{i=1}^{k} \frac{2t - T - \theta_i}{2} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v_n(s)ds \\ &- \frac{1}{2} \int_{\theta_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds + \int_{\theta_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds + \left(t - \frac{T}{2}\right) \frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta) - \lambda T^{1-\beta}} \\ &\times \left(\int_{\theta_k}^{T} \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v_n(s)ds + \sum_{i=1}^{k} \frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)} \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v_n(s)ds \right) \end{aligned}$$

Then we have to show that there exists $v_* \in S_{F,\, y_*}$ in order to prove that $h_* \in \mathcal{N}(y_*)$ such that for each $t \in J$

$$\begin{aligned} h_*(t) &= M_*(t) + \frac{1}{2}\sum_{i=1}^{k}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)}v_*(s)ds + \sum_{i=1}^{k}\frac{2t - T - \theta_i}{2}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)}v_*(s)ds \\ &\quad -\frac{1}{2}\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v_*(s)ds + \int_{\theta_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_*(s)ds + \left(t - \frac{T}{2}\right)\frac{\lambda\Gamma(2-\beta)}{\Gamma(2-\beta) - \lambda T^{1-\beta}} \\ &\quad \times\left(\int_{\theta_k}^{T}\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}v_*(s)ds + \sum_{i=1}^{k}\frac{(T-\theta_i)^{1-\beta}}{\Gamma(2-\beta)}\int_{\theta_{i-1}}^{\theta_i}\frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)}v_*(s)ds\right) \end{aligned} \quad (8)$$

Let us consider the continuous linear operator $\mathcal{L} : L^1(J, R_+) \to C(J, R)$,

$$v \to (\mathcal{L}v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds.$$

Obviously, by the continuity of $M(t)$, we have

$$\|h_n(t) - M_n(t) - [h_*(t) - M_*(t)]\| \to 0$$

as $n \to \infty$.

It results from Lemma 2.4 that $\mathcal{L} \circ S_F$ is a closed graph operator. Furthermore, since $(h_n(t) - M_n(t)) \in \mathcal{L}(S_{F,y_n})$ and $y_n \to y_*$, Lemma 2.4 implies that the relation (8) holds for some $v_* \in S_{F,\, y_*}$.

As a consequence of Proposition 2.5, $\mathcal{N}$ is an upper semi-continuous compact map with convex closed values. Therefore, thanks to Bohnenblust-Karlin's fixed theorem (Lemma 2.6), we conclude that $\mathcal{N}$ has a fixed point $y \in PC^1(J, R)$ which is a solution of the problem (1-3). □

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